Chapter 4

An operational matrix approach to solve a two-dimension variable-order reaction advection diffusion equation with Vieta-Fibonacci polynomials

4.1 Introduction

Groundwater pollution is an interesting and important area of research in the context of science and engineering, and it directly affects living species. The substances like industrial wastes, fertilizers, pesticides, and road salt diffuse into the groundwater through the natural porous mechanism and become the main sources of groundwater contamination. Generally, there are two main approaches to studying the contamination of groundwater. The first approach is based on the development of software for numerical simulation of the system of groundwater, for example, the geographic information system [133]. The second approach depends on the formulation and solution of mathematical models of solute transport in groundwater through porous media. The mathematical models of contaminant/solute transport in groundwater are governed by the well-known partial differential equation, i.e., reaction advection-diffusion equation. The development of various models associated with solute transport can be seen in ([134], [135], [136], [137]). Some other applications of reation advection diffusion equation can be found in chemical reactions, water vapor transport in the atmosphere, weather prediction, energy, and mass transfer, etc. Several studies ([138], [139]) illustrate that the standard (classical) reaction advection diffusion equation is unable to capture many real situations due to the occurrence of the anomalous diffusion process. Anomalous diffusion equation is governed by the fractional diffusion equation, and it is observed that the particles spread faster in the case of anomalous diffusion process than the classical diffusion process ([140]). In the area of flows in porous media, several researchers reported papers related to constant-order fractional reaction advection diffsion equation, some of them are given in ([141], [142], [143], [144], [145], [128]). [146] and [91] presented fractional-order reaction advection diffuion equations arising in porous media and discussed its solution by using operational matrix method and collocation scheme. In this field, the time derivative of fractional-order is used to model the particles which remain static for an extended period of time, and the space-derivative of fractional order describes the bulk motions passing through extremely conductive layers or fractures ([147]).

Recently, variable-order reaction advection diffusion equation gained a great importance because it is the generalization of constant-order fractional reaction advection diffusion equation ([148], [149], [150]), and it has the potential to describe many real situations which are not covered by the standard reaction advection diffuion equations. Mostly, variable-order reaction advection diffuion equations do not have analytical solutions, therefore several different numerical schemes have been developed ([151], [152], [153]) to solve these types of problems. In 2019, [154] discussed a numerical solution of non-linear variable-order two-dimensions reaction-diffusion equation involving Mittag-Leffer non-singular kernel. [155] utilized a meshless scheme to present a numerical solution to the two-dimensions variable-order reaction advection diffuion equations. In [156], the authors used the radial basis function for solving two-dimensional variable-order version of advection-diffusion equation. [157] used Haar wavelet and finite differences to present a numerical solutions of one-dimensions and two-dimensions advection-dispersion and diffusion equation with variable-order derivative. In 2021, [158] discussed a messless approach to solve advection-diffusion equations involving variable-order time-fractional derivative. [127] developed a fractional version of two-dimensions reaction advection diffuion equations by using variable-order fractional derivatives in the Heydari-Hosseininia sense and its solution is calculated by a hybrid approach. [159] introduced orthonormal shifted discrete Legendre polynomials to solve the problem of coupled system variable-order reaction-advection diffusion equations. [126] presented a numerical solution with the aid of finite difference and Fibonacci collocation approach to the multi-term variable-order fractional reaction advection diffuion equations in the heterogeneous medium. [129] discussed a sufficiently accurate numerical method to solve variable-order time fractional advection-reaction-subdiffusion equations. [160] presented a messfree scheme to deal the variable-order advection-diffusion equations.

In this study, motivated by previous literature of fractional reaction advection diffuion equations, the authors consider the following time-space reaction advection diffuion equations of variable-order:

$$\begin{aligned}\frac{\partial^{\alpha(\mathcal{S},t)}u(\mathcal{S},t)}{\partial t^{\alpha(\mathcal{S},t)}} =& - a(u,\mathcal{S},t)\frac{\partial}{\partial x}\left(-\frac{\partial^{\beta(\mathcal{S},t)}u(\mathcal{S},t)}{\partial x^{\beta(\mathcal{S},t)}}\right) - c(u,\mathcal{S},t)\frac{\partial^{\delta(\mathcal{S},t)}u(\mathcal{S},t)}{\partial x^{\delta(\mathcal{S},t)}}\\ & - b(u,\mathcal{S},t)\frac{\partial}{\partial y}\left(-\frac{\partial^{\gamma(\mathcal{S},t)}u(\mathcal{S},t)}{\partial y^{\gamma(\mathcal{S},t)}}\right) - d(u,\mathcal{S},t)\frac{\partial^{\sigma(\mathcal{S},t)}u(\mathcal{S},t)}{\partial y^{\sigma(\mathcal{S},t)}}\\ & + \psi(u,\mathcal{S},t),\end{aligned} \tag{4.1}$$

with the following conditions:

$$\begin{aligned}
u(x,y,0) &= f_1(x,y),\\
u(0,y,t) &= f_2(y,t),\\
u(1,y,t) &= f_3(y,t),\\
u(x,0,t) &= f_4(x,t),\\
u(x,1,t) &= f_5(x,t), \qquad (4.2)
\end{aligned}$$

where $0 < \alpha(\mathcal{S},t),\ \beta(\mathcal{S},t),\ \gamma(\mathcal{S},t),\ \delta(\mathcal{S},t),\ \sigma(\mathcal{S},t) \leq 1$, $\mathcal{S} = (x,y) \in$ [0,1] × [0,1], $0 < t$ and $u(\mathcal{S},t)$ is the concentration of solute in fluid, $\alpha(\mathcal{S},t)$, $\beta(\mathcal{S},t)$, $\gamma(\mathcal{S},t)$, $\delta(\mathcal{S},t)$ and $\sigma(\mathcal{S},t)$ are respectively time and space fractional derivatives of variable-order, $\psi(\mathcal{S},t)$ is the forced term, $f_1(x,y)$ is a known function which represents the initial solute concentration, $f_2(y,t)$, $f_3(y,t)$, $f_4(x,t)$ and $f_5(x,t)$ are the known functions which represent the solute concentration at the boundary points at any time t. When eq 4.1 is in conservation form (advection term is govern by fractional Fick's law) then $\beta(\mathcal{S},t) = \delta(\mathcal{S},t)$ and $\gamma(\mathcal{S},t) = \sigma(\mathcal{S},t)$.

The following is a breakdown of the paper's structure. In section 4.2, the approximation of an arbitrary function and the operational matrices of differentiation for the shifted Vieta-Fibonacci polynomials are described. A brief description of the scheme for the general case is discussed in section 4.3. Section 4.4 will wrap up the conversation on convergence and error analysis. The numerical computation of this study is described in section 4.5, and finally, the conclusion is presented in section 4.6.

4.2 Properties of Shifted Vieta-Fibonacci Polynomials

A brief discussion about shifted Vieta-Fibonacci polynomials has been given in chapter 2.

4.2.1 Approximation of an Arbitrary Function

Let us suppose $\varphi(t) = \left[VF_1^*(t), ..., VF_{m+1}^*(t)\right]^T \in L^2[0,1]$ is the set of shifted Vieta-Fibonacci polynomials. Then a function $u(t) \in L^2[0,1]$ can be written in terms of shifted Vieta-Fibonacci polynomials as:

$$u(t) = \sum_{i=1}^{\infty} c_i VF_i^*(t), \tag{4.3}$$

where the coefficients c_i are given by

$$c_i = \frac{8}{\pi} \int_0^1 u(t) VF_i^*(t) \chi(t) dt. \tag{4.4}$$

To approximate $u(t)$, one can truncate the above infinte series as follows:

$$u(t) \simeq \sum_{i=1}^{m+1} c_i VF_i^*(t) = C^T \varphi(t), \tag{4.5}$$

where notation T means transpose and

$$C = \left[c_1, c_2, ..., c_{m+1}\right]^T, \tag{4.6}$$

$$\varphi(t) = \left[VF_1^*(t), ..., VF_{m+1}^*(t)\right]^T. \tag{4.7}$$

Similiarly, an arbitrary function $u(\mathcal{S},t) \in L^2[0,1] \times L^2[0,1] \times L^2[0,1]$ can be written in terms of shifted Vieta-Fibonacci polynomials as:

$$u(\mathcal{S},t) \simeq \sum_{i=1}^{m+1}\sum_{j=1}^{m+1}\sum_{k=1}^{m+1} c_{ijk} VF_i^*(x) VF_j^*(y) VF_k^*(t) = \varphi^T(t)\mathbb{A}(\varphi(x)\otimes\varphi(y)), \tag{4.8}$$

where $\mathbb{A}$=$[c_{ijk}]$ is the $(m+1)\times(m+1)^2$ matrix whose entries are

$$c_{ijk} = \frac{512}{\pi^3}\int_0^1\int_0^1\int_0^1 u(\mathcal{S},t) VF_i^*(x) VF_j^*(y) VF_t^* \chi(x)\chi(y)\chi(t)dxdydt. \tag{4.9}$$

The first-order derivative of the shifted Vieta-Fibonacci vector $\varphi(t)$ is defined as follows:

$$\frac{d\varphi(t)}{dt} = \mathbf{D}\varphi(t), \tag{4.10}$$

where $\varphi(.)$ is defined in Eq. (4.7) and $\mathbf{D}$ is the $(m+1)\times(m+1)$ operational matrix of the shifted Vieta-Fibonacci vector $\varphi(t)$ for first-order derivative with the following entries:

$$\mathbf{D} = \begin{cases} 4j, & j = 1,2,...,i-1, \;\; i = 2,3,...(m+1), \;\; i+j \text{ is odd}, \\ 0, & \text{otherwise}, \end{cases} \tag{4.11}$$

4.2.2 The Operational Matrix for the Derivatives of Variable-Order

Lemma. Let $VF_i^*(x)$ be a shifted Vieta-Fibonacci polynomials then

$$D_x^{\mu(\mathcal{S},t)} VF_i^*(x) = 0, \quad i = 1, ..., q-1, \quad q-1 < \mu(\mathcal{S},t) \leq q, \quad q \in \mathbb{N}. \tag{4.12}$$

In the following theorem, we generalize the operational matrix of shifted Vieta-Fibonacci polynomials for the fractional derivative of variable-order.

Theorem 4.1. *Let $\varphi(x)$ be shifted Vieta-Fibonacci vector defined as Eq. (4.7) and also suppose $\mu(\mathcal{S},t) > 0$ then*

$$D_x^{\mu(\mathcal{S},t)} \varphi(x) = \Psi_x^{\mu(\mathcal{S},t)} \varphi(x),$$

where $\Psi_x^{\mu(\mathcal{S},t)}$ is the $(m+1) \times (m+1)$ operational matrix of Caputo fractional derivative of variable-order, $\mu(\mathcal{S},t)$ that is defined as follows:

$$\Psi_x^{\mu(\mathcal{S},t)} = x^{-\mu(\mathcal{S},t)} \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum_{i=q}^{q} \Theta_{q,l,i,1}^{\mu(\mathcal{S},t)} & \sum_{i=q}^{q} \Theta_{q,l,i,2}^{\mu(\mathcal{S},t)} & \cdots & \sum_{i=q}^{q} \Theta_{q,l,i,m+1}^{\mu(\mathcal{S},t)} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{i=q}^{n} \Theta_{n,l,i,1}^{\mu(\mathcal{S},t)} & \sum_{i=q}^{n} \Theta_{n,l,i,2}^{\mu(\mathcal{S},t)} & \cdots & \sum_{i=q}^{n} \Theta_{n,l,i,m+1}^{\mu(\mathcal{S},t)} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{i=q}^{m+1} \Theta_{m+1,l,i,1}^{\mu(\mathcal{S},t)} & \sum_{i=q}^{m+1} \Theta_{m+1,l,i,2}^{\mu(\mathcal{S},t)} & \cdots & \sum_{i=q}^{m+1} \Theta_{m+1,l,i,m+1}^{\mu(\mathcal{S},t)} \end{pmatrix}$$

where

$$\Theta^{\mu(\mathcal{S},t)}_{n,l,i.j} = \frac{4}{\sqrt{\pi}} \sum_{l=0}^{j} \frac{(-1)^{(n+j-l-i)}(2)^{2(l+i)}\Gamma(n+j+l+i+2)\Gamma(i+1)\Gamma(i+l+3/2)}{\Gamma(n-i)\Gamma(j-l)\Gamma(2l+2)\Gamma(2i+2)\Gamma(i+1-\mu(\mathcal{S},t))(\Gamma(i+l+3/2))}.$$

Proof. By using the operator $D_x^{\mu(\mathcal{S},t)}$ on the function $VF_n^*(x)$, we get

$$\begin{aligned} D_x^{\mu(\mathcal{S},t)} VF_n^*(x) &= \sum_{i=0}^{n} \frac{(-1)^{n-i-1}(2)^{2i}\Gamma(n+i+1)}{\Gamma(n-i)\Gamma(2i+2)} D_x^{\mu(\mathcal{S},t)}(x^i), \quad n = q, ..., m+1, \\ &= \sum_{i=q}^{n} \frac{(-1)^{n-i-1}(2)^{2i}\Gamma(n+i+1)\Gamma(i+1)}{\Gamma(n-i)\Gamma(2i+2)\Gamma(i+1-\mu(\mathcal{S},t))} x^{i-\mu(\mathcal{S},t)}, \\ &= x^{-\mu(\mathcal{S},t)} \sum_{i=q}^{n} \frac{(-1)^{n-i-1}(2)^{2i}\Gamma(n+i+1)\Gamma(i+1)}{\Gamma(n-i)\Gamma(2i+2)\Gamma(i+1-\mu(\mathcal{S},t))} x^{i}. \end{aligned} \tag{4.13}$$

Now, we approximate x^i as

$$x^i = \sum_{j=1}^{m+1} b_{ij} VF_j^*(x), \tag{4.14}$$

where

$$\begin{aligned} b_{ij} &= \frac{8}{\pi} \int_0^1 x^i \sqrt{x-x^2} VF_j^*(x) dx, \\ &= \frac{8}{\pi} \sum_{l=0}^{j} \frac{(-1)^{j-l-1}(2)^{2l}\Gamma(j+l+1)}{\Gamma(j-l)\Gamma(2l+2)} \int_0^1 x^{i+l}\sqrt{x-x^2}dx, \\ &= \frac{8}{\pi} \sum_{l=0}^{j} \frac{(-1)^{j-l-1}(2)^{2l}\Gamma(j+l+1)}{\Gamma(j-l)\Gamma(2l+2)} \left(\frac{\sqrt{\pi}}{2} \frac{\Gamma(i+l+3/2)}{\Gamma(i+l+3)} \right). \end{aligned} \tag{4.15}$$

Substituting Eq. (4.14) into Eq. (4.15), we get

$$D_x^{\mu(\mathcal{S},t)} VF_n^*(x) = x^{-\mu(\mathcal{S},t)} \sum_{i=q}^{n} \sum_{j=1}^{m+1} \frac{(-1)^{n-i-1}(2)^{2i}\Gamma(n+i+1)\Gamma(i+1)}{\Gamma(n-i)\Gamma(2i+2)\Gamma(i+1-\mu(\mathcal{S},t))} b_{ij} VF_j^*(x), \tag{4.16}$$

where $n = q, ..., m+1$.

Now, let us write

$$D_x^{\mu(\mathcal{S},t)} VF_n^*(x) = \sum_{j=1}^{m+1} \left(\sum_{i=q}^{n} \Theta_{n,l,i,j}^{\mu(\mathcal{S},t)} \right) VF_j^*(x), \quad n = q, ..., m+1, \tag{4.17}$$

where

$$\Theta_{n,l,i.j}^{\mu(\mathcal{S},t)} = \frac{4}{\sqrt{\pi}} \sum_{l=0}^{j} \frac{(-1)^{(n+j-l-i)}(2)^{2(l+i)}\Gamma(n+j+l+i+2)\Gamma(i+1)\Gamma(i+l+3/2)}{\Gamma(n-i)\Gamma(j-l)\Gamma(2l+2)\Gamma(2i+2)\Gamma(i+1-\mu(\mathcal{S},t))(\Gamma(i+l+3/2))}. \tag{4.18}$$

Let us write Eq. (4.17) in a vector form as

$$D_x^{\mu(\mathcal{S},t)} VF_n^*(x) \simeq \left[\sum_{i=q}^{n} \Theta_{n,l,i,1}^{\mu(\mathcal{S},t)}, \sum_{i=q}^{n} \Theta_{n,l,i,2}^{\mu(\mathcal{S},t)}, ..., \sum_{i=q}^{n} \Theta_{n,l,i,m+1}^{\mu(\mathcal{S},t)} \right] \varphi(x), \tag{4.19}$$

where $n = q, ..., m+1$.

Also according to Lemma, we have

$$D_x^{\mu(\mathcal{S},t)} VF_n^*(x) = [0, 0, ..., 0]\, \varphi(x), \quad n = 1, 2, ..., q-1. \tag{4.20}$$

The Eqs. (4.19) and (4.20) gives the desired result. □

4.3 Discription of the Present Method

To solve the Eqs. (4.1)-(4.2), we approximate $u(\mathcal{S}, t)$ shifted Vieta-Fibonacci polynomials as given below:

$$u(\mathcal{S}, t) \simeq \sum_{i=1}^{m+1} \sum_{j=1}^{m+1} \sum_{k=1}^{m+1} c_{ijk} VF_i^*(x) VF_j^*(y) VF_k^*(t), \tag{4.21}$$

where c_{ijk} are the unknown coefficients for $i = 1, ...m+1$, $j = 1, ...m+1$ and $k = 1, ...m+1$.

Now, we write

$$u(\mathcal{S}, t) = \varphi^T(t)\mathbb{A}(\varphi(x) \otimes \varphi(y)), \tag{4.22}$$

where $\mathbb{A} = [c_{ijk}]$ is the $(m+1) \times (m+1)^2$ matrix whose entries are unknowns and $\varphi(t) = [VF_1^*(t), VF_2^*(t), ..., VF_{m+1}^*(t)]^T$ is a column vector. Now, substituting

$$\begin{aligned}
\frac{\partial^{\alpha(\mathcal{S},t)}u}{\partial t^{\alpha(\mathcal{S},t)}} &= (D^{\alpha(\mathcal{S},t)}\varphi^T(t))\mathbb{A}(\varphi(x) \otimes \varphi(y)),\\
\frac{\partial}{\partial x}\left(\frac{\partial^{\beta(\mathcal{S},t)}u(\mathcal{S},t)}{\partial x^{\beta(\mathcal{S},t)}}\right) &= \varphi^T(t)\mathbb{A}(D^{1+\beta(\mathcal{S},t)}\varphi(x) \otimes \varphi(y)),\\
\frac{\partial}{\partial y}\left(\frac{\partial^{\gamma(\mathcal{S},t)}u(\mathcal{S},t)}{\partial y^{\gamma(\mathcal{S},t)}}\right) &= \varphi^T(t)\mathbb{A}(\varphi(x) \otimes D^{1+\gamma(\mathcal{S},t)}\varphi(y)),\\
\frac{\partial^{\delta(\mathcal{S},t)}u(\mathcal{S},t)}{\partial x^{\delta(\mathcal{S},t)}} &= \varphi^T(t)\mathbb{A}(D^{\delta(\mathcal{S},t)}\varphi(x) \otimes \varphi(y)),\\
\frac{\partial^{\sigma(\mathcal{S},t)}u(\mathcal{S},t)}{\partial y^{\sigma(\mathcal{S},t)}} &= \varphi^T(t)\mathbb{A}(\varphi(x) \otimes D^{\sigma(\mathcal{S},t)}\varphi(y)),
\end{aligned}$$

in the Eq. (4.1), we get

$$\begin{aligned}
&(D^{\alpha(\mathcal{S},t)}\varphi^T(t))\mathbb{A}(\varphi(x) \otimes \varphi(y))\\
&\qquad = a(\varphi^T(t)\mathbb{A}(\varphi(x) \otimes \varphi(y)), \mathcal{S}, t)\varphi^T(t)\mathbb{A}(D^{1+\beta(\mathcal{S},t)}\varphi(x) \otimes \varphi(y))\\
&\qquad + b(\varphi^T(t)\mathbb{A}(\varphi(x) \otimes \varphi(y)), \mathcal{S}, t)\varphi^T(t)\mathbb{A}(\varphi(x) \otimes D^{1+\gamma(\mathcal{S},t)}\varphi(y))\\
&\qquad - c(\varphi^T(t)\mathbb{A}(\varphi(x) \otimes \varphi(y)), \mathcal{S}, t)\varphi^T(t)\mathbb{A}(D^{\delta(\mathcal{S},t)}\varphi(x) \otimes \varphi(y))\\
&\qquad - d(\varphi^T(t)\mathbb{A}(\varphi(x) \otimes \varphi(y)), \mathcal{S}, t)\varphi^T(t)\mathbb{A}(\varphi(x) \otimes D^{\sigma(\mathcal{S},t)}\varphi(y))\\
&\qquad + \psi(\varphi^T(t)\mathbb{A}(\varphi(x) \otimes \varphi(y)), \mathcal{S}, t).
\end{aligned} \tag{4.23}$$

From the conditions (4.22) and the Eq. (4.2), we get

$$\begin{aligned}
\varphi^T(0)\mathbb{A}(\varphi(x)\otimes\varphi(y)) =& f_1(x,y),\\
\varphi^T(t)\mathbb{A}(\varphi(0)\otimes\varphi(y)) =& f_2(y,t),\\
\varphi^T(t)\mathbb{A}(\varphi(1)\otimes\varphi(y)) =& f_3(y,t),\\
\varphi^T(t)\mathbb{A}(\varphi(x)\otimes\varphi(0)) =& f_4(x,t),\\
\varphi^T(t)\mathbb{A}(\varphi(x)\otimes\varphi(1)) =& f_5(x,t).
\end{aligned} \tag{4.24}$$

Now, we collocate Eq. (4.23) at $(m-1)\times(m-1)\times m$ points and Eq. (4.24) at points $x_i = y_i = t_i = \frac{2i-1}{2m+1}$ for $i = 1, 2, ..., m$. This step produces $(m+1)^3$ set of nonlinear algebraic equations.The solution of that non-linear system produces matrix $\mathbb{A}$. In this way, we can get the numerical solution of our proposed variable-order reaction advection diffuion equations (4.1)-(4.2).

4.4 Error and Convergence Analysis

Theorem 4.2. *Suppose that* $\varphi^T(t)\mathbb{A}(\varphi(x)\otimes\varphi(y))$ *is the approximation of* $u(\mathcal{S},t)$ *by shifted Vieta-Fibonacci polynomials. If the function* $u(\mathcal{S},t)$ *has sixth-order continuous derivatives, then*

$$\begin{aligned}
&\|c_{111}\| \le L_{0,0,0},\ \|c_{211}\| \le L_{0,0,0},\ \|c_{121}\| \le L_{0,0,0},\ \|c_{112}\| \le L_{0,0,0},\\
&\|c_{221}\| \le L_{0,0,0},\ \|c_{122}\| \le L_{0,0,0},\ \|c_{212}\| \le L_{0,0,0},\ \|c_{222}\| \le L_{0,0,0},\\
&\|c_{i11}\| \le L_{2,0,0},\ \|c_{i12}\| \le L_{2,0,0},\ \|c_{i21}\| \le L_{2,0,0},\ \|c_{i22}\| \le L_{2,0,0},\\
&\|c_{1j1}\| \le L_{0,2,0},\ \|c_{2j1}\| \le L_{0,2,0},\ \|c_{1j2}\| \le L_{0,2,0},\ \|c_{2j2}\| \le L_{0,2,0},\\
&\|c_{11k}\| \le L_{0,0,2},\ \|c_{12k}\| \le L_{0,0,2},\ \|c_{21k}\| \le L_{0,0,2},\ \|c_{22k}\| \le L_{0,0,2},\\
&\|c_{ij1}\| \le L_{2,2,0},\ \|c_{ij2}\| \le L_{2,2,0},\ \|c_{1jk}\| \le L_{0,2,2},\ \|c_{2jk}\| \le L_{0,2,2},\\
&\|c_{i1k}\| \le L_{2,0,2},\ \|c_{i2k}\| \le L_{2,0,2},\ \|c_{ijk}\| \le L_{2,2,2},\quad for\quad i,j,k>2,
\end{aligned}$$

where

$$\begin{aligned}
L_{0,0,0} &= max\left\{\|u(x,y,t)\| : (x,y,t)\in[0,1]\times[0,1]\times[0,1]\right\},\\
L_{2,0,0} &= max\left\{\|u_{xx}(x,y,t)\| : (x,y,t)\in[0,1]\times[0,1]\times[0,1]\right\},\\
L_{0,2,0} &= max\left\{\|u_{yy}(x,y,t)\| : (x,y,t)\in[0,1]\times[0,1]\times[0,1]\right\},\\
L_{0,0,2} &= max\left\{\|u_{tt}(x,y,t)\| : (x,y,t)\in[0,1]\times[0,1]\times[0,1]\right\},\\
L_{2,2,0} &= max\left\{\|u_{xxyy}(x,y,t)\| : (x,y,t)\in[0,1]\times[0,1]\times[0,1]\right\},\\
L_{0,2,2} &= max\left\{\|u_{yytt}(x,y,t)\| : (x,y,t)\in[0,1]\times[0,1]\times[0,1]\right\},\\
L_{2,0,2} &= max\left\{\|u_{xxtt}(x,y,t)\| : (x,y,t)\in[0,1]\times[0,1]\times[0,1]\right\},\\
L_{2,2,2} &= max\left\{\|u_{xxyytt}(x,y,t)\| : (x,y,t)\in[0,1]\times[0,1]\times[0,1]\right\}.
\end{aligned}$$

Proof. Applying Eq. (4.9), the coefficient c_{111} is computed as

$$c_{111} = \frac{512}{\pi^3} \int_0^1 \int_0^1 \int_0^1 u(\mathcal{S}, t)\chi(x)\chi(y)\chi(t)dxdydt, \tag{4.25}$$

and by change of variables $2x - 1 = (\xi)$, $2y - 1 = \cos(\omega)$, and $2t - 1 = \cos(\kappa)$, we obtain

$$c_{111} = \frac{8}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi u\left(\frac{(\xi)+1}{2}, \frac{\cos(\omega)+1}{2}, \frac{\cos(\kappa)+1}{2}\right) \times \sin^2(\xi)\sin^2(\omega)\sin^2(\kappa)d\xi d\omega d\kappa. \tag{4.26}$$

From Eq. (4.26), we have $\|c_{111}\| \leq L_{0,0,0}$.

Next, from Eq. (4.9) we have

$$c_{211} = \frac{512}{\pi^3} \int_0^1 \int_0^1 \int_0^1 u(\mathcal{S}, t)VF_2^*(x)\chi(x)\chi(y)\chi(t)dxdydt. \tag{4.27}$$

The same change of variables and the following property

$$VF_{r+1}^*(s) = \frac{\sin((r+1)\varrho)}{\sin(\varrho)}, \quad s = \frac{\cos(\varrho)+1}{2}, \tag{4.28}$$

leads to

$$c_{211} = \frac{8}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi u\left(\frac{cos(\xi)+1}{2}, \frac{\cos(\omega)+1}{2}, \frac{\cos(\kappa)+1}{2}\right) \sin^2(\xi)\sin^2(\omega)\sin^2(\kappa)\sin 2\xi d\xi d\omega d\kappa, \tag{4.29}$$

which implies $\|c_{211}\|, \leq L_{0,0,0}$. Similarly one can conclude that $\|c_{121}\| \leq L_{0,0,0}$, $\|c_{112}\| \leq L_{0,0,0}$, $\|c_{221}\| \leq L_{0,0,0}$, $\|c_{122}\| \leq L_{0,0,0}$, $\|c_{212}\| \leq L_{0,0,0}$ and $\|c_{222}\| \leq L_{0,0,0}$.

The coefficients c_{i11} for $i > 2$ are computed as:

$$c_{i11} = \frac{512}{\pi^3} \int_0^1 \int_0^1 \int_0^1 u(\mathcal{S}, t) V F_i^*(x) \chi(x) \chi(y) \chi(t) dx dy dt. \tag{4.30}$$

With the help of previous change of variables and property (4.28), Eq. (4.30) becomes

$$c_{i11} = \frac{8}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi u\left(\frac{cos(\xi)+1}{2}, \frac{\cos(\omega)+1}{2}, \frac{\cos(\kappa)+1}{2}\right) \\ \times \sin(\xi)\sin(i\xi)\sin^2(\omega)\sin^2(\kappa) d\xi d\omega d\kappa, \tag{4.31}$$

or equivalently

$$c_{i11} = \frac{4}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi u\left(\frac{cos(\xi)+1}{2}, \frac{\cos(\omega)+1}{2}, \frac{\cos(\kappa)+1}{2}\right) \\ \times [\cos(i-1)\xi - \cos(i+1)\xi] \sin^2(\omega)\sin^2(\kappa) d\xi d\omega d\kappa. \tag{4.32}$$

Now, integrating Eq. (4.33) twice with respect to ξ, we get

$$c_{i11} = \frac{1}{2\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi u_{\xi\xi}\left(\frac{cos(\xi)+1}{2}, \frac{\cos(\omega)+1}{2}, \frac{\cos(\kappa)+1}{2}\right) \\ \times \Lambda_i(\xi)\sin^2(\omega)\sin^2(\kappa) d\xi d\omega d\kappa, \tag{4.33}$$

where

$$\Lambda_i(\xi) = \left(\frac{\sin(i-2)\xi}{(i-1)(i-2)} - \frac{\sin(i\xi)}{(i-1)i} - \frac{\sin(i\xi)}{(i+1)i} + \frac{\sin(i+2)\xi}{(i+1)(i+2)}\right)\sin(\xi). \tag{4.34}$$

Now from Eqs. (4.33)-(4.34), we have $\|c_{i11}\| \leq L_{2,0,0}$. Similiarly, it can be computed that $\|c_{i12}\| \leq L_{2,0,0}$, $\|c_{i21}\| \leq L_{2,0,0}$, $\|c_{i22}\| \leq L_{2,0,0}$, $\|c_{1j1}\| \leq L_{0,2,0}$, $\|c_{2j1}\| \leq L_{0,2,0}$, $\|c_{1j2}\| \leq L_{0,2,0}$, $\|c_{2j2}\| \leq L_{0,2,0}$, $\|c_{11k}\| \leq L_{0,0,2}$, $\|c_{12k}\| \leq L_{0,0,2}$, $\|c_{21k}\| \leq L_{0,0,2}$ and

$\|c_{22k}\| \leq L_{0,0,2}$. Moreover, for $i, j > 2$, we have

$$c_{ij1} = \frac{8}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi u \left(\frac{cos(\xi)+1}{2}, \frac{\cos(\omega)+1}{2}, \frac{\cos(\kappa)+1}{2} \right) \\ \times \sin(\xi)\sin(i\xi)\sin(\omega)\sin(j\omega)\sin^2(\kappa) d\xi d\omega d\kappa. \tag{4.35}$$

Now, integrating Eq. (4.35) twice with respect to ξ, we get

$$c_{ij1} = \frac{1}{2\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi u_{\xi\xi} \left(\frac{cos(\xi)+1}{2}, \frac{\cos(\omega)+1}{2}, \frac{\cos(\kappa)+1}{2} \right) \\ \times \Lambda_i(\xi)\sin(\omega)\sin(j\omega)\sin^2(\kappa) d\xi d\omega d\kappa. \tag{4.36}$$

Again, integrating Eq. (4.36) twice with respect to ω, we obtain

$$c_{ij1} = \frac{1}{32\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi u_{\xi\xi\omega\omega} \left(\frac{cos(\xi)+1}{2}, \frac{\cos(\omega)+1}{2}, \frac{\cos(\kappa)+1}{2} \right) \\ \times \Lambda_i(\xi)\Lambda_j(\omega)\sin^2(\kappa) d\xi d\omega d\kappa. \tag{4.37}$$

Taking the absolute value of Eq. (4.37), we obtain

$$\|c_{ij1}\| = || \frac{1}{32\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi u_{\xi\xi\omega\omega} \left(\frac{cos(\xi)+1}{2}, \frac{\cos(\omega)+1}{2}, \frac{\cos(\kappa)+1}{2} \right) \\ \times \Lambda_i(\xi)\Lambda_j(\omega)\sin^2(\kappa) d\xi d\omega d\kappa ||. \tag{4.38}$$

After some mathematical manipulation, we get

$$\|c_{ij1}\| \leq L_{2,2,0}. \tag{4.39}$$

Similiarly $\|c_{ij2}\| \leq L_{2,2,0}$, $\|c_{1jk}\| \leq L_{0,2,2}$, $\|c_{2jk}\| \leq L_{0,2,2}$, $\|c_{i1k}\| \leq L_{2,0,2}$, and $\|c_{i2k}\| \leq L_{2,0,2}$.

Moreover, for $i, j, k > 2$, we have

$$c_{ijk} = \frac{8}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi u\left(\frac{cos(\xi)+1}{2}, \frac{\cos(\omega)+1}{2}, \frac{\cos(\kappa)+1}{2}\right)$$

$$\times \sin(\xi)\sin(i\xi)\sin(\omega)\sin(j\omega)\sin(\kappa)\sin(k\kappa)d\xi d\omega d\kappa, \quad (4.40)$$

and by using the above process, we obtain

$$c_{ijk} = \frac{1}{512\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi u_{\xi\xi\omega\omega\kappa\kappa}\left(\frac{cos(\xi)+1}{2}, \frac{\cos(\omega)+1}{2}, \frac{\cos(\kappa)+1}{2}\right)$$

$$\times \Lambda_i(\xi)\Lambda_j(\omega)\Lambda_k(\kappa)d\xi d\omega d\kappa, \quad (4.41)$$

and consequently

$$\|c_{ijk}\| \leq L_{2,2,2}, \quad (4.42)$$

which completes the proof. □

Theorem 4.3. *Let* $\varphi^T(t)\mathbb{A}(\varphi(x)\otimes\varphi(y))$ *be the approximation of* $u(\mathcal{S}, t)$ *in the terms of shifted Vieta-Fibonacci polynomials. If the function* $u(\mathcal{S}, t)$ *has continuous derivatives of the sixth order, then*

$$\left\|u(\mathcal{S}, t) - \varphi^T(t)\mathbb{A}(\varphi(x)\otimes\varphi(y))\right\|_{L^2} \leq \sqrt{G(i,j,k)}, \quad (4.43)$$

where

$$\begin{aligned}
G(i,j,k) &= \frac{\pi^3}{32}\sum_{k=m+2}^{\infty}\frac{L_{0,0,2}^2}{(k-2)^4}+\frac{\pi^3}{32}\sum_{j=m+2}^{\infty}\frac{L_{0,2,0}^2}{(j-2)^4}+\frac{\pi^3}{32}\sum_{i=m+2}^{\infty}\frac{L_{2,0,0}^2}{(i-2)^4}\\
&+\frac{\pi^3}{1024}\sum_{j=3}^{m+1}\sum_{k=m+2}^{\infty}\frac{L_{0,2,2}^2}{(j-2)^4(k-2)^4}+\frac{\pi^3}{1024}\sum_{j=m+2}^{\infty}\sum_{k=3}^{\infty}\frac{L_{0,2,2}^2}{(j-2)^4(k-2)^4}\\
&+\frac{\pi^3}{1024}\sum_{i=3}^{m+1}\sum_{k=m+2}^{\infty}\frac{L_{2,0,2}^2}{(i-2)^4(j-2)^4}+\frac{\pi^3}{1024}\sum_{i=m+2}^{\infty}\sum_{k=3}^{\infty}\frac{L_{2,0,2}^2}{(i-2)^4(k-2)^4}\\
&+\frac{\pi^3}{1024}\sum_{i=3}^{m+1}\sum_{j=m+2}^{\infty}\frac{L_{2,2,0}^2}{(i-2)^4(j-2)^4}+\frac{\pi^3}{1024}\sum_{i=m+2}^{\infty}\sum_{j=3}^{\infty}\frac{L_{2,2,0}^2}{(i-2)^4(j-2)^4}\\
&+\frac{\pi^3}{512\times 64}\sum_{i=3}^{m+1}\sum_{j=3}^{m+1}\sum_{k=m+2}^{\infty}\frac{L_{2,2,2}^2}{(i-2)^4(j-2)^4(k-2)^4}\\
&+\frac{\pi^3}{512\times 64}\sum_{i=3}^{m+1}\sum_{j=m+2}^{\infty}\sum_{k=3}^{\infty}\frac{L_{2,2,2}^2}{(i-2)^4(j-2)^4(k-2)^4}\\
&+\frac{\pi^3}{512\times 64}\sum_{i=m+2}^{\infty}\sum_{j=3}^{\infty}\sum_{k=3}^{\infty}\frac{L_{2,2,2}^2}{(i-2)^4(j-2)^4(k-2)^4}.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
&||u(\mathcal{S},t)-\varphi^T(t)\mathbb{A}(\varphi(x)\otimes\varphi(y))||_{L^2}^2\\
&\quad=\int_0^1\int_0^1\int_0^1(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}c_{ijk}VF_i^*(x)VF_j^*(y)VF_k^*(t)\\
&\quad-\sum_{i=1}^{m+1}\sum_{j=1}^{m+1}\sum_{k=1}^{m+1}c_{ijk}VF_i^*(x)VF_j^*(y)VF_k^*(t))^2\chi(x)\chi(y)\chi(t)dxdydt,
\end{aligned}$$

$$
\begin{aligned}
=\int_0^1\int_0^1\int_0^1 (&\sum_{i=1}^{m+1}\sum_{j=1}^{m+1}\sum_{k=m+2}^{\infty} c_{ijk} VF_i^*(x)VF_j^*(y)VF_k^*(t)\\
&+\sum_{i=1}^{m+1}\sum_{j=m+2}^{\infty}\sum_{k=1}^{\infty} c_{ijk} VF_i^*(x)VF_j^*(y)VF_k^*(t)\\
&+\sum_{i=m+2}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} c_{ijk} VF_i^*(x)VF_j^*(y)VF_k^*(t))^2\chi(x)\chi(y)\chi(t)dxdydt,
\end{aligned}
$$

$$
\begin{aligned}
=&\sum_{i=1}^{m+1}\sum_{j=1}^{m+1}\sum_{k=m+2}^{\infty} c_{ijk}\int_0^1 VF_i^{*^2}(x)\chi(x)dx\int_0^1 VF_j^{*^2}(y)\chi(y)dy\int_0^1 VF_k^{*^2}(t)\chi(t)dt\\
&+\sum_{i=1}^{m+1}\sum_{j=m+2}^{\infty}\sum_{k=m+2}^{\infty} c_{ijk}\int_0^1 VF_i^{*^2}(x)\chi(x)dx\int_0^1 VF_j^{*^2}(y)\chi(y)dy\int_0^1 VF_k^{*^2}(t)\chi(t)dt\\
&+\sum_{i=m+2}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} c_{ijk}\int_0^1 VF_i^{*^2}(x)\chi(x)dx\int_0^1 VF_j^{*^2}(y)\chi(y)dy\int_0^1 VF_k^{*^2}(t)\chi(t)dt.
\end{aligned}
\tag{4.44}
$$

Eq. (4.44) and orthogonality of shifted Vieta-Fibonacci polynomials yield

$$
=\sum_{i=1}^{m+1}\sum_{j=1}^{m+1}\sum_{k=m+2}^{\infty} c_{ijk}^2\frac{\pi^3}{512}+\sum_{i=1}^{m+1}\sum_{j=m+2}^{\infty}\sum_{k=m+2}^{\infty} c_{ijk}\frac{\pi^3}{512}+\sum_{i=m+2}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} c_{ijk}\frac{\pi^3}{512}. \tag{4.45}
$$

From Theorem 4.2 and Eq. (4.45), we have

$$
\begin{aligned}
&||u(\mathcal{S},t)-\varphi^T(t)\mathbb{A}(\varphi(x)\otimes\varphi(y))||_{L^2}^2 \\
&\qquad\le \frac{\pi^3}{32}\sum_{k=m+2}^{\infty}\frac{L_{0,0,2}^2}{(k-2)^4}+\frac{\pi^3}{32}\sum_{j=m+2}^{\infty}\frac{L_{0,2,0}^2}{(j-2)^4}+\frac{\pi^3}{32}\sum_{i=m+2}^{\infty}\frac{L_{2,0,0}^2}{(i-2)^4} \\
&\qquad+\frac{\pi^3}{1024}\sum_{j=3}^{m+1}\sum_{k=m+2}^{\infty}\frac{L_{0,2,2}^2}{(j-2)^4(k-2)^4}+\frac{\pi^3}{1024}\sum_{j=m+2}^{\infty}\sum_{k=3}^{\infty}\frac{L_{0,2,2}^2}{(j-2)^4(k-2)^4} \\
&\qquad+\frac{\pi^3}{1024}\sum_{i=3}^{m+1}\sum_{k=m+2}^{\infty}\frac{L_{2,0,2}^2}{(i-2)^4(j-2)^4}+\frac{\pi^3}{1024}\sum_{i=m+2}^{\infty}\sum_{k=3}^{\infty}\frac{L_{2,0,2}^2}{(i-2)^4(k-2)^4} \\
&\qquad+\frac{\pi^3}{1024}\sum_{i=3}^{m+1}\sum_{j=m+2}^{\infty}\frac{L_{2,2,0}^2}{(i-2)^4(j-2)^4}+\frac{\pi^3}{1024}\sum_{i=m+2}^{\infty}\sum_{j=3}^{\infty}\frac{L_{2,2,0}^2}{(i-2)^4(j-2)^4} \\
&\qquad+\frac{\pi^3}{512\times 64}\sum_{i=3}^{m+1}\sum_{j=3}^{m+1}\sum_{k=m+2}^{\infty}\frac{L_{2,2,2}^2}{(i-2)^4(j-2)^4(k-2)^4} \\
&\qquad+\frac{\pi^3}{512\times 64}\sum_{i=3}^{m+1}\sum_{j=m+2}^{\infty}\sum_{k=3}^{\infty}\frac{L_{2,2,2}^2}{(i-2)^4(j-2)^4(k-2)^4} \\
&\qquad+\frac{\pi^3}{512\times 64}\sum_{i=m+2}^{\infty}\sum_{j=3}^{\infty}\sum_{k=3}^{\infty}\frac{L_{2,2,2}^2}{(i-2)^4(j-2)^4(k-2)^4}.
\end{aligned}
\tag{4.46}
$$

Finally, by taking the square roots of both sides of Eq. (4.46), the proof is completed.

□

4.5 Numerical Examples and Discussion

In this section, first, we take the following three examples to demonstrate the correctness of the proposed approach:

Table 4.1: Obtained errors for example 4.5.1.

m	L_∞ error	convergence order	L_2 error	convergence order
5	6.57496E-4	-	2.59743E-4	-
10	3.85759E-5	2.33925	1.97385E-5	2.12586
16	4.93857E-6	2.36098	2.85824E-6	2.21949
24	5.84763E-7	2.76618	4.85869E-7	2.29737
34	8.57936E-8	2.85204	1.89564E-8	2.69435

Example 4.5.1 Let us take the following reaction advection diffuion equations with fractional derivative [161]:

$$\begin{aligned}\frac{\partial u(\mathcal{S},t)}{\partial t} =& a(u,\mathcal{S},t)\frac{\partial^{1.8}u(\mathcal{S},t)}{\partial x^{1.8}} + b(u,\mathcal{S},t)\frac{\partial^{1.6}u(\mathcal{S},t)}{\partial y^{1.6}} \\ &- c(u,\mathcal{S},t)\frac{\partial^{0.8}u(\mathcal{S},t)}{\partial x^{0.8}} - d(u,\mathcal{S},t)\frac{\partial^{0.6}u(\mathcal{S},t)}{\partial y^{0.6}} + f(u,\mathcal{S},t), \qquad (4.47)\end{aligned}$$

with the conditions

$$\begin{aligned}&u(x,y,0) = x^{4.8}y^3,\ u(0,y,t) = u(x,0,t) = 0, \\ &u(1,y,t) = y^3e^{-t},\ u(x,1,t) = x^{4.8}e^{-t}, \qquad (4.48)\end{aligned}$$

for all $x \in [0,1], \quad y \in [0,1], \quad t \geq 0$, where

$$\begin{aligned}&a(u,\mathcal{S},t) = \frac{6}{\Gamma(5.8)}x^{1.8}, \quad b(u,\mathcal{S},t) = \frac{\Gamma(2.4)}{6}y^{1.6}, \quad c(u,\mathcal{S},t) = \frac{12}{\Gamma(5.8)}x^{0.8}, \\ &d(u,\mathcal{S},t) = \frac{\Gamma(3.4)}{12}y^{0.6}, \quad f(u,\mathcal{S},t) = -2x^{4.8}y^3e^{-t}.\end{aligned}$$

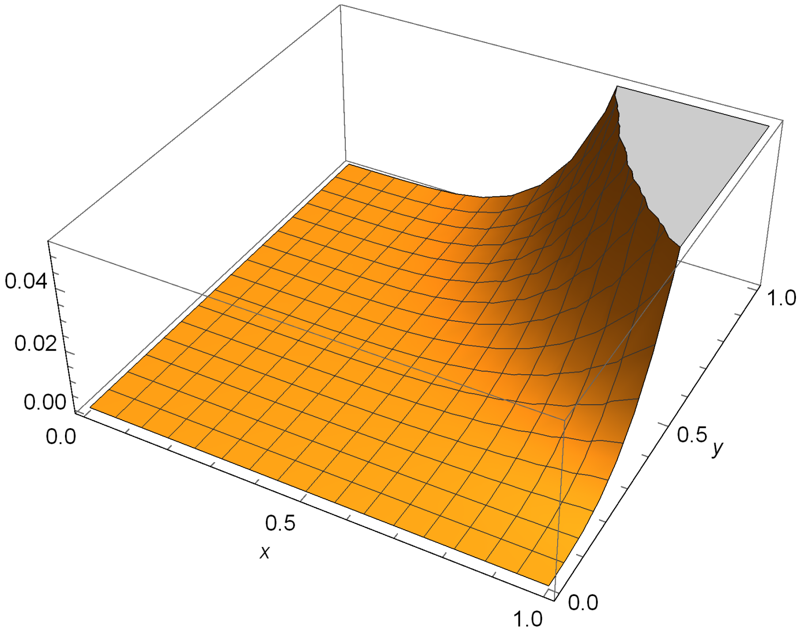

FIGURE 4.1: Plots of numerical solution for Example 4.5.1 at $m = 11$.

TABLE 4.2: Comparison of maximum absolute errors for example 4.5.2.

$\alpha(\mathcal{S}, t)$	error of [162]	our maximum absolute error
0.5	2.4690E-4	1.2096E-4
0.6	2.2923E-4	1.0571E-4
0.7	2.0829E-4	8.3137E-5
0.8	1.8402E-4	5.9135E-5
0.9	1.5642E-4	3.9162E-5

As given in [159], we define the convergence order (CO) as

$$CO = \frac{log\left(\frac{R_{n_1}}{R_{n_2}}\right)}{log\left(\frac{n_2}{n_1}\right)},$$

where R_{n_1} and R_{n_2} are the first and second L_2 or L_∞ error values respectively. Moreover, n_1 and n_2 are the number of the orthonormal shifted Vieta-Fibonacci polynomials utilized in the first and second implementations, respectively.

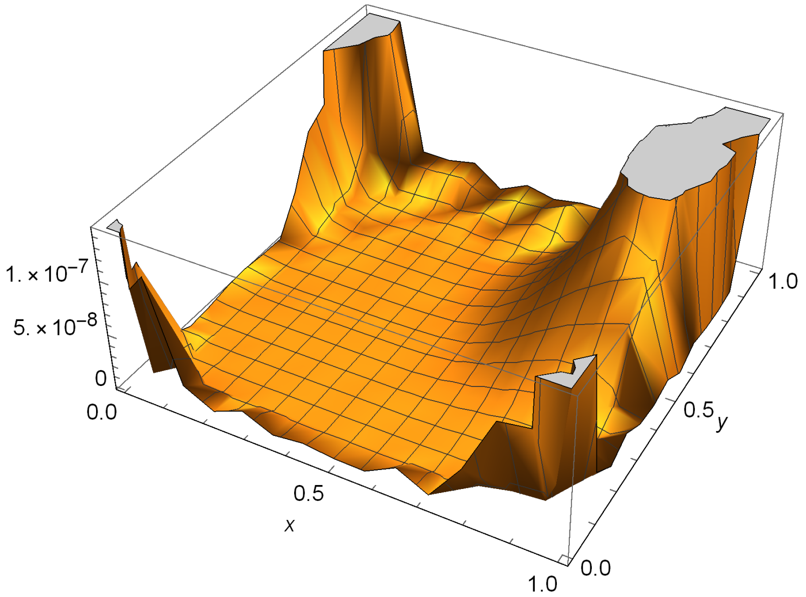

FIGURE 4.2: Plots of absolute error for Example 4.5.1 at $m = 11$.

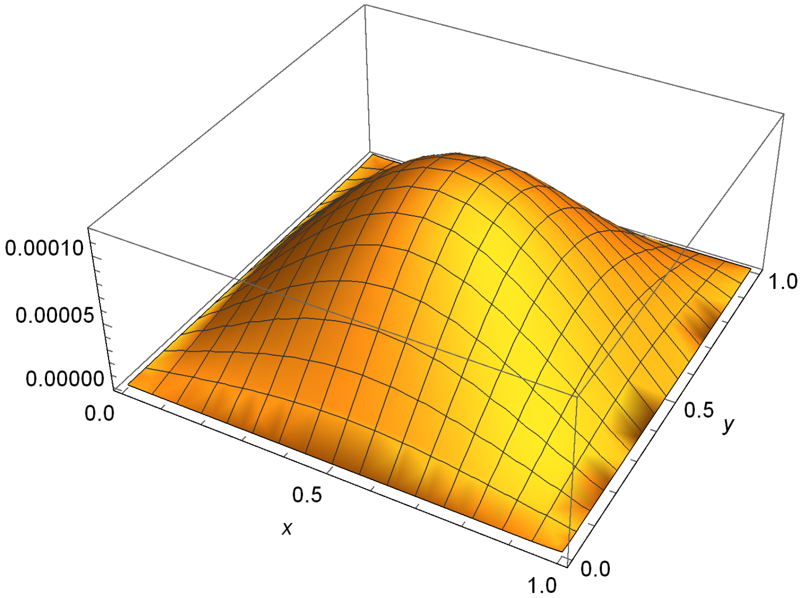

FIGURE 4.3: Absolute error for example 4.5.2 at $\alpha(\mathcal{S}, t) = 0.5$.

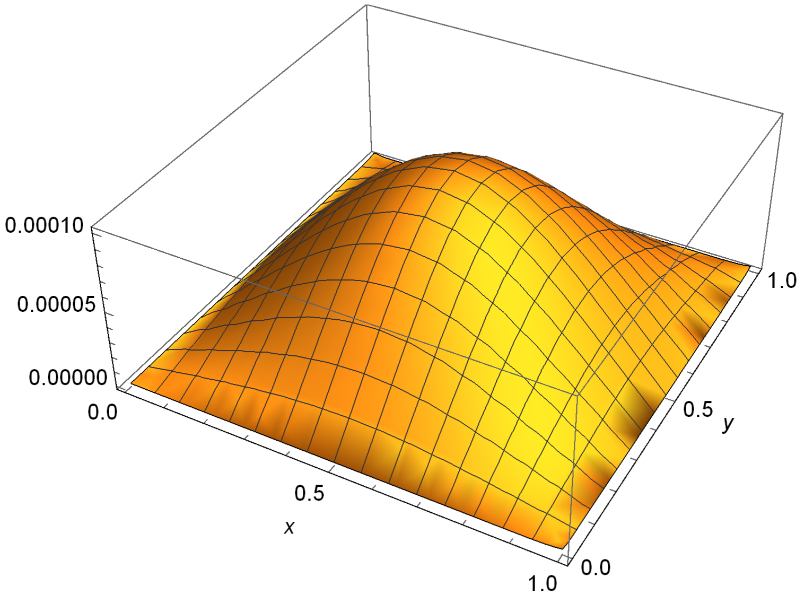

FIGURE 4.4: Absolute error for example 4.5.2 at $\alpha(\mathcal{S},t) = 0.6$.

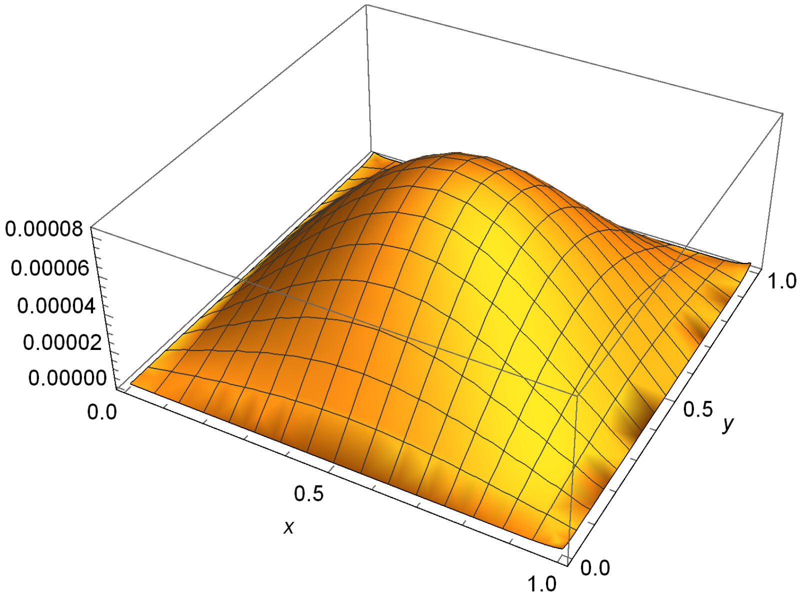

FIGURE 4.5: Absolute error for example 4.5.2 at $\alpha(\mathcal{S},t) = 0.7$.

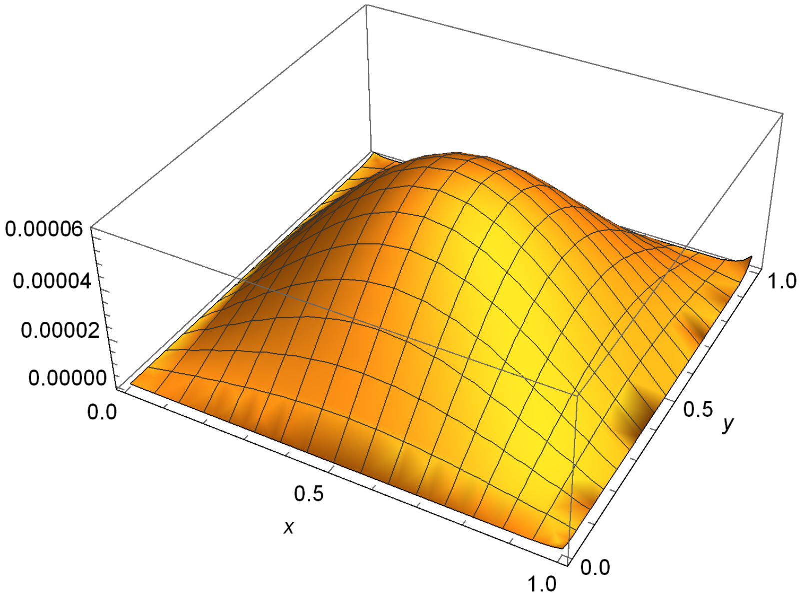

Figure 4.6: Absolute error for example 4.5.2 at $\alpha(\mathcal{S}, t) = 0.8$.

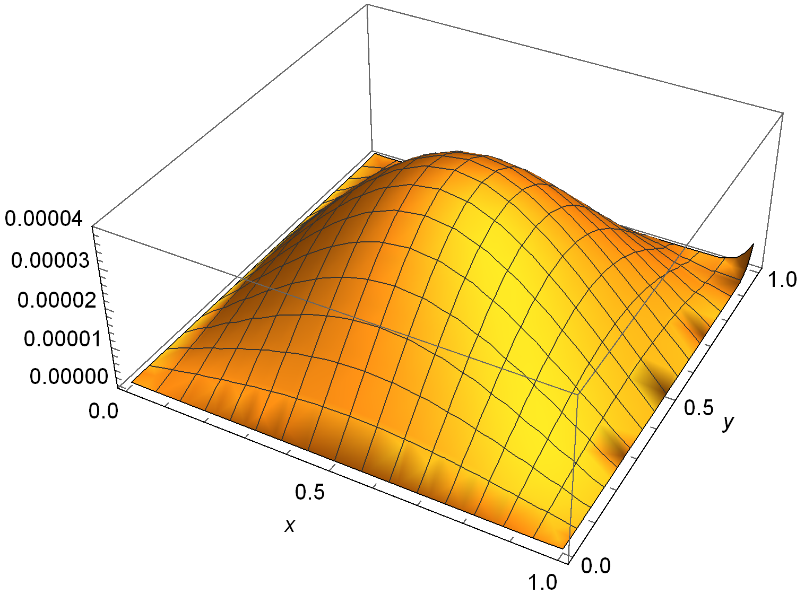

Figure 4.7: Absolute error for example 4.5.2 at $\alpha(\mathcal{S}, t) = 0.9$.

TABLE 4.3: Comparison of maximum absolute errors for example 4.5.3.

$\beta(\mathcal{S},t)$	$\gamma(\mathcal{S},t)$	$\delta(\mathcal{S},t)$	$\sigma(\mathcal{S},t)$	Wen and Tang (2013)	L_∞ error
xyt	xyt	xyt	xyt	2.1454E-3	1.6743E-5
$\sin(\frac{xyt}{2})$	$\cos(\frac{xyt}{2})$	$\sin(\frac{xyt}{2})$	$\cos(\frac{xyt}{2})$	2.0714E-3	1.7015E-5
$\frac{e^{xyt}-sin(xyt)}{10}$	$\frac{e^{xyt}-cos(xyt)}{10}$	$\frac{e^{xyt}-sin(xyt)}{10}$	$\frac{e^{xyt}-cos(xyt)}{10}$	2.2288E-3	2.4237E-5
$\frac{e^{xyt}-xyt}{8}$	$\frac{e^{xyt}+xyt}{8}$	$\frac{e^{xyt}-xyt}{8}$	$\frac{e^{xyt}+xyt}{8}$	1.9361E-3	2.2124E-5

The analytical solution of the example 4.5.1 is $u(\mathcal{S},t) = x^{4.8}y^3e^{-t}$. Table (4.1) shows the L_∞ errors and L_2 errors with increasing values of approximating polynomials, and it also depicts the high order of convergence of this numerical scheme. In Fig. (4.1), the approximate solution of Eq. (4.47) and its absolute errors at time $t = 1$ and $m = 11$ are demonstrated. For this problem the L_2 and L_∞ errors for $u(\mathcal{S},t)$ are tabulated in Table (4.1), which are defined as.

$$L_2 = \sqrt{\int_0^1 \int_0^1 |u(\mathcal{S},t) - u'(\mathcal{S},t)|^2 dxdy}, \tag{4.49}$$

$$L_\infty = \max_{\{0<x<1\}} \max_{\{0<y<1\}} |u(\mathcal{S},t) - u^{'}(\mathcal{S},t)|, \tag{4.50}$$

where $u(\mathcal{S},t)$ and $u^{'}(\mathcal{S},t)$ are the approximate and exact solutions.

Example 4.5.2 Consider the following non-linear fractional-order ADE ([113], [162]):

$$D_t^{\alpha(\mathcal{S},t)}u(\mathcal{S},t) = \frac{\partial^2 u(\mathcal{S},t)}{\partial x^2} + \frac{\partial^2 u(\mathcal{S},t)}{\partial y^2} + \frac{2}{\Gamma(2-\alpha(\mathcal{S},t))} t^{2-\alpha(\mathcal{S},t)}(x-x^2)^2(y-y^2)^2 \\ - 2t^2(1-6x-6x^2)(y-y^2)^2 - 2t^2(1-6y-6y^2)(x-x^2)^2, \quad (4.51)$$

with the conditions

$$u(x,y,0) = 0, \quad (4.52)$$

$$u(0,y,t) = u(1,y,t) = 0, \quad (4.53)$$

$$u(x,0,t) = u(x,1,t) = 0, \quad (4.54)$$

The analytical solution to the example 4.5.2 is given by,

$$u(\mathcal{S},t) = t^2(x-x^2)^2(y-y^2)^2. \quad (4.55)$$

Figures (4.2)-(4.6) show the absolute errors of our approximate solution graphically for the example 4.5.2 at $m = 9$ for $\alpha(\mathcal{S},t)$=0.5, 0.6, 0.7, 0.8, 0.9 at time $t = 1$. Table (4.2) presents the comparison of the maximum absolute error of our proposed scheme with the scheme given in [162] for different values of $\alpha(\mathcal{S},t)$. From this table, it is conclude that our results have better accuracy than the result of Ali et al. (2017).

Example 4.5.3 Let us consider the third example on a finite rectangular domain [0,1]×[0,1] for Eqs. (4.1) including the dispersion coefficients [163]

$$a(u,\mathcal{S},t)=\frac{\Gamma(3-\beta(\mathcal{S},t))}{8}x^{1+\beta(\mathcal{S},t)},\quad b(u,\mathcal{S},t)=\frac{\Gamma(3-\gamma(\mathcal{S},t))}{8}y^{1+\gamma(\mathcal{S},t)},$$

and the advection coefficients

$$c(u,\mathcal{S},t)=\frac{\Gamma(4-\delta(\mathcal{S},t))}{4}x^{\delta(\mathcal{S},t)},\quad d(u,\mathcal{S},t)=\frac{\Gamma(4-\sigma(\mathcal{S},t))}{4}y^{1+\sigma(\mathcal{S},t)},$$

with the initial condition

$$u(\mathcal{S},0)=x^3y^3,$$

and essential boundary conditions in the form $u(0,y,t)=u(x,0,t)=0$, $u(1,y,t)=e^{-t}y^3, u(x,1,t)=e^{-t}x^3,\quad t>0.$

The analytical solution to this problem is given by,

$$u(\mathcal{S},t)=e^{-t}x^3y^3. \tag{4.56}$$

The presented method is applied for this example for some value of $\beta(\mathcal{S},t)$, $\gamma(\mathcal{S},t)$ $\delta(\mathcal{S},t)$ and $\sigma(\mathcal{S},t)$. The maximum absolute errors obtained by the presented scheme are listed in Table (4.3). From this table, one can be observe that for selected variable orders $\beta(\mathcal{S},t)$, $\gamma(\mathcal{S},t)$, $\delta(\mathcal{S},t)$ and $\sigma(\mathcal{S},t)$ the achieved results are more accurate than Wen and Tang (2013).

Example 4.5.4 Consider the following fractional reaction advection diffusion equation on a finite domain [0,1]×[0,1]

$$\begin{aligned}\frac{\partial^{\alpha(\mathcal{S},t)}u(\mathcal{S},t)}{\partial t^{\alpha(\mathcal{S},t)}} = \frac{\partial^2 u(\mathcal{S},t)}{\partial x^2} + \frac{\partial^2 u(\mathcal{S},t)}{\partial y^2} - \frac{\partial u(\mathcal{S},t)}{\partial x} - \frac{\partial u(\mathcal{S},t)}{\partial y} \\ + e^x\left(\frac{t^{1-\alpha(\mathcal{S},t)}}{\Gamma(2-\alpha(\mathcal{S},t))} + 1\right), \qquad (4.57)\end{aligned}$$

under the conditions

$$\begin{aligned} u(x,y,0) &= e^x y \\ u(0,y,t) &= t + y \\ u(1,y,t) &= e(t+y) \\ u(x,0,t) &= e^x t \\ u(x,1,t) &= e^x(t+1) \end{aligned}$$

The analytical solution for this example is

$$u(x,y,t) = (t+y)e^x$$

From Table 4.4, it is clear that the obtained maximum absolute error for this example by our proposed method is almost equal to the method used in previous chapter. The maximum absolute errorcan be reduced with the increase in m, which shows higher convergence rate of our proposed scheme.

Table 4.4: The computational error for example 4.5.4

$\alpha(\mathcal{S},t)$	m	L_2 error(Chap 4)	L_∞ error(Chap 4)	L_2 error(Chap 3)	L_∞ error(Chap 3)
$\frac{2+\sin(t)}{4}$	6	2.46387×10^{-5}	4.75835×10^{-5}	2.46387×10^{-5}	4.75835×10^{-5}
	8	6.53795×10^{-6}	8.49205×10^{-6}	6.53795×10^{-6}	8.49205×10^{-6}
	10	4.81568×10^{-7}	7.59386×10^{-7}	4.81568×10^{-7}	7.59386×10^{-7}
	12	7.26473×10^{-8}	9.46853×10^{-8}	7.26473×10^{-8}	9.46853×10^{-8}
	14	3.04735×10^{-9}	5.49325×10^{-9}	3.04735×10^{-9}	5.49325×10^{-9}
$\alpha(\mathcal{S},t)$	m	L_2 error(Chap 4)	L_∞ error(Chap 4)	L_2 error(Chap 3)	L_∞ error(Chap 3)
$\frac{20e^{\frac{t}{2}}-12}{20e^{\frac{t}{2}}-10}$	6	2.64835×10^{-5}	4.94635×10^{-5}	2.64835×10^{-5}	4.94635×10^{-5}
	8	6.84739×10^{-6}	8.74635×10^{-6}	6.84739×10^{-6}	8.74635×10^{-6}
	10	5.02463×10^{-7}	7.83564×10^{-7}	5.02463×10^{-7}	7.83564×10^{-7}
	12	7.46835×10^{-8}	9.68352×10^{-8}	7.46835×10^{-8}	9.68352×10^{-8}
	14	3.14623×10^{-9}	5.68362×10^{-9}	3.14623×10^{-9}	5.68362×10^{-9}
$\alpha(\mathcal{S},t)$	m	L_2 error(Chap 4)	L_∞ error(Chap 4)	L_2 error(Chap 3)	L_∞ error(Chap 3)
$0.90+\frac{t}{150}$	6	2.85735×10^{-5}	4.83648×10^{-5}	2.85735×10^{-5}	4.83648×10^{-5}
	8	6.69453×10^{-6}	8.69357×10^{-6}	6.69453×10^{-6}	8.69357×10^{-6}
	10	4.95735×10^{-7}	7.74693×10^{-7}	4.95735×10^{-7}	7.74693×10^{-7}
	12	7.37468×10^{-8}	9.57364×10^{-8}	7.37468×10^{-8}	9.57364×10^{-8}
	14	3.27834×10^{-9}	5.57936×10^{-9}	3.27834×10^{-9}	5.57936×10^{-9}

4.6 Conclusion

In our work, the authors have developed an approach based on an operational matrix of Vieta-Fibonacci polynomials and the collocation method to solve two-dimensions space-time variable-order reaction advection diffuion equations. The fractional derivative of variable order are described in the Caputo sense. It is seen that the proposed scheme is sufficiently efficient and accurate, and the accuracy of the scheme enhances as the degree of Vieta-Fibonacci polynomials increases. It is analytically shown that the obtained approximate solution converges to the exact solution. Moreover, it is also numerically found that the rate of convergence of the scheme becomes fast when we improve the degree of Vieta-Fibonacci polynomials (m), and these facts are verified for the three considered examples.

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