

## Chapter 4

# An intermediate Voronovskaja type theorem for new $\alpha$ -Bernstein Kantorovich type operators

Throughout this chapter, we consider  $X$  to be the set  $[0, 1]$ .

### 4.1 Introduction

An important concern for most of the researchers about the Bernstein operators is to improve their speed of convergence. In the literature, although there is no such operator till now which has always a better convergence rate than the Bernstein polynomials for all continuous functions, we can see that the  $\lambda$ -Bernstein operators introduced by Cai et al. [38] has better convergence than the classical Bernstein operators for at least some test functions with wise selection of the shape parameter  $\lambda$ .

In [44], Chen et al. generalized Bernstein operators with the help of a parameter  $\alpha \geq 0$ , defined as

$$T_{m,\alpha}\Psi(x) = \sum_{j=0}^m p_{m,j}^{(\alpha)}(x)\Psi\left(\frac{j}{m}\right), \quad \Psi \in \mathfrak{C}(X), \quad (4.1)$$

where  $p_{m,j}^{(\alpha)}(x)$  are the  $\alpha$ -Bernstein polynomials defined as

$$\begin{cases} p_{1,0}^{(\alpha)}(x) = 1 - x, \\ p_{1,1}^{(\alpha)}(x) = x, \\ p_{m,j}^{(\alpha)}(x) = \left[ \binom{m-2}{j} (1-\alpha)x + \binom{m-2}{j-2} (1-\alpha)(1-x) + \binom{m}{j} \alpha x(1-x) \right] x^{j-1} (1-x)^{m-j-1}, \quad m \geq 2. \end{cases}$$

We observe that for  $\alpha = 1$ , the  $\alpha$ -Bernstein operator (4.1) is reduced to the classical Bernstein operators (1.1). In that paper, the authors showed the relevance of the operators (4.1) over the Bernstein operators by proving some convergence properties. Furthermore, Mohiuddine et al. [90] have defined a Kantorovich generalization of the operators (4.1) as

$$K_{m,\alpha}\Psi(x) = (m+1) \sum_{j=0}^m p_{m,j}^{(\alpha)}(x) \int_{\frac{j}{m+1}}^{\frac{j+1}{m+1}} \Psi(t) dt, \quad (4.2)$$

and studied uniform convergence of the operators and rate of convergence in a local and global sense in terms of first- and second-order modulus of continuity. They have also investigated the bivariate generalization of Bernstein-Kantorovich operators and their approximation behaviors.

Moreover, long back in 1963, P. P. Korovkin brought a revolution by stating “If a positive linear operator on a compact set preserves constant, linear, and quadratic polynomials at least in limiting sense, then it will approximate any continuous function on that set”. Using this, the examination of the convergence of Positive linear operators to their test functions became easier. In [72], J. P. King defined the Lototsky-Bernstein operator, which preserves  $x^2$ . Recently, in [100], Dumitru Popa has introduced the Lototsky-Kantorovich operators and studied Korovkin-type results and intermediate Voronovskaja-type results for a generalized variant in the

probability measure space. In recent years, Kantorovich-type generalizations have been a popular area of research both in the sense of defining new operators as well as in respect of exponential sampling series. For further investigation, one can see, [4, 8, 9, 10, 23, 27, 69, 75, 77, 79, 80, 104] ] and the reference cited therein.

Motivated by the above-defined work, in this chapter, we introduce a new Kantorovich variant of the  $\alpha$ -Bernstein operator and study the Korovkin type convergence result and Voronovskaja type results for probability measure space. Also, we investigate some of its important convergence properties using the relationship between the modulus of continuity and Peetre's K-functional. At the end, we estimate the error for the class of functions whose derivatives are of bounded variation.

Throughout this chapter, the spaces  $\mathfrak{C}(X)$  and  $\mathfrak{C}_B^2(X)$  are equipped with the sup norm:  $\|\Psi\| = \sup\{|\Psi(x)| : x \in X\}$ .

In the next section, we introduce a generalized version of  $\alpha$ -Bernstein-Kantorovich operators (4.2) and establish a Korovkin-type result.

## 4.2 A new modification and Auxiliary results

The generalization of our interest in this research work is defined for the function  $\Psi \in \mathfrak{C}(X)$ ,  $m \in \mathbb{N}$ , and  $x \in (0, 1)$  by

$$K_{m,\alpha}^* \Psi(x) = \frac{m+1}{b_m - a_m} \sum_{j=0}^m p_{m,j}^{(\alpha)}(x) \int_{\frac{j+a_m}{m+1}}^{\frac{j+b_m}{m+1}} \Psi(u) du, \quad (4.3)$$

where  $(a_m)_{m \geq 1}$  and  $(b_m)_{m \geq 1}$  are two sequences of real numbers with  $0 \leq a_m < b_m \leq 1$ , for every  $m \geq 1$ .

Again, by substituting  $u = \frac{j+t}{m+1}$  in (4.3), we get the following variant:

$$K_{m,\alpha}^* \Psi(x) = \frac{1}{b_m - a_m} \sum_{j=0}^m p_{m,j}^{(\alpha)}(x) \int_{a_m}^{b_m} \Psi \left( \frac{j+t}{m+1} \right) dt. \quad (4.4)$$

**Remark 4.1.** One can note that if  $a_m = 0$  &  $b_m = 1$  are constant sequences, then the above-defined operator boils down to the natural  $\alpha$ -Bernstein-Kantorovich operators defined in [90].

In what follows, we consider some results, notations, and notions. We consider the real valued functions  $e_i(x) = x^i$  and  $e_{i,x}(t) = (t-x)^i$  for all  $x \in X$  and  $i \in \mathbb{N} \cup \{0\}$ . It was proved by Chen et al. [44] that:  $T_{m,\alpha}(e_0)(x) = \sum_{j=0}^m p_{m,j}^{(\alpha)}(x) = 1$ ,  $T_{m,\alpha}(e_1)(x) = \sum_{j=0}^m p_{m,j}^{(\alpha)}(x) \left(\frac{j}{m}\right) = x$  and  $T_{m,\alpha}(e_2)(x) = \sum_{j=0}^m p_{m,j}^{(\alpha)}(x) \left(\frac{j}{m}\right)^2 = x^2 + \frac{m+2(1-\alpha)}{m^2} x(1-x)$ .

As a consequence, we also have

$$T_{m,\alpha}(e_{1,x})(x) = \sum_{j=0}^m p_{m,j}^{(\alpha)}(x) \left(\frac{j}{m} - x\right) = 0 \quad (4.5)$$

$$T_{m,\alpha}(e_{2,x})(x) = \sum_{j=0}^m p_{m,j}^{(\alpha)}(x) \left(\frac{j}{m} - x\right)^2 = \frac{m+2(1-\alpha)}{m^2} x(1-x). \quad (4.6)$$

Throughout this chapter, by  $E_m$ ,  $\mathcal{F}_m$ , and  $\xi_m$ , we denote the sequence of sample space, sigma algebra, and probability measure, respectively. Now, we establish an important convergence result for a natural extension of the new  $\alpha$ -Bernstein-Kantorovich operators (4.4). For a particular case, one can see [44, 90].

**Theorem 4.2.** Let  $(a_m)_{m \geq 1}$  and  $(b_m)_{m \geq 1}$  are two sequences of real numbers with  $0 \leq a_m < b_m \leq 1$ , for every  $m \geq 1$ . Let  $(E_m, \mathcal{F}_m, \xi_m)$  be a sequence of probability measure spaces and  $h_m : E_m \rightarrow \mathbb{R}$  be a sequence of  $\xi_m$ -integrable functions with the property that for every  $m \in \mathbb{N}$ , there exists  $\beta_m \geq 0$ , such that  $0 \leq h_m(\omega_m) \leq \beta_m$ , for

all  $\omega_m \in E_m$ . Let the operator  $A_{m,\alpha}^* : \mathfrak{C}(X) \rightarrow \mathfrak{C}(X)$  be defined by

$$A_{m,\alpha}^* \Psi(x) = \frac{1}{b_m - a_m} \sum_{j=0}^m p_{m,j}^{(\alpha)}(x) \int_{E_m} \Psi\left(\frac{j + h_m(\omega_m)}{m + \beta_m}\right) d\mu_m(\omega_m), \quad \forall \alpha \geq 0,$$

where  $m \in \mathbb{N}$  and  $\{\mu_m\}_{m \geq 1}$  is the new sequence of measure  $\{(b_m - a_m)\xi_m\}_{m \geq 1}$ .

Then  $\lim_{m \rightarrow \infty} A_{m,\alpha}^* \Psi = \Psi$  uniformly on  $X$  for every  $\Psi \in \mathfrak{C}(X)$ , iff  $\lim_{m \rightarrow \infty} \frac{\beta_m}{(m + \beta_m)} = 0$ .

*Proof.* ( $\Leftarrow$ )

Let us note that all  $\mu_m(\Omega_m) = b_m - a_m$ ,  $\forall m \in \mathbb{N}$  and the operators  $A_{m,\alpha}^*$  are linear and positive. So, it is obvious that  $A_{m,\alpha}^*(e_0)(x) = 1$ ,  $\forall m \in \mathbb{N}$ . One can conclude that  $\lim_{m \rightarrow \infty} A_{m,\alpha}^*(e_0) = e_0$ .

Next, for every  $x \in X$ , we get

$$\begin{aligned} A_{m,\alpha}^*(e_1)(x) &= \frac{1}{b_m - a_m} \sum_{j=0}^m p_{m,j}^{(\alpha)}(x) \int_{E_m} \frac{j + h_m(\omega_m)}{m + \beta_m} d\mu_m(\omega_m) \\ &= \frac{mT_{m,\alpha}(e_1)(x)}{m + \beta_m} + \frac{\int_{E_m} h_m(\omega_m) d\mu_m}{(b_m - a_m)(m + \beta_m)} \end{aligned} \quad (4.7)$$

or equivalently

$$\begin{aligned} A_{m,\alpha}^*(e_1)(x) - x &= \frac{mx}{m + \beta_m} + \frac{\int_{E_m} h_m(\omega_m) d\mu_m}{(b_m - a_m)(m + \beta_m)} - x \\ \implies A_{m,\alpha}^*(e_1)(x) - x &= x \left[ \frac{m}{m + \beta_m} - 1 \right] + \frac{\int_{E_m} h_m(\omega_m) d\mu_m}{(b_m - a_m)(m + \beta_m)} \\ \implies \|A_{m,\alpha}^*(e_1) - e_1\| &\leq \left| \frac{m}{m + \beta_m} - 1 \right| + \frac{\beta_m}{m + \beta_m} = \frac{2\beta_m}{m + \beta_m}. \end{aligned} \quad (4.8)$$

So, by using the limiting condition taken in the hypothesis, we deduce that

$$\lim_{m \rightarrow \infty} \|A_{m,\alpha}^*(e_1) - e_1\| = 0.$$

Again,  $\forall x \in X$ , we have

$$\begin{aligned} A_{m,\alpha}^*(e_2)(x) &= \frac{1}{b_m - a_m} \sum_{j=0}^m p_{m,j}^{(\alpha)}(x) \int_{E_m} \left( \frac{j + h_m(\omega_m)}{m + \beta_m} \right)^2 d\mu_m(\omega_m) \\ &= \frac{m^2 T_{m,\alpha}(e_2)(x)}{(m + \beta_m)^2} + \frac{2m T_{m,\alpha}(e_1)(x)}{(b_m - a_m)(m + \beta_m)^2} \int_{E_m} h_m(\omega_m) d\mu_m \\ &\quad + \frac{\int_{E_m} (h_m(\omega_m))^2 d\mu_m}{(b_m - a_m)(m + \beta_m)^2} \\ &\leq \frac{m^2 T_{m,\alpha}(e_2)(x)}{(m + \beta_m)^2} + \frac{2m T_{m,\alpha}(e_1)(x)}{(m + \beta_m)^2} \beta_m + \frac{\beta_m^2}{(m + \beta_m)^2} \end{aligned}$$

or equivalently

$$\|A_{m,\alpha}^*(e_2)(x) - e_2(x)\| \leq \frac{m^2 x^2 + [m + 2(1 - \alpha)]x(1 - x) + 2mx\beta_m + \beta_m^2}{(m + \beta_m)^2} - x^2.$$

It is easy to see that  $\lim_{m \rightarrow \infty} \frac{\beta_m}{(m + \beta_m)} = 0$  implies  $\lim_{m \rightarrow \infty} \frac{m}{(m + \beta_m)} = 1$  and hence it follows that  $\lim_{m \rightarrow \infty} A_{m,\alpha}^*(e_2) = e_2$ . Thus, the result follows by applying the well-known Korovkin's theorem [21].

( $\Rightarrow$ )

Let us suppose that  $\forall \Psi \in \mathfrak{C}(X)$ ,  $\lim_{m \rightarrow \infty} A_{m,\alpha}^*(\Psi) = \Psi$  uniformly on the set  $X$ . Clearly then  $\lim_{m \rightarrow \infty} A_{m,\alpha}^*(e_1) = e_1$ .

Now, from (4.8), we have

$$\begin{aligned} A_{m,\alpha}^*(e_1)(x) - x &= \frac{-\beta_m x}{(m + \beta_m)} + \frac{\int_{E_m} h_m(\omega_m) d\mu_m}{(b_m - a_m)(m + \beta_m)} \\ \Rightarrow \frac{\beta_m}{m + \beta_m} \left\| \frac{\int_{E_m} h_m(\omega_m) d\mu_m}{(b_m - a_m)\beta_m} - x \right\| &= \|A_{m,\alpha}^*(e_1)(x) - x\|. \end{aligned} \quad (4.9)$$

It is easy to notice that  $\frac{\int_{E_m} h_m(\omega_m) d\mu_m}{(b_m - a_m)\beta_m} \in [0, 1]$ ,  $\forall m \in \mathbb{N}$ . Hence,

$$\frac{1}{2} \leq \left\| \frac{\int_{\Omega_m} h_m(\omega_m) d\mu_m}{(b_m - a_m)\beta_m} - x \right\| = \sup_{x \in X} \left| \frac{\int_{\Omega_m} h_m(\omega_m) d\mu_m}{(b_m - a_m)\beta_m} - x \right| \leq 1.$$

This, along with the fact that right side of (4.9) tends to zero as  $m \rightarrow \infty$ , we conclude  $\lim_{m \rightarrow \infty} \frac{\beta_m}{(m + \beta_m)} = 0$ .  $\square$

### 4.3 A Voronovskaja-type result

We'll use the following theorem, in order to prove an intermediate Voronovskaja-type result.

**Theorem 4.3.** [99] *Let  $V_m : \mathfrak{C}(X) \rightarrow \mathfrak{C}(X)$  be a sequence of positive linear operators such that  $\lim_{m \rightarrow \infty} V_m(\Psi) = \Psi$  uniformly on  $X$  for every  $\Psi \in \mathfrak{C}(X)$ . Let  $x \in X$  and suppose that there exists a sequence  $(\lambda_m)_{m \in \mathbb{N}} \subset (0, \infty)$  with  $\lim_{m \rightarrow \infty} \lambda_m = \infty$  such that the sequences  $(\lambda_m V_m(e_{1,x})(x))_{m \in \mathbb{N}}$  and  $(\lambda_m^2 V_m(e_{2,x})(x))_{m \in \mathbb{N}}$  are bounded. Then  $\forall \Psi \in \mathfrak{C}(X)$ , which is differentiable at a point  $x \in X$  we must have*

$$\lim_{m \rightarrow \infty} \lambda_m [V_m(\Psi)(x) - \Psi(x)V_m(e_0)(x) - \Psi'(x)V_m(e_{1,x})(x)] = 0.$$

D. Popa [100] discussed an intermediate Voronovskaja type theorem for Lototsky-Bernstein type operators. We establish an intermediate Voronovskaja type theorem for the  $\alpha$ -Bernstein operators defined in (4.1).

**Theorem 4.4.** *Let  $T_{m,\alpha} : \mathfrak{C}(X) \rightarrow \mathfrak{C}(X)$  be the sequence of  $\alpha$ -Bernstein operators and  $x \in X$ . Suppose  $\exists$  a sequence  $(\lambda_m)_{m \in \mathbb{N}} \subset (0, \infty)$  with the properties that  $\lim_{m \rightarrow \infty} \lambda_m = \infty$  and  $\lim_{m \rightarrow \infty} \frac{\lambda_m^2}{m} \in \mathbb{R}$ . Then the following equation holds for every*

$\Psi \in \mathfrak{C}(X)$  differentiable at  $x \in X$ .

$$\lim_{m \rightarrow \infty} \lambda_m [T_{m,\alpha}(\Psi)(x) - \Psi(x)] = 0.$$

*Proof.* From (4.5), we have  $\lambda_m T_{m,\alpha}(e_{1,x})(x) = 0$ , this implies that  $(\lambda_m T_{m,\alpha}(e_{1,x})(x))_{m \in \mathbb{N}}$  is a bounded sequence. From (4.6), we obtain

$$\lambda_m^2 T_{m,\alpha}(e_{2,x})(x) = \frac{\lambda_m^2}{m} + 2 \frac{\lambda_m^2}{m^2} (1 - \alpha) x(1 - x).$$

Since  $\lim_{m \rightarrow \infty} \frac{\lambda_m^2}{m} = 0$  is also bounded. This implies that  $\lambda_m^2 T_{m,\alpha}(e_{2,x})(x)$  is also bounded. Thus, using Theorem 4.3, the desired result can be obtained.  $\square$

Now, we discuss an intermediate Voronovskaja type result for a generalized  $\alpha$ -Bernstein type operators.

**Theorem 4.5.** *Let  $(a_m)_{m \geq 1}$  and  $(b_m)_{m \geq 1}$  be two sequences of real numbers with  $0 \leq a_m < b_m \leq 1$ , for every  $m \geq 1$ . Suppose  $\exists$  a sequence  $(\lambda_m)_{m \in \mathbb{N}} \subset (0, \infty)$  with the properties that  $\lim_{m \rightarrow \infty} \lambda_m = \infty$  and  $\lim_{m \rightarrow \infty} \frac{\lambda_m^2}{m} \in \mathbb{R}$ . Let  $(E_m, \xi_m)$  be a sequence of probability measure spaces and  $h_m : E_m \rightarrow \mathbb{R}$  be  $\xi_m$ -integrable functions having the property that  $\exists \theta_m \geq 0$  such that  $0 \leq h_m(\omega_m) \leq \theta_m$ ,  $\forall m \in \mathbb{N}$  and for every  $\omega_m \in E_m$ . Let  $A_{m,\alpha}^* : \mathfrak{C}(X) \rightarrow \mathfrak{C}(X)$  be the linear positive operator defined by*

$$A_{m,\alpha}^* \Psi(x) = \frac{1}{(b_m - a_m)} \sum_{j=0}^m p_{m,j}^{(\alpha)}(x) \int_{E_m} \Psi \left( \frac{j + h_m(\omega_m)}{m + \theta_m} \right) d\mu_m(\omega_m) \quad \forall m \in \mathbb{N}, \quad (4.10)$$

where  $\{\mu_m\}_{m \geq 1}$  is the new sequence of measure  $\{(b_m - a_m)\xi_m\}_{m \geq 1}$ . If  $\lim_{m \rightarrow \infty} \frac{\theta_m}{m + \theta_m} = 0$ , then the following limiting condition holds for every function  $\Psi \in \mathfrak{C}(X)$ , which is

differentiable at  $x \in X$ .

$$\lim_{m \rightarrow \infty} \lambda_m [A_{m,\alpha}^* \Psi(x) - \Psi(x)] = 0.$$

*Proof.* Let  $x \in X$  and  $m \in \mathbb{N}$ . From (4.10), we have

$$\begin{aligned} A_{m,\alpha}^*(e_{1,x})(x) &= \frac{1}{(b_m - a_m)} \sum_{j=0}^m p_{m,j}^{(\alpha)}(x) \int_{E_m} \left( \frac{j + h_m(\omega_m)}{m + \theta_m} - x \right) d\mu_m(\omega_m) \\ &= \frac{m}{(m + \theta_m)} \sum_{j=0}^m (j/m - x) p_{m,j}^{(\alpha)}(x) \\ &\quad + \frac{1}{(b_m - a_m)(m + \theta_m)} \int_{E_m} (h_m(\omega_m) - \theta_m x) d\mu_m(\omega_m) \\ &= \frac{m}{(m + \theta_m)} T_{m,\alpha}(e_{1,x})(x) + \frac{\int_{E_m} (h_m(\omega_m) - \theta_m x) d\mu_m(\omega_m)}{(b_m - a_m)(m + \theta_m)}. \end{aligned}$$

Since  $\lim_{m \rightarrow \infty} \frac{\lambda_m}{\sqrt{m}} \in \mathbb{R}$ , then it follows that  $\lim_{m \rightarrow \infty} \frac{\lambda_m}{m} = 0$  and thus we have

$$\lim_{m \rightarrow \infty} \lambda_m A_{m,\alpha}^*(e_{1,x})(x) = \lim_{m \rightarrow \infty} \lambda_m T_{m,\alpha}(e_{1,x})(x) = 0.$$

Next,

$$\begin{aligned} A_{m,\alpha}^*(e_{2,x})(x) &= \frac{1}{(b_m - a_m)} \sum_{j=0}^m p_{m,j}^{(\alpha)}(x) \int_{E_m} \left( \frac{j + h_m(\omega_m)}{m + \theta_m} - x \right)^2 d\mu_m(\omega_m), \quad \forall m \in \mathbb{N} \\ &= \frac{m^2}{(m + \theta_m)^2} \sum_{j=0}^m (j/m - x)^2 p_{m,j}^{(\alpha)}(x) \\ &\quad + \frac{2m}{(b_m - a_m)(m + \theta_m)^2} \sum_{j=0}^m (j/m - x) p_{m,j}^{(\alpha)}(x) \int_{E_m} (h_m(\omega_m) - \theta_m x) d\mu_m(\omega_m) \\ &\quad + \frac{\int_{E_m} (h_m(\omega_m) - \theta_m x)^2 d\mu_m(\omega_m)}{(b_m - a_m)(m + \theta_m)^2} \\ &= \frac{m^2}{(m + \theta_m)^2} T_{m,\alpha}(e_{2,x})(x) \\ &\quad + \frac{2m}{(b_m - a_m)(m + \theta_m)^2} T_{m,\alpha}(e_{1,x})(x) \int_{E_m} (h_m(\omega_m) - \theta_m x) d\mu_m(\omega_m) \\ &\quad + \frac{\int_{E_m} (h_m(\omega_m) - \theta_m x)^2 d\mu_m(\omega_m)}{(b_m - a_m)(m + \theta_m)^2} \end{aligned}$$

Multiplying  $\lambda_m^2$  on both sides of the above equation, we have

$$\begin{aligned} \lambda_m^2 A_{m,\alpha}^*(e_{2,x})(x) &= \frac{m^2}{(m + \theta_m)^2} \lambda_m^2 T_{m,\alpha}(e_{2,x})(x) \\ &+ \frac{2m^2}{(b_m - a_m)(m + \theta_m)^2} \frac{\lambda_m}{m} \cdot \lambda_m T_{m,\alpha}(e_{1,x})(x) \int_{E_m} (h_m(\omega_m) - \theta_m x) d\mu_m(\omega_m) \\ &+ \lambda_m^2 \frac{\int_{E_m} (h_m(\omega_m) - \theta_m x)^2 d\mu_m(\omega_m)}{(b_m - a_m)(m + \theta_m)^2}. \end{aligned}$$

In the proof of the Theorem 4.4, we have shown that  $(\lambda_m^2 T_{m,\alpha}(e_{2,x})(x))$  is a bounded. Next, applying the limit  $m \rightarrow \infty$  in the above equation, we deduce,  $(\lambda_m^2 A_{m,\alpha}^*(e_{2,x})(x))$  is also bounded. The required proof is obtained by applying Theorem 4.3.  $\square$

## 4.4 Rate of Convergence

First, we calculate the central moments of the operators (4.3).

**Lemma 4.6.** *The central moments of the newly defined operators (4.3) are evaluated to be:*

$$\begin{aligned} K_{m,\alpha}^*(e_{1,x})(x) &= \frac{-x}{m+1} + \frac{a_m + b_m}{2(m+1)}, \\ K_{m,\alpha}^*(e_{2,x})(x) &= \frac{x^2}{(m+1)^2} + \frac{\{m + 2(1 - \alpha)\}x(1-x) - x(a_m + b_m)}{(m+1)^2} + \frac{a_m^2 + a_m b_m + b_m^2}{3(m+1)^2}. \end{aligned}$$

In the next result, we shall discuss the rate of convergence of the operators  $K_{m,\alpha}^*$  for the Lipschitz type function space.

**Theorem 4.7.** *Let  $\Psi \in Lip_{M,\eta}^*(X)$ . Then for each  $x \in (0, 1]$ ,*

$$|K_{m,\alpha}^*(\Psi; x) - \Psi(x)| \leq \frac{M}{x^{\eta/2}} (K_{m,\alpha}^*(t-x)^2; x)^{\eta/2}.$$

*Proof.* In view of the fact that  $K_{m,\alpha}^*(e_0; x) = 1$ , we have

$$\begin{aligned} |K_{m,\alpha}^*(\Psi; x) - \Psi(x)| &\leq |K_{m,\alpha}^*(\Psi(t) - \Psi(x); x)| \\ &\leq K_{m,\alpha}^*(|\Psi(t) - \Psi(x)|; x) \end{aligned} \quad (4.11)$$

On the right hand side of (4.11), we apply Hölder's inequality by setting  $p = 2/\eta$  and  $q = 2/(2 - \eta)$  and obtain

$$\begin{aligned} |K_{m,\alpha}^*(\Psi; x) - \Psi(x)| &\leq M \left( K_{m,\alpha}^*(|\Psi(t) - \Psi(x)|^{2/\eta}; x) \right)^{\eta/2} \left( K_{m,\alpha}^*(e_0^{2/(2-\eta)}; x) \right)^{(2-\eta)/2} \\ &= \frac{M}{x^{\eta/2}} \left( K_{m,\alpha}^* \left( \frac{(t-x)^2}{t+x}; x \right) \right)^{\eta/2} \\ &\leq \frac{M}{x^{\eta/2}} \left( K_{m,\alpha}^*((t-x)^2; x) \right)^{\eta/2}. \end{aligned}$$

This is the required result. □

**Theorem 4.8.** *If  $\alpha \geq 0$  and  $\Psi \in \mathfrak{C}(X)$ . Then for all  $x \in X$ ,*

$$|K_{m,\alpha}^*(\Psi; x) - \Psi(x)| \leq 2\Omega_1 \left( \Psi; \sqrt{K_{m,\alpha}^*((t-x)^2; x)} \right).$$

*Proof.* Using the fact that  $K_{m,\alpha}^*(1; x) = 1$ , we have

$$|K_{m,\alpha}^*(\Psi; x) - \Psi(x)| \leq K_{m,\alpha}^*(|\Psi(t) - \Psi(x)|; x) \quad (4.12)$$

Now, applying the following property of modulus of continuity

$$|\Psi(t) - \Psi(x)| \leq \Omega_1(\Psi; \delta) \left( \frac{|t-x|^2}{\delta^2} + 1 \right),$$

in (4.12) to get

$$|K_{m,\alpha}^*(\Psi; x) - \Psi(x)| \leq \Omega_1(\Psi; \delta) \left( \frac{K_{m,\alpha}^*((t-x)^2; x)}{\delta^2} + 1 \right).$$

If we select  $\delta = \sqrt{K_{m,\alpha}^*((t-x)^2; x)}$ , we get the desired result.  $\square$

**Theorem 4.9.** *Let  $\Psi \in \mathfrak{C}(X)$  and  $m \in \mathbb{N}$ . Then  $\exists \mathcal{M} > 0$  such that*

$$|K_{m,\alpha}^*\Psi(x) - \Psi(x)| \leq \mathcal{M}\Omega_2 \left( \Psi, \frac{\sqrt{\delta_m^{(\alpha)}(x)}}{2} \right) + \Omega_1(\Psi, \tau_m^{(\alpha)}(x)),$$

where  $\delta_m^{(\alpha)}(x) = \frac{1}{2} [K_{m,\alpha}^*(e_{2,x})(x) + \{K_{m,\alpha}^*(e_{1,x})(x)\}^2]$  and  $\tau_m^{(\alpha)}(x) = K_{m,\alpha}^*(e_{1,x})(x)$ .

*Proof.* We define an auxiliary operator  $\tilde{K}_{m,\alpha}^* : \mathfrak{C}(X) \rightarrow \mathfrak{C}(X)$  by

$$\tilde{K}_{m,\alpha}^*\Psi(x) = K_{m,\alpha}^*\Psi(x) - \Psi(\tau_m^{(\alpha)}(x)) + \Psi(x).$$

It is easy to observe that  $\tilde{K}_{m,\alpha}^*(e_{1,x})(x) = 0$ . Now, for any  $\Phi \in C_B^2(X)$  and  $u, x \in X$  the Taylor series expansion yields

$$\Phi(u) = \Phi(x) + (u-x)\Phi'(x) + \int_x^u (u-z)\Phi''(z)dz. \quad (4.13)$$

We apply the auxiliary operator on both sides of (4.13) to get

$$\begin{aligned} |\tilde{K}_{m,\alpha}^*\Phi(x) - \Phi(x)| &\leq \tilde{K}_{m,\alpha}^*|\Phi(t) - \Phi(x)|(x) \leq \tilde{K}_{m,\alpha}^* \left| \int_x^u (u-z)\Phi''(z)dz \right| (x) \\ &\leq K_{m,\alpha}^* \left| \int_x^u (u-z)\Phi''(z)dz \right| (x) + \left| \int_x^{\tau_m^{(\alpha)}(x)} (\tau_m^{(\alpha)}(x) - z)\Phi''(z)dz \right| \\ &\leq \frac{1}{2} [K_{m,\alpha}^*(e_{2,x})(x)\|\Phi''\| + (\tau_m^{(\alpha)}(x) - x)^2 \|\Phi''\|] \\ &= \delta_m^{(\alpha)}(x)\|\Phi''\|. \end{aligned}$$

One should notice that  $\|K_{m,\alpha}^* \Psi(x)\| \leq \|\Psi\|$  for any  $\Psi \in \mathfrak{C}(X)$  and hence applying triangle inequality, we can get

$$\|\tilde{K}_{m,\alpha}^* \Psi(x)\| \leq 3\|\Psi\|, \text{ for any } \Psi \in \mathfrak{C}(X).$$

Using the above inequality we have

$$\begin{aligned} & |K_{m,\alpha}^* \Psi(x) - \Psi(x)| \\ & \leq |\tilde{K}_{m,\alpha}^* (\Psi - \Phi)(x) - (\Psi - \Phi)(x)| + |\tilde{K}_{m,\alpha}^* \Phi(x) - \Phi(x)| + |\Psi(x) - \Psi(\tau_m^{(\alpha)}(x))| \\ & \leq 4\|\Psi - \Phi\| + \delta_m^{(\alpha)}(x)\|\Phi''\| + \Omega_1(\Psi, \tau_m^{(\alpha)}(x)) \\ & = 4 \left[ \|\Psi - \Phi\| + \frac{\delta_m^{(\alpha)}(x)}{4}\|\Phi''\| \right] + \Omega_1(\Psi, \tau_m^{(\alpha)}(x)). \end{aligned}$$

Lastly, taking infimum over all  $\Phi \in C_B^2(X)$  and then using the relation (1.9), we get the desired result.  $\square$

## 4.5 Direct estimation

In this section, we'll discuss a convergence result for the class of absolutely continuous functions whose first derivatives are of bounded variation on  $X$ .

**Remark 4.10.** Taking into consideration of the above Lemma 4.6, the following bounds can be obtained for large value of  $m$ :

$$\begin{aligned} |K_{m,\alpha}^*(e_{1,x})(x)| & \leq 1 \frac{(1+x)}{m+1}, \\ |K_{m,\alpha}^*(e_{2,x})(x)| & \leq \frac{r}{m+1} \gamma_{m,\alpha}^2(x), \end{aligned}$$

where  $r \geq 1$ ,  $\gamma_{m,\alpha}^2(x) = \left[ \phi^2(x) + \frac{3+2|1-\alpha|}{m+1} \right]$  and  $\phi^2(x) = x(1-x)$ .

Next, we set the kernel  $\mathfrak{K}_{m,\alpha}(x,t) = \frac{m+1}{b_m-a_m} \sum_{j=0}^m p_{m,j}^{(\alpha)}(x) \chi_{m,j}(t)$ , where  $\chi_{m,j}(t)$  is the characteristics function on  $\left[ \frac{j+a_m}{m+1}, \frac{j+b_m}{m+1} \right]$  with respect to  $X$ . Clearly then

$$K_{m,\alpha}^* \Psi(x) = \int_0^1 \mathfrak{K}_{m,\alpha}(x,t) \Psi(t) dt. \quad (4.14)$$

**Lemma 4.11.** *For  $x \in (0, 1]$ ,  $r \geq 1$  and taking  $m$  large enough, the following relations hold:*

1. *If  $0 \leq y < x$ , then*

$$\zeta_{m,\alpha}(x,y) = \int_0^y \mathfrak{K}_{m,\alpha}(x,t) dt \leq \frac{r}{(m+1)} \frac{\gamma_{m,\alpha}^2(x)}{(x-y)^2}.$$

2. *If  $x < z < 1$ , then*

$$1 - \zeta_{m,\alpha}(x,z) = \int_z^1 \mathfrak{K}_{m,\alpha}(x,t) dt \leq \frac{r}{(m+1)} \frac{\gamma_{m,\alpha}^2(x)}{(z-x)^2}.$$

*Proof.* We use (4.14) and Remark 4.10 and observe that, for  $0 \leq y < x$ ,

$$\begin{aligned} \zeta_{m,\alpha}(x,y) &= \int_0^y \mathfrak{K}_{m,\alpha}(x,t) dt \leq \int_0^y \left( \frac{t-x}{y-x} \right)^2 \mathfrak{K}_{m,\alpha}(x,t) dt \\ &\leq \frac{K_{m,\alpha}^*((t-x)^2; x)}{(x-y)^2} \leq \frac{r}{(m+1)} \frac{\gamma_{m,\alpha}^2(x)}{(x-y)^2}. \end{aligned}$$

In a similar way, the other part can be done. □

**Theorem 4.12.** *Let  $\Psi \in BV'(0, 1)$  and  $x \in (0, 1)$ . Then for  $r \geq 1$  and sufficiently large  $m$ , the following inequality holds:*

$$|K_{m,\alpha}^* \Psi(x) - \Psi(x)| \leq \frac{1}{2} |\Psi'(x+) + \Psi'(x-)| \frac{(1+x)}{m+1} + \frac{1}{2} \sqrt{\frac{r}{m+1}} |\Psi'(x+) - \Psi'(x-)| \gamma_{m,\alpha}(x)$$

$$\begin{aligned}
 & + \frac{r\gamma_{m,\alpha}^2(x)}{x(m+1)} \sum_{j=1}^{[\sqrt{m}]} \mathfrak{T}_{x-\frac{x}{j}}^x(\Psi'_x) + \frac{x}{\sqrt{m}} \mathfrak{T}_{x-\frac{x}{\sqrt{m}}}^x(\Psi'_x) \\
 & + \frac{r\gamma_{m,\alpha}^2(x)}{(1-x)(m+1)} \sum_{j=1}^{[\sqrt{m}]} \mathfrak{T}_{x+\frac{(1-x)}{j}}^x(\Psi'_x) + \frac{(1-x)}{\sqrt{m}} \mathfrak{T}_{x+\frac{(1-x)}{\sqrt{m}}}^x(\Psi'_x),
 \end{aligned}$$

where  $\mathfrak{T}_q^r(\Psi)$  denotes the total variation of the function  $\Psi$  on  $[q, r] \subset X$  and the function  $\Psi'_x$  is defined as

$$\Psi'_x(t) = \begin{cases} \Psi'(t) - \Psi'(x-), & 0 \leq t < x, \\ \Psi'(t) - \Psi'(x+), & x < t \leq 1, \\ 0, & t = x. \end{cases}$$

*Proof.* We observe the fact that  $K_{m,\alpha}^*(e_0)(x) = e_0$  and hence

$$\begin{aligned}
 K_{m,\alpha}^* \Psi(x) - \Psi(x) &= \int_0^1 (\Psi(t) - \Psi(x)) \mathfrak{K}_{m,\alpha}(x, t) dt \\
 &= \int_0^1 \mathfrak{K}_{m,\alpha}(x, t) \left( \int_x^t \Psi'(z) dz \right) dt. \tag{4.15}
 \end{aligned}$$

Now, for any  $\Psi \in BV'(0, 1)$ , the following relation holds:

$$\begin{aligned}
 \Psi'(z) &= \frac{1}{2} \{ \Psi'(x+) + \Psi'(x-) \} + \Psi'_x(z) + \frac{1}{2} \{ \Psi'(x+) - \Psi'(x-) \} \operatorname{sgn}(z - x) \\
 &+ \delta_x^z [ \Psi'(z) - \frac{1}{2} \{ \Psi'(x+) + \Psi'(x-) \} ], \tag{4.16}
 \end{aligned}$$

where  $\delta_x^z$  is the Kronecker delta function defined as

$$\delta_x^z = \begin{cases} 1, & z = x \\ 0, & z \neq x. \end{cases}$$

We can easily observe that

$$\int_0^1 \left( \int_x^t \left[ \Psi'(z) - \frac{1}{2} \{ \Psi'(x+) + \Psi'(x-) \} \right] \delta_x^z dz \right) \mathfrak{K}_{m,\alpha}(x, t) dt = 0. \quad (4.17)$$

Now, using the relation (4.14), we get

$$\int_0^1 \left( \int_x^t \frac{1}{2} \{ \Psi'(x+) + \Psi'(x-) \} dz \right) \mathfrak{K}_{m,\alpha}(x, t) dt = \frac{1}{2} \{ \Psi'(x+) + \Psi'(x-) \} K_{m,\alpha}^*(t-x)(x). \quad (4.18)$$

Again, some calculations and the well-known Cauchy-Schwarz inequality yields

$$\begin{aligned} & \int_0^1 \left( \int_x^t \frac{1}{2} \{ \Psi'(x+) - \Psi'(x-) \} \operatorname{sgn}(z-x) dz \right) \mathfrak{K}_{m,\alpha}(x, t) dt \\ & \leq \frac{1}{2} \{ \Psi'(x+) - \Psi'(x-) \} \int_0^1 |t-x| \mathfrak{K}_{m,\alpha}(x, t) dt \\ & = \frac{1}{2} \{ \Psi'(x+) - \Psi'(x-) \} K_{m,\alpha}^*(|t-x|)(x) \\ & \leq \frac{1}{2} \{ \Psi'(x+) - \Psi'(x-) \} [K_{m,\alpha}^*((t-x)^2)(x)]^{1/2}. \end{aligned} \quad (4.19)$$

Now, using (4.16)-(4.19) and Remark 4.10 in (4.15), we have

$$\begin{aligned} K_{m,\alpha}^* \Psi(x) - \Psi(x) & \leq \frac{1}{2} \{ \Psi'(x+) + \Psi'(x-) \} K_{m,\alpha}^*(t-x)(x) \\ & \quad + \frac{1}{2} \{ \Psi'(x+) - \Psi'(x-) \} [K_{m,\alpha}^*((t-x)^2)(x)]^{1/2} \\ & \quad + \int_0^1 \left( \int_x^t \Psi'_x(z) dz \right) \mathfrak{K}_{m,\alpha}(x, t) dt. \end{aligned}$$

This yields

$$\begin{aligned} |K_{m,\alpha}^* \Psi(x) - \Psi(x)| & \leq \frac{1}{2} |\Psi'(x+) + \Psi'(x-)| \frac{(1+x)}{m+1} + \frac{1}{2} \sqrt{\frac{r}{m+1}} \gamma_{m,\alpha}(x) |\Psi'(x+) - \Psi'(x-)| \\ & \quad + \left| \int_0^x \left( \int_x^t \Psi'_x(z) dz \right) \mathfrak{K}_{m,\alpha}(m, t) dt \right| + \left| \int_x^1 \left( \int_x^t \Psi'_x(z) dz \right) \mathfrak{K}_{m,\alpha}(x, t) dt \right| \\ & = \frac{1}{2} |\Psi'(x+) + \Psi'(x-)| \frac{(1+x)}{m+1} + \frac{1}{2} \sqrt{\frac{r}{m+1}} |\Psi'(x+) - \Psi'(x-)| \gamma_{m,\alpha}(x) \\ & \quad + \mathfrak{J}_1 + \mathfrak{J}_2, \end{aligned} \quad (4.20)$$

where  $\mathfrak{J}_1 = \left| \int_0^x \left( \int_x^t \Psi'_x(z) dz \right) \mathfrak{K}_{m,\alpha}(x,t) dt \right|$ , and  $\mathfrak{J}_2 = \left| \int_x^1 \left( \int_x^t \Psi'_x(z) dz \right) \mathfrak{K}_{m,\alpha}(x,t) dt \right|$ .

In order to establish our claim, we need to find estimations for the integrals  $\mathfrak{J}_1$  and  $\mathfrak{J}_2$ . It is worth noting that for  $t < x$ , we have  $\mathfrak{K}_{m,\alpha}(x,t) dt = d_t \zeta_{m,\alpha}(x,t)$ , where  $d_t$  denotes the differential operator with respect to the variable  $t$ .

Using the by-parts rule of integration, we obtain

$$\begin{aligned} \mathfrak{J}_1 &= \left| \int_0^x \left( \int_x^t \Psi'_x(z) dz \right) d_t \zeta_{m,\alpha}(x,t) \right| \\ &= \left| \left[ \left( \int_x^t \Psi'_x(z) dz \right) \zeta_{m,\alpha}(x,t) \right]_0^x - \int_0^x \Psi'_x(t) \zeta_{m,\alpha}(x,t) dt \right| \\ &= \left| \int_0^x \zeta_{m,\alpha}(x,t) \Psi'_x(t) dt \right| \leq \int_0^y |\zeta_{m,\alpha}(x,t)| |\Psi'_x(t)| dt + \int_y^x |\zeta_{m,\alpha}(x,t)| |\Psi'_x(t)| dt. \end{aligned}$$

Choosing  $y = x - \frac{x}{\sqrt{m}}$ ,

$$\begin{aligned} \mathfrak{J}_1 &\leq \int_0^{x - \frac{x}{\sqrt{m}}} |\zeta_{m,\alpha}(x,t)| |\Psi'_x(t)| dt + \int_{x - \frac{x}{\sqrt{m}}}^x |\zeta_{m,\alpha}(x,t)| |\Psi'_x(t)| dt \\ &:= \mathfrak{J}_3 + \mathfrak{J}_4. \end{aligned}$$

To estimate  $\mathfrak{J}_4$ , we use the facts that  $\Psi'_x(x) = 0$  and  $|\zeta_{m,\alpha}(x,t)| \leq 1$  and then

$$\begin{aligned} \mathfrak{J}_4 &= \int_{x - \frac{x}{\sqrt{m}}}^x |\zeta_{m,\alpha}(x,t)| |\Psi'_x(t)| dt = \int_{x - \frac{x}{\sqrt{m}}}^x |\zeta_{m,\alpha}(x,t)| |\Psi'_x(t) - \Psi'_x(x)| dt \\ &\leq \int_{x - \frac{x}{\sqrt{m}}}^x \mathfrak{T}_t^x(\Psi'_x) dt \\ &\leq \frac{x}{\sqrt{m}} \mathfrak{T}_{x - \frac{x}{\sqrt{m}}}^x(\Psi'_x). \end{aligned} \tag{4.21}$$

Further using the Lemma 4.11 and then the substitution  $t = x - \frac{x}{z}$  yields the following inequality for  $\mathfrak{J}_3$

$$\begin{aligned}
 \mathfrak{J}_3 &= \int_0^{x - \frac{x}{\sqrt{m}}} |\zeta_{m,\alpha}(x, t)| |\Psi'_x(t)| dt \leq \frac{r\gamma_{m,\alpha}^2(x)}{(m+1)} \int_0^{x - \frac{x}{\sqrt{m}}} \frac{|\Psi'_x(t)|}{(x-t)^2} dt \\
 &= \frac{r\gamma_{m,\alpha}^2(x)}{(m+1)} \int_0^{x - \frac{x}{\sqrt{m}}} \frac{|\Psi'_x(t) - \Psi'_x(x)|}{(x-t)^2} dt \\
 &\leq \frac{r\gamma_{m,\alpha}^2(x)}{x(m+1)} \int_1^{\sqrt{m}} \mathfrak{F}_{x - \frac{x}{z}}^x(\Psi'_x) dz \\
 &\leq \frac{r\gamma_{m,\alpha}^2(x)}{x(m+1)} \sum_{j=1}^{[\sqrt{m}]} \int_j^{j+1} \mathfrak{F}_{x - \frac{x}{z}}^x(\Psi'_x) dz \\
 &\leq \frac{r\gamma_{m,\alpha}^2(x)}{x(m+1)} \sum_{j=1}^{[\sqrt{m}]} \mathfrak{F}_{x - \frac{x}{j}}^x(\Psi'_x). \tag{4.22}
 \end{aligned}$$

The inequalities (4.21) and (4.22) combiningly imply

$$\mathfrak{J}_1 \leq \frac{r\gamma_{m,\alpha}^2(x)}{x(m+1)} \sum_{j=1}^{[\sqrt{m}]} \mathfrak{F}_{x - \frac{x}{j}}^x(\Psi'_x) + \frac{x}{\sqrt{m}} \mathfrak{F}_{x - \frac{x}{\sqrt{m}}}^x(\Psi'_x). \tag{4.23}$$

Again, we notice that for  $\mathfrak{K}_{m,\alpha}(x, t) dt = d_t(1 - \zeta_{m,\alpha}(x, t))$ ,  $t > x$  and so applying integration by parts we obtain

$$\begin{aligned}
 \mathfrak{J}_2 &= \left| \int_x^1 \left( \int_x^t \Psi'_x(z) dz \right) d_t(1 - \zeta_{m,\alpha}(x, t)) \right| \\
 &= \left| \left[ \left( \int_x^t \Psi'_x(z) dz \right) (1 - \zeta_{m,\alpha}(x, t)) \right]_x^1 - \int_x^1 \Psi'_x(t) (1 - \zeta_{m,\alpha}(x, t)) dt \right| \\
 &= \left| \int_x^1 (1 - \zeta_{m,\alpha}(x, t)) \Psi'_x(t) dt \right| \\
 &\leq \int_x^u |1 - \zeta_{m,\alpha}(x, t)| |\Psi'_x(t)| dt + \int_u^1 |1 - \zeta_{m,\alpha}(x, t)| |\Psi'_x(t)| dt.
 \end{aligned}$$

We choose  $u = x + \frac{(1-x)}{\sqrt{m}}$ , so that

$$\begin{aligned} \mathfrak{J}_2 &\leq \int_x^{x+\frac{(1-x)}{\sqrt{m}}} |1 - \zeta_{m,\alpha}(x, t)| |\Psi'_x(t)| dt + \int_{x+\frac{(1-x)}{\sqrt{m}}}^1 |1 - \zeta_{m,\alpha}(x, t)| |\Psi'_x(t)| dt \\ &:= \mathfrak{J}_5 + \mathfrak{J}_6. \end{aligned}$$

For  $\mathfrak{J}_6$ , we use Lemma 4.11 followed by the substitution  $t = x + \frac{(1-x)}{z}$  to get

$$\begin{aligned} \mathfrak{J}_6 &= \int_{x+\frac{(1-x)}{\sqrt{m}}}^1 |1 - \zeta_{m,\alpha}(x, t)| |\Psi'_x(t)| dt \leq \frac{r\gamma_{m,\alpha}^2(x)}{(m+1)} \int_{x+\frac{(1-x)}{\sqrt{m}}}^1 \frac{|\Psi'_x(t)|}{(t-x)^2} dt \\ &= \frac{r\gamma_{m,\alpha}^2(x)}{(m+1)} \int_{x+\frac{(1-x)}{\sqrt{m}}}^1 \frac{|\Psi'_x(t) - \Psi'_x(x)|}{(t-x)^2} dt \\ &\leq \frac{r\gamma_{m,\alpha}^2(x)}{(x-1)(m+1)} \int_{\sqrt{m}}^1 \mathfrak{F}_x^{x+\frac{(1-x)}{z}}(\Psi'_x) dz \\ &= \frac{r\gamma_{m,\alpha}^2(x)}{(1-x)(m+1)} \int_1^{\sqrt{m}} \mathfrak{F}_x^{x+\frac{(1-x)}{z}}(\Psi'_x) dz \\ &\leq \frac{r\gamma_{m,\alpha}^2(x)}{(1-x)(m+1)} \sum_{j=1}^{[\sqrt{m}]} \int_j^{j+1} \mathfrak{F}_x^{x+\frac{(1-x)}{z}}(\Psi'_x) dz \\ &\leq \frac{r\gamma_{m,\alpha}^2(x)}{(1-x)(m+1)} \sum_{j=1}^{[\sqrt{m}]} \mathfrak{F}_x^{x+\frac{(1-x)}{j}}(\Psi'_x). \end{aligned}$$

Using the fact  $|1 - \zeta_{m,\alpha}(x, t)| \leq 1$ , we can further deduce

$$\begin{aligned} \mathfrak{J}_5 &= \int_x^{x+\frac{(1-x)}{\sqrt{m}}} |1 - \zeta_{m,\alpha}(x, t)| |\Psi'_x(t)| dt = \int_x^{x+\frac{(1-x)}{\sqrt{m}}} |1 - \zeta_{m,\alpha}(x, t)| |\Psi'_x(t) - \Psi'_x(x)| dt \\ &\leq \int_x^{x+\frac{(1-x)}{\sqrt{m}}} \mathfrak{F}_x^t(\Psi'_x) dt \leq \frac{(1-x)}{\sqrt{m}} \mathfrak{F}_x^{x+\frac{(1-x)}{\sqrt{m}}}(\Psi'_x). \end{aligned}$$

Thus, we have the following estimation for the integral  $\mathfrak{J}_2$ :

$$\mathfrak{J}_2 \leq \frac{r\gamma_{m,\alpha}^2(x)}{(1-x)(m+1)} \sum_{j=1}^{[\sqrt{m}]} \mathfrak{F}_x^{x+\frac{(1-x)}{j}}(\Psi'_x) + \frac{(1-x)}{\sqrt{m}} \mathfrak{F}_x^{x+\frac{(1-x)}{\sqrt{m}}}(\Psi'_x). \quad (4.24)$$

Finally, putting (4.23) and (4.24) in (4.20), the proof is concluded.  $\square$

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