Chapter 4

Difference Equation with Minima based Reaching Laws with Rate-Regulatory Function

4.1 Introduction

This chapter introduces a modified reaching law based on a difference equation with minima for discrete-time sliding mode control. The proposed approach combines characteristics of the modified Gao's reaching law when the system is far from the origin and Utkin's equivalent control law when it is close to the origin. It effectively eliminates chattering, ensuring the system states perfectly adhere to the sliding hyperplane. Additionally, the approach regulates the rate of change of the sliding variable through a carefully tuned design parameter, resulting in a less aggressive control law compared to Gao's reaching law for very large initial conditions. The proposed reaching law is applicable to both unperturbed and perturbed systems. In the case of the unperturbed system, the sliding variable converges to zero in finite time, while for the perturbed system, it remains in the vicinity of the sliding hyperplane. The efficacy of the proposed reaching law is demonstrated through numerical and practical examples, particularly in the simulation of a pendulum system. A detailed comparative analysis with existing methods and those proposed in previous chapters provides a comprehensive perspective.

The rest of the chapter is structured as follows: Section II discusses the motivation behind the use of rate-regulatory functions (RRF). In Section III, the main results of this chapter are discussed. Section IV validates the efficacy of the proposed scheme using two illustrative examples of a numerical system and a physical system (pendulum). Finally, Section V concludes this chapter.

4.2 Rate-Regulatory Function

The following results will be helpful in order to understand the concept of variable gain based rate of change of sliding variable.

With the help of rate-regulatory function, we intend to change the evolution rate of the sliding variable by incorporating the former with the proportional term in the reaching law. An illustration of this is shown in Figure 4.1. We can see that reaching law without RRF tend to converge much slowly as compared with that of the RRF based reaching law.

The corresponding curve for such type of functions is shown in Figure 4.2. One can notice that, the curve in the graph has value close to 1 for large values of the sliding variable, which results in the less rate of change of the sliding variable to reduce the amount of control effort required. Similarly, for the lower values of sliding variable, the curve tends to move towards 0, which makes the rate of change of sliding variable much faster.

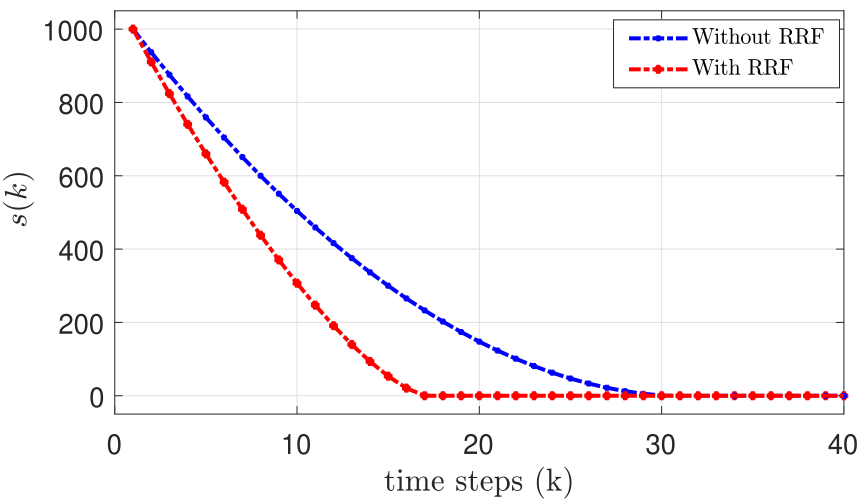


Figure 4.1: A comparative illustration of evolution of sliding variable

A comparative analysis is conducted between the reaching law proposed in previous chapter and the RRF based reaching law is done. By means of a simulation we have

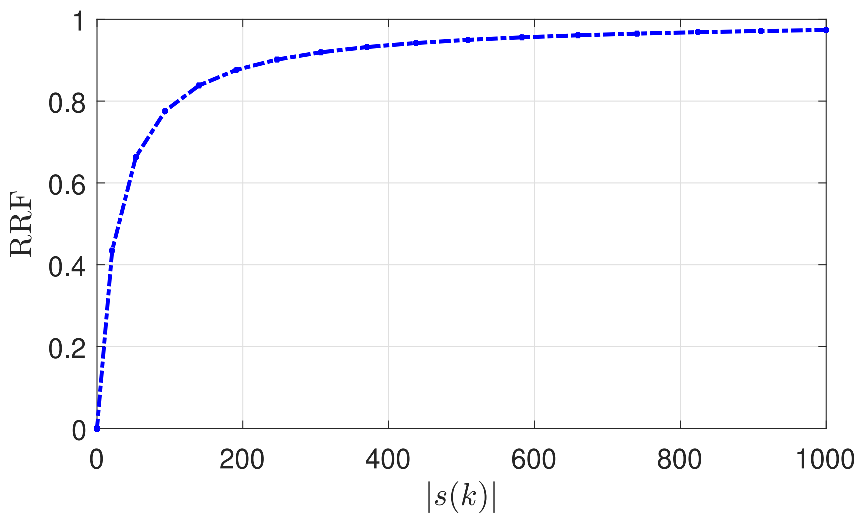


Figure 4.2: Variation of rate-regulatory function with respect to $s(k)$

shown that the RRF based reaching law has better convergence properties with respect to the algorithm proposed in previous chapter. For simulation purpose, all the common parameters were kept same, i.e., $\gamma = 2$. The initial conditions were also taken same, i.e., $s(0) = 1000$. From Figure 4.1, one can see that proposed reaching law has better convergence properties as compared to the one proposed in previous chapter. This enhancement has occurred due to the presence of rate regulatory function RRF. Since in algorithm from previous chapter, there is no rate regulatory function, instead it has a constant coefficient of 1, the convergence speed is much slower. One can also see the characteristics of RRF as a function of s in Figure 4.2. It is important to note that with the presence of RRF, we can steer the coefficient of the proportional term and utilize the full range of $[0, 1)$. This leads to higher value of RRF towards 1 for large initial conditions, yielding a restricted rate of change of the sliding variable. Similarly, for values in the neighbourhood of sliding manifold, RRF decreases towards 0 and produces faster response. This gives quite better convergence properties to RRF based reaching law.

In the next section, we exploit such RRF functions to design the reaching laws based on difference equation with minima for better transient behavior.

4.3 Rate-Regulatory Function based Reaching Laws for DSMC

In this section, we present the reaching laws for both unperturbed and perturbed DTS based on difference equation with minima which utilize the rate-regulatory function.

4.3.1 RL1 for Unperturbed Discrete-Time Systems

Let us consider the following RL for unperturbed DTS

$$s(k+1) = \Xi(s)s(k) - \text{sgn}[s(k)]\min\{\Xi(s)|s(k)|, \gamma\} \tag{4.1}$$

where $\Xi(s) = 1 - \lambda(s)$, the mapping $\lambda : \mathbb{R} \to (0, 1]$ and is defined here as, $\lambda(s) = \frac{\varrho}{\varrho+|s(k)|}$ and $\varrho, \gamma \in \mathbb{R}_+$ are scalar constants, chosen by the designer.

Remark 4.1 *The mapping λ is such that, the image $\Xi(s)$ increases for the larger values of $|s(k)|$ and decreases for the smaller values of $|s(k)|$, as is depicted in the Figure 4.3. This yields a fast rate of change of $s(k)$ for small $|s(k)|$ and a slow rate of change for large $|s(k)|$.*

Remark 4.2 *The selection of the function $\Xi(s)$ is taken from the [28], where the authors have modified the proportional term in the [25] and, in turn, met the system's constraints on the closed-loop signals.*

Remark 4.3 *The image $\Xi(s)$ for different values of ϱ is more flat towards 0 for smaller $|s(k)|$ and more flat towards 1 for larger $|s(k)|$, as can be seen in Figure 4.3. Selecting the small values of ϱ yields the $\Xi(s)$ close to 1 and makes the system response slower, while for the large values of ϱ, $\Xi(s)$ remains quite less than 1 and decreases fastly towards zero, making the system response comparatively faster.*

As per the proposed RL (4.1), the rate of change of the sliding variable can be calculated as

$$\begin{aligned}\Delta s(k) =& |s(k+1) - s(k)| \\ =& |\Xi(s)s(k) - \text{sgn}[s(k)]\min\{\Xi(s)|s(k)|, \gamma\} - s(k)| \\ =& | - \lambda(s)s(k) - \text{sgn}[s(k)]\min\{\Xi(s)|s(k)|, \gamma\}|. \end{aligned} \tag{4.2}$$

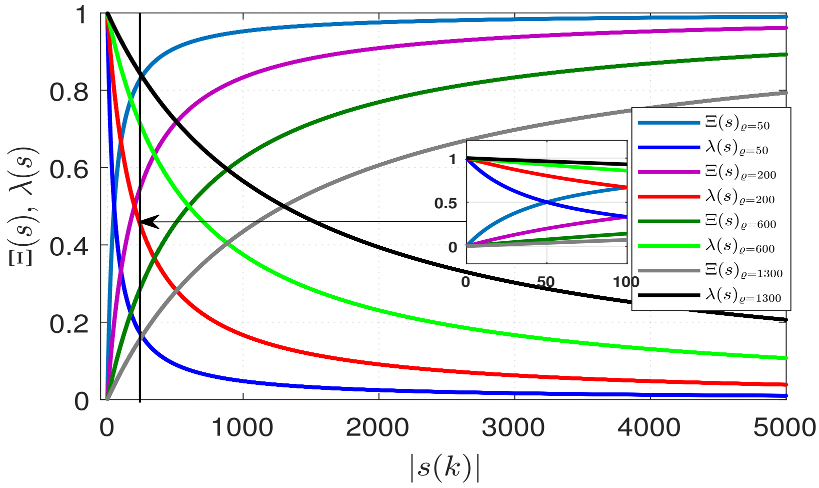


Figure 4.3: $\Xi(s)$ and $\lambda(s)$ for different values of ϱ

Considering the case when system states are far away from the sliding hyperplane, then $\min\{\Xi(s)|s(k)|, \gamma\} = \gamma$. Further, it follows that

$$
\begin{aligned}
\Delta s(k) &= | -\lambda(s)s(k) - \text{sgn}[s(k)]\gamma| \\
&= \left| -\frac{\varrho}{\varrho + |s(k)|} s(k) - \text{sgn}[s(k)]\gamma \right| \\
&\leq \varrho + \gamma. \qquad (4.3)
\end{aligned}
$$

Hence, the rate of change of the sliding variable is always bounded by the design parameters.

Remark 4.4 *In the proposed RL* (4.1)*, one can notice that for any k, the $s(k)$ is always bounded by the design parameters, i.e., $\varrho+\gamma$, regardless of the size of the initial conditions. This is a beneficial feature of the proposed RL, as the aforementioned bound is specified by the user.*

Let us consider the following DTS

$$
z(k+1) = Az(k) + bu(k) \qquad (4.4)
$$

where $z \in D \subseteq \mathbb{R}^n$ with D as an open subset, $k \in \mathbb{Z}_{\geq 0}$, $u \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^{n\times 1}$. We define the switching function as $s(k) := \wp(z(k))$. The level set $\wp^{-1}(0) := \{z \in D : \wp(z) = 0\}$ defines the sliding hyperplane, which we assume to be sufficiently smooth. We further consider that $\wp(z(k)) = c^\top z(k)$ for our analysis. Consider the following switching function as

$$s(k) = c^\top z(k) \tag{4.5}$$

where $c \in \mathbb{R}^{n\times 1}$ with its elements chosen such that $c^\top b \neq 0$.

Definition 4.5 *We say that sliding mode exists on the level set* $\wp^{-1}(0)$ *if* $s(k) = 0$.

Remark 4.6 *Definition 4.5 also has a similarity with the ideal sliding mode, which is defined for CTS. As compared with the existing definitions of the QSM [25, 27], in our case the sliding variable exactly lands on the positively invariant set* $\wp^{-1}(0)$.

Remark 4.7 *Moreover, Definition 4.5 does not follow the standard definition (the one proposed in [25]) as well, since the method proposed here does not require the system state to cross and recross the sliding hyperplane* $\wp^{-1}(0)$ *due to the structure of the proposed RL. As a result, chattering is eliminated, which is quite prevalent in [25] and hence inessential control effort is avoided. Further, the risk of exciting unmodelled high-frequency dynamics of the controlled system is also eliminated.*

Definition 4.8 *DTS* (4.4) *satisfies the reaching condition of the sliding mode if and only if, for some* $k \geq 0$, *the following holds:*

$$\Xi(s)|s(k)| > \gamma \implies |s(k+1)| < |s(k)| - \nu, \tag{4.6}$$

$$\Xi(s)|s(k)| \leq \gamma \implies s(k+1) = 0 \tag{4.7}$$

where ν *is a sufficiently small positive constant.*

To compute $u(k)$ such that (4.1) holds, we utilize (4.4) and (4.5) to compute

$$s(k+1) = c^\top A z(k) + c^\top b u(k) \tag{4.8}$$

On comparing (4.1) and (4.8), the control law is obtained as

$$u(k) = (c^\top b)^{-1}\{-c^\top A z(k) + \Xi(s)s(k) - \mathrm{sgn}[s(k)] \min\{\Xi(s)|s(k)|, \gamma\}\} \tag{4.9}$$

Next, we analyze the proposed RL and the properties it exhibits.

Theorem 4.9 *If in the RL (4.1), $\Xi(s_0)|s(0)| > \gamma$, with $\gamma > 0$, then for some $k \geq \mathcal{K}(s_0)$, $s(k) = 0$, where $\mathcal{K}(s_0) = \lceil \frac{|s_0|}{\gamma} \rceil$.*

Proof: We rewrite the RL proposed in (4.1): $s(k+1) = \Xi(s)s(k) - \text{sgn}[s(k)]$ $\min\{\Xi(s)|s(k)|, \gamma\}$. If we take the case when $\Xi(s_0)|s(0)| \leq \gamma$, then $s(k+1) = \Xi(s)s(k) - \text{sgn}[s(k)]\Xi(s)|s(k)| \implies s(k+1) = 0$ in just one sample instant, satisfying the conditions given in the Definition 4.8, with $\mathcal{K}(s_0) = 1$ time step. Further, $\forall\ k \geq \mathcal{K}(s_0)$, $s(k) = 0$, depicting the existence of the sliding mode as per the Definition 4.5. Moving on to the case when $\Xi(s_0)|s(0)| > \gamma \implies s(k+1) = \Xi(s)s(k) - \text{sgn}[s(k)]\gamma$. We can further write, $|s(k+1)| = |\Xi(s)s(k) - \text{sgn}[s(k)]\gamma|$. Since $\Xi(s)|s(k)| > \gamma$ and $|\Xi(s)| \leq 1$, it follows that, $|s(k+1)| \leq |s(k)| - \gamma$. Further,

$$\begin{aligned} |s(k)| &\leq |s(k-1)| - \gamma \\ &\leq |s(k-2)| - 2\gamma \\ &\vdots \\ &\leq |s(0)| - k\gamma \end{aligned} \tag{4.10}$$

If $(|s(0)| - k\gamma) \leq \gamma$ and since $|\Xi(s)| \leq 1$, then from (4.10), $|s(k)| \leq \gamma \implies \Xi(s)|s(k)| \leq \gamma \implies s(k+1) = 0$ for all $k \geq \lceil \frac{|s_0|}{\gamma} \rceil$, with the settling time function denoted as $\mathcal{K}(s_0) = \lceil \frac{|s_0|}{\gamma} \rceil$. Hence, it follows that the absolute value of the sliding function becomes zero after some finite time. This completes the proof. □

In the next part, we follow the case where the system is subjected to matched type bounded perturbation.

4.3.2 RL1 for Perturbed Discrete-Time Systems

Consider the following RL for perturbed DTS

$$s(k+1) = \Xi(s)s(k) - \text{sgn}[s(k)]\min\{\Xi(s)|s(k)|, \gamma\} + c^\top b\delta(k) \tag{4.11}$$

where $\delta(k)$ is the matched type bounded perturbation which satisfies $\underline{\delta} \leq \delta(k) \leq \bar{\delta}$, where $\underline{\delta}, \bar{\delta} \in \mathbb{R}$ are known constants. Additionally, the following holds: $\delta_0 = \frac{\underline{\delta}+\bar{\delta}}{2}$, is the mean of $\delta(k)$, and $\delta_d = \frac{\bar{\delta}-\underline{\delta}}{2}$, is the maximum deviation from the mean of $\delta(k)$. $\gamma \in \mathbb{R}_+$ is a scalar constant which is chosen such that it satisfies $\gamma \geq \max\{|\underline{\delta}|, |\bar{\delta}|\}$. To analyze the properties of the RL (4.11), consider the following perturbed DTS,

$$z(k+1) = Az(k) + bu(k) + b\delta(k) \tag{4.12}$$

where $z \in D \subseteq \mathbb{R}^n$ with D as an open subset, $k \in \mathbb{Z}_{\geq 0}$, $u \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^{n\times 1}$ and $\delta(k)$ is the matched type bounded perturbation. Let us consider the sliding hyperplane as defined earlier by the level set $\wp^{-1}(0)$ and switching function as in (4.5) with $c \in \mathbb{R}^{n\times 1}$ having its elements selected such that $c^\top b \neq 0$ and $|c^\top b| \leq 1$. For the analysis purpose, consider $\max\{|\underline{\delta}|, |\bar{\delta}|\} = \delta_m$, where $\delta_m \in \mathbb{R}_+$.

Definition 4.10 *We say that the QSM exists in the neighborhood of the level set* $\wp^{-1}(0)$ *if* $|s(k)| \leq |c^\top b|\delta_m$.

Definition 4.11 *DTS* (4.12) *satisfies the reaching condition of the QSM in the neighborhood of* $\wp^{-1}(0)$ *if and only if, for some* $k \geq 0$*, the following holds:*

$$\Xi(s)|s(k)| > \gamma \implies |s(k+1)| < |s(k)| - \nu, \tag{4.13}$$

$$\Xi(s)|s(k)| \leq \gamma \implies 0 \leq |s(k+1)| \leq |c^\top b|\delta_m \tag{4.14}$$

where ν *is a sufficiently small positive constant.*

Remark 4.12 *The reaching law proposed in* (4.1) *is without considering any perturbation term, which results in the existence of perfect sliding mode. On the contrary, for the perturbed system* (4.27)*, the reaching law* (4.11) *also considers the effect of perturbation by adding an extra term* $c^\top b\delta(k)$*.*

To compute $u(k)$ such that (4.11) holds, we utilize (4.12) and (4.5) to compute

$$s(k+1) = c^\top Az(k) + c^\top bu(k) + c^\top b\delta(k) \tag{4.15}$$

On comparing (4.11) and (4.15), the control law is obtained as

$$u(k) = (c^\top b)^{-1}\{-c^\top Az(k) + \Xi(s)s(k) - \text{sgn}[s(k)]\min\{\Xi(s)|s(k)|, \gamma\}\} \tag{4.16}$$

In the next theorem, we describe the properties of the aforementioned RL.

Theorem 4.13 *If in the RL* (4.11)*,* $\Xi(s_0)|s(0)| > \gamma$*, with* $\gamma > 0$*,* $|c^\top b| \leq 1$*, then for some* $k \geq \mathcal{K}(s_0)$*,* $|s(k)| \leq |c^\top b|\delta_m$*, where* $\mathcal{K}(s_0) = \lceil \frac{|s_0| - |c^\top b|\delta_m}{\gamma - |c^\top b|\delta_m} \rceil$.

Proof: Rewriting the RL proposed in (4.11): $s(k+1) = \Xi(s)s(k) - \text{sgn}[s(k)]$ $\min\{\Xi(s)|s(k)|, \gamma\} + c^\top b\delta(k)$. If we take the case when $\Xi(s_0)|s(0)| \leq \gamma$, then $s(k+1) = \Xi(s)s(k) - \text{sgn}[s(k)]\Xi(s)|s(k)| + c^\top b\delta(k) \implies s(k+1) = c^\top b\delta(k)$ in just one sample instant. Further, it follows that $|s(k+1)| \leq |c^\top b|\delta_m$, satisfying the conditions given in the Definition 4.11, with $\mathcal{K}(s_0) = 1$ time step. Further, for all $k \geq \mathcal{K}(s_0)$, $|s(k)| \leq |c^\top b|\delta_m$, depicting the existence of the QSM as per the Definition 4.10. We now move on to the case when $\Xi(s_0)|s(0)| > \gamma \implies s(k+1) = \Xi(s)s(k) - \mathrm{sgn}[s(k)]\gamma + c^\top b\delta(k)$. We can further write, $|s(k+1)| = |\Xi(s)s(k) - \mathrm{sgn}[s(k)]\gamma + c^\top b\delta(k)|$. Since $\Xi(s)|s(k)| > \gamma$, $|\Xi(s)| \leq 1$ and $|c^\top b| \leq 1$, it follows that, $|s(k+1)| \leq |s(k)| - \gamma + |c^\top b|\delta_m$. Further, it can be written that

$$\begin{aligned}
|s(k)| &\leq |s(k-1)| - \gamma + |c^\top b|\delta_m \\
&\leq |s(k-2)| - 2(\gamma - |c^\top b|\delta_m) \\
&\vdots \\
&\leq |s(0)| - k(\gamma - |c^\top b|\delta_m)
\end{aligned} \tag{4.17}$$

If $(|s(0)| - k(\gamma - |c^\top b|\delta_m)) \leq \gamma$ and since $|\Xi(s)| \leq 1$, then from (4.17), $|s(k)| \leq \gamma \implies \Xi(s)|s(k)| \leq \gamma \implies |s(k+1)| \leq |c^\top b|\delta_m$, for all $k \geq \left\lceil \frac{|s_0| - |c^\top b|\delta_m}{\gamma - |c^\top b|\delta_m} \right\rceil$, with the settling time function denoted as $\mathcal{K}(s_0) = \left\lceil \frac{|s_0| - |c^\top b|\delta_m}{\gamma - |c^\top b|\delta_m} \right\rceil$. Hence, it follows that the absolute value of the sliding function is ultimately bounded by $|c^\top b|\delta_m$ after some finite time. This completes the proof. □

4.3.3 RL2 for Unperturbed Discrete-Time Systems

Let us consider the following RL for unperturbed DTS

$$s(k+1) = \Xi(s)s(k) - \gamma \mathrm{sgn}[s(k)] \min\left\{ \frac{\Xi(s)|s(k)|}{\gamma}, |s(k)|^{\varpi} \right\} \tag{4.18}$$

where $\Xi(s) = 1 - \lambda(s(k))$, the mapping $\lambda : \mathbb{R} \to (0, 1]$ and defined here as, $\lambda(s(k)) = \frac{\varrho}{\varrho + |s(k)|}$ and $\varrho, \gamma \in \mathbb{R}_+$ are scalar constants, chosen by the designer and $\varpi \in (0, 1)$. $\Xi(s)$ acts as a rate regulatory function, which regulates the rate of change of the sliding variable $s(k)$. Let us consider the following DTS

$$z(k+1) = Az(k) + bu(k) \tag{4.19}$$

where $z \in D \subseteq \mathbb{R}^n$ with D as an open subset, $k \in \mathbb{Z}_{\geq 0}$, $u \in \mathbb{R}$, $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^{n \times 1}$. It is assumed that the pair (A, b) is controllable. Further, the switching function is defined as $s(k) := \wp(z(k))$.

Definition 4.14 *The level set $\wp^{-1}(0)$ is the collection of all the points z for which the the value of the switching function is zero, i.e., $\wp^{-1}(0) := \{z \in D : \wp(z) = 0\}$.*

The level set $\wp^{-1}(0)$ defines the sliding hyperplane which we assume to be sufficiently smooth. We further consider a linear switching function for our analysis. Let us consider the following switching function as

$$s(k) = c^\top z(k) \tag{4.20}$$

where $c \in \mathbb{R}^{n\times 1}$ with its entries selected so that $c^\top b \neq 0$.

Definition 4.15 *We say that sliding mode exists on the level set $\wp^{-1}(0)$, if for any $k \geq \mathcal{K}(s_0)$, $s(k) = 0$, where $\mathcal{K}(s_0)$ is the settling time function.*

Definition 4.16 *The reaching condition of the sliding mode for DTS (4.19) is satisfied iff for some $k \geq 0$, the following holds:*

$$\frac{\Xi(s)|s(k)|}{\gamma} > |s(k)|^{\varpi} \implies |s(k+1)| < |s(k)| - \nu, \tag{4.21}$$

$$\frac{\Xi(s)|s(k)|}{\gamma} \leq |s(k)|^{\varpi} \implies s(k+1) = 0 \tag{4.22}$$

where ν is a sufficiently small positive constant.

To compute $u(k)$ such that (4.18) holds, we utilise (4.19) and (4.20) to compute

$$s(k+1) = c^\top Az(k) + c^\top bu(k) \tag{4.23}$$

On comparing (4.18) and (4.23), control law is obtained as

$$\begin{aligned} u(k) =& (c^\top b)^{-1}\Big\{ - c^\top Az(k) + \varpi(s)s(k) - \gamma\mathrm{sgn}[s(k)] \\ & \min\left\{\frac{\Xi(s)|s(k)|}{\gamma}, |s(k)|^{\varpi}\right\}\Big\} \end{aligned} \tag{4.24}$$

Next, we analyse the proposed RL and the properties it exhibits.

Theorem 4.17 *If in the RL (4.18), $\frac{\Xi(s_0)|s_0|}{\gamma} > |s_0|^{\varpi}$, with $\gamma > 0$, $\varpi \in (0,1)$ then for some $k \geq \mathcal{K}(s_0)$, $s(k) = 0$, where $\mathcal{K}(s_0) \leq \left\lceil \log_{[1-\gamma|s_0|^{\varpi-1}]} \frac{\gamma^{\frac{1}{1-\varpi}}}{|s_0|} \right\rceil + 1$.*

Proof: We rewrite the RL proposed in (4.18): $s(k+1) = \Xi(s)s(k) - \gamma\mathrm{sgn}[s(k)]$ $\min\left\{\frac{\Xi(s)|s(k)|}{\gamma}, |s(k)|^{\varpi}\right\}$. If we take the case when $\frac{\Xi(s_0)|s_0|}{\gamma} \leq |s_0|^{\varpi}$, then $s(k+1) = \Xi(s)s(k) - \mathrm{sgn}[s(k)]\Xi(s)|s(k)| \implies s(k+1) = 0$ taking 1 sample instant only and satisfying the conditions as per Definition 4.16, with $\mathcal{K}(s_0) = 1$ time step. Further, $\forall\, k \geq \mathcal{K}(s_0)$, $s(k) = 0$, depicting the existence of the sliding mode as per the Definition 4.15. Moving on to the case when $\frac{\Xi(s_0)|s_0|}{\gamma} > |s_0|^{\varpi} \implies s(k+1) = \Xi(s)s(k) - \gamma\text{sgn}[s(k)]|s(k)|^{\varpi}$. We can further write, $|s(k+1)| = |\Xi(s)s(k) - \gamma\text{sgn}[s(k)]|s(k)|^{\varpi}|$. Since $\Xi(s)|s(k)| > \gamma|s(k)|^{\varpi}$ and $|\Xi(s)| \leq 1$, it follows that, $|s(k+1)| \leq |s(k)| - \gamma|s(k)|^{\varpi}$. Further,

$$\begin{aligned}|s(k)| &\leq |s(k-1)|(1-\gamma|s(k-1)|^{\varpi-1})\\ &\leq |s(k-2)|(1-\gamma|s(k-2)|^{\varpi-1})(1\\ &-\gamma|s(k-1)|^{\varpi-1})\\ &\vdots\\ &\leq |s_0|(1-\gamma|s_0|^{\varpi-1})\cdots(1-\gamma|s(k-1)|^{\varpi-1})\end{aligned}$$

Since, $\varpi \in (0,1)$, $(1-\gamma|s(k-1)|^{\varpi-1}) < (1-\gamma|s(k-2)|^{\varpi-1})$ and so on. Then, it follows that

$$|s(k)| \leq |s_0|(1-\gamma|s_0|^{\varpi-1})^k \tag{4.25}$$

If $\left|(|s_0|(1-\gamma|s_0|^{\varpi-1})^k)\right| \leq \gamma|s_0|^{\varpi}$ and since $|\varpi(s)| \leq 1$, then from (4.25), $|s(k)| \leq \gamma|s(k)|^{\varpi}$ $\implies \Xi(s)|s(k)| \leq \gamma|s(k)|^{\varpi} \implies s(k+1) = 0$ for all $k \geq \left\lceil \log_{[1-\gamma|s_0|^{\varpi-1}]} \frac{\gamma^{\frac{1}{1-\varpi}}}{|s_0|} \right\rceil + 1$, with the settling time function denoted as $\mathcal{K}(s_0) \leq \left\lceil \log_{[1-\gamma|s_0|^{\varpi-1}]} \frac{\gamma^{\frac{1}{1-\varpi}}}{|s_0|} \right\rceil + 1$. Hence, it follows that, absolute value of the switching variable becomes 0 after some finite time. □

In the following subsection, we carry out the analysis on perturbed DTS.

4.3.4 RL2 for Perturbed Discrete-Time Systems

Consider the following RL for perturbed DTS

$$s(k+1) = \Xi(s)s(k) - \gamma\text{sgn}[s(k)]\min\left\{\frac{\Xi(s)|s(k)|}{\gamma}, |s(k)|^{\varpi}\right\} + c^{\top}b\delta(k) \tag{4.26}$$

where $\delta(k)$ is the matched type bounded perturbation such that $\underline{\delta} \leq \delta(k) \leq \bar{\delta}$ holds, where $\underline{\delta}, \bar{\delta} \in \mathbb{R}$ are known a priori. Furthermore, the following holds: $\delta_0 = \frac{\underline{\delta}+\bar{\delta}}{2}$, is the mean of $\delta(k)$, and $\delta_d = \frac{\bar{\delta}-\underline{\delta}}{2}$, is the maximum deviation from the mean of $\delta(k)$. $\gamma \in \mathbb{R}_+$ is a design parameter such that $\gamma \geq \max\{|\underline{\delta}|, |\bar{\delta}|\}$.

To analyse the behavior of the RL (4.26), we consider the following system

$$z(k+1) = Az(k) + bu(k) + b\delta(k) \tag{4.27}$$

where $z \in D \subseteq \mathbb{R}^n$ with D as an open subset, $k \in \mathbb{Z}_{\geq 0}$, $u \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^{n\times 1}$ and $\delta(k)$ is the matched type bounded perturbation. Similarly, sliding manifold is defined by $\wp^{-1}(0)$ and switching function as in (4.20) with $c \in \mathbb{R}^{n\times 1}$ having its elements selected such that $c^\top b \neq 0$ and an additional condition $|c^\top b| \leq 1$ (its significance will be observed later). We carry out the analysis considering $\max\{|\underline{\delta}|, |\bar{\delta}|\} = \delta_m$, where $\delta_m \in \mathbb{R}_+$.

Definition 4.18 *We say that the QSM exists in the neighbourhood of the level set $\wp^{-1}(0)$, if for any $k \geq \mathcal{K}(s_0)$, $|s(k)| \leq |c^\top b|\delta_m$, where $\mathcal{K}(s_0)$ is the settling time function.*

Definition 4.19 *The reaching condition of the QSM in the neighbourhood of $\wp^{-1}(0)$ for system (4.27) is satisfied iff for some $k \geq 0$, the following holds:*

$$\frac{\Xi(s)|s(k)|}{\gamma} > |s(k)|^{\varpi} \implies |s(k+1)| < |s(k)| - \nu, \tag{4.28}$$

$$\frac{\Xi(s)|s(k)|}{\gamma} \leq |s(k)|^{\varpi} \implies 0 \leq |s(k+1)| \leq |c^\top b|\delta_m \tag{4.29}$$

where ν is a sufficiently small positive constant.

To compute $u(k)$ such that (4.26) holds, we utilise (4.27) and (4.20) to compute

$$s(k+1) = c^\top Az(k) + c^\top bu(k) + c^\top b\delta(k) \tag{4.30}$$

On comparing (4.26) and (4.30), control law is obtained as

$$\begin{aligned} u(k) =&(c^\top b)^{-1}\Big\{ - c^\top Az(k) + \Xi(s)s(k) - \gamma\text{sgn}[s(k)] \\ &\min\left\{\frac{\Xi(s)|s(k)|}{\gamma}, |s(k)|^{\varpi}\right\}\Big\} \end{aligned} \tag{4.31}$$

The following theorem expresses the behavior of the aforementioned law for perturbed case.

Theorem 4.20 *Consider the RL (4.26) with $|c^\top b| \leq 1$. If in RL (4.26), $\frac{\Xi(s_0)|s_0|}{\gamma} > |s_0|^{\varpi}$, with $\gamma > 0$, $\varpi \in (0,1)$ then for some $k \geq \mathcal{K}(s_0)$, $|s(k)| \leq |c^\top b|\delta_m$, where $\mathcal{K}(s_0) \leq \left\lceil \log_{[1-\gamma|s_0|^{\varpi-1}+|c^\top b|\delta_m|s_0|^{-1}]} \frac{\gamma^{\frac{1}{1-\varpi}}}{|s_0|} \right\rceil + 1$.*

Proof: Rewriting the RL proposed in (4.26): $s(k+1) = \Xi(s)s(k) - \gamma\text{sgn}[s(k)]$ $\min\left\{\frac{\Xi(s)|s(k)|}{\gamma}, |s(k)|^{\varpi}\right\} + c^\top b\delta(k)$. If we take the case when $\frac{\Xi(s_0)|s_0|}{\gamma} \leq |s_0|^{\varpi}$, then $s(k+1) = \Xi(s)s(k) - \text{sgn}[s(k)]\Xi(s)|s(k)| + c^\top b\delta(k) \implies s(k+1) = c^\top b\delta(k)$ in 1 sample instant only. Hence, it follows that $|s(k+1)| \leq |c^\top b|\delta_m$, satisfying the conditions given in the Definition 4.19, with $\mathcal{K}(s_0) = 1$ time step. Further, $\forall\ k \geq \mathcal{K}(s_0)$, $|s(k)| \leq |c^\top b|\delta_m$, depicting the existence of the QSM as per the Definition 4.18. We now move on to the case when

$\frac{\Xi(s_0)|s_0|}{\gamma} > |s_0|^{\varpi} \implies s(k+1) = \Xi(s)s(k) - \gamma \text{sgn}[s(k)]|s(k)|^{\varpi} + c^{\top}b\delta(k)$. We can further write, $|s(k+1)| = |\Xi(s)s(k) - \gamma\text{sgn}[s(k)]|s(k)|^{\varpi} + c^{\top}b\delta(k)|$. Since $\Xi(s)|s(k)| > \gamma|s(k)|^{\varpi}$ and $|\Xi(s)| \leq 1$, it follows that, $|s(k+1)| \leq |s(k)| - \gamma|s(k)|^{\varpi} + |c^{\top}b|\delta_m$. Further, it can be written that

$$
\begin{aligned}
|s(k)| &\leq |s(k-1)|(1-\gamma|s(k-1)|^{\varpi-1} + |c^{\top}b|\delta_m|s(k-1)|^{-1}) \\
&\leq |s(k-2)|(1-\gamma|s(k-2)|^{\varpi-1} + |c^{\top}b|\delta_m|s(k-2)|^{-1}) \\
&(1-\gamma|s(k-1)|^{\varpi-1} + |c^{\top}b|\delta_m|s(k-1)|^{-1}) \\
&\vdots \\
&\leq |s_0|(1-\gamma|s_0|^{\varpi-1} + |c^{\top}b|\delta_m|s_0|^{-1})\cdots \\
&(1-\gamma|s(k-1)|^{\varpi-1} + |c^{\top}b|\delta_m|s(k-1)|^{-1})
\end{aligned}
$$

Since, $\varpi \in (0,1)$, $(1-\gamma|s(k-1)|^{\varpi-1} + |c^{\top}b|\delta_m|s(k-1)|^{-1}) < (1-\gamma|s(k-2)|^{\varpi-1} + |c^{\top}b|\delta_m|s(k-2)|^{-1})$ and so on. Then, it follows that

$$
|s(k)| \leq |s_0|(1-\gamma|s_0|^{\varpi-1} + |c^{\top}b|\delta_m|s_0|^{-1})^k \tag{4.32}
$$

If $\left|(|s_0|(1-\gamma|s_0|^{\varpi-1} + |c^{\top}b|\delta_m|s_0|^{-1})^k)\right| \leq \gamma|s_0|^{\varpi}$ and since $|\varpi(s)| \leq 1$, then from (4.32), $|s(k)| \leq \gamma|s(k)|^{\varpi} \implies \varpi(s)|s(k)| \leq \gamma|s(k)|^{\varpi} \implies |s(k+1)| \leq |c^{\top}b|\delta_m$, for all $k \geq \left\lceil \log_{[1-\gamma|s_0|^{\varpi-1}+|c^{\top}b|\delta_m|s_0|^{-1}]} \frac{\gamma^{\frac{1}{1-\varpi}}}{|s_0|} \right\rceil + 1$, with the settling time function expressed as $\mathcal{K}(s_0) \leq \left\lceil \log_{[1-\gamma|s_0|^{\varpi-1}+|c^{\top}b|\delta_m|s_0|^{-1}]} \frac{\gamma^{\frac{1}{1-\varpi}}}{|s_0|} \right\rceil + 1$. Therefore, it follows that, absolute value of the sliding function is ultimately bounded by $|c^{\top}b|\delta_m$ after finite-time steps. □

Remark 4.21 *Again, we notice that the width of the ultimate band ($|s(k)| \leq |c^{\top}b|\delta_m \leq \delta_m$) can be significantly reduced by proper selection of vector c ensuring $|c^{\top}b| \leq 1$.*

In most of the discrete variable structure control literature, authors have considered disturbance with zero mean value. For such cases, $\delta_m = \delta_d$, where δ_d is the maximum deviation of the disturbance from its mean value. The width of the ultimate band for [25] is given as, $|s(k)| \leq 2\delta_d + \epsilon$, for [27] as, $|s(k)| \leq \delta_d$ and for [60] as, $|s(k)| \leq \delta_d + \epsilon$. In this regard, the proposed reaching law yields lesser ultimate band width, which speaks for its benefits over others.

4.4 Simulation Results and Discussion

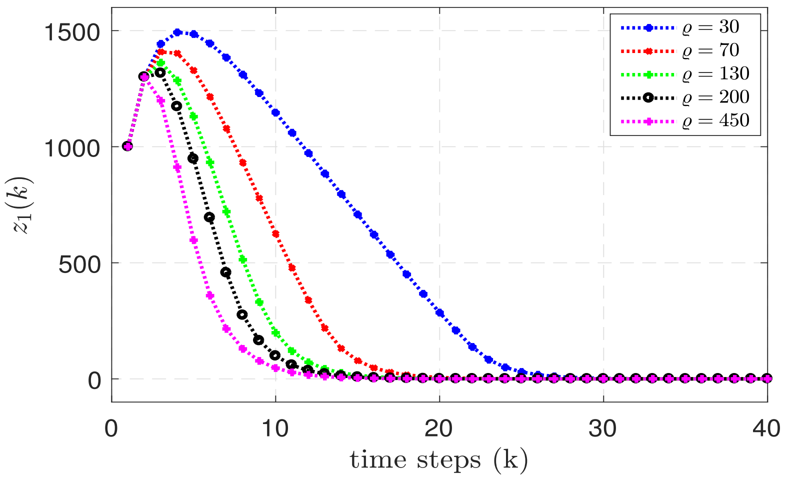
In this section, we discuss a numerical example and a practical example to demonstrate the effectiveness of the proposed reaching laws. Moreover, we conduct a comparative study to show the superiority of the proposed designs. We show that the sliding variable is made zero in finite time for unperturbed DTS and for the perturbed case, the sliding variable reaches an invariant set and remains there for all future time.

4.4.1 Example 1

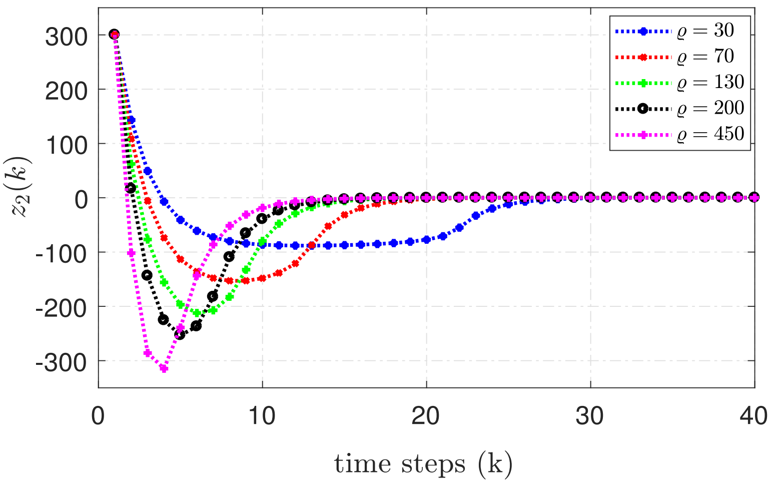
Consider the following dynamical system

$$
\begin{aligned}
z_1(k+1) &= z_1(k) + z_2(k) \\
z_2(k+1) &= u(k) + \delta(k)
\end{aligned}
\tag{4.33}
$$

where $z = [z_1 \ z_2]^\top \in D \subset \mathbb{R}^2$, $u(k)$ is the control input, and $\delta(k)$ is the matched type bounded perturbation, which is taken as $\sin(0.5k)$. The difference equation with minima based RL (RL1) is designed as per (4.1) and (4.11). The simulation is conducted for 5 values of ϱ, which are $30, 70, 130, 200,$ and 450. The system's initial conditions are taken as $z_1(0) = 1000$ and $z_2(0) = 300$. The design parameters are chosen as $c_1 = 0.4$, $c_2 = 1$, and $\gamma = 8$. The states of the unperturbed DTS are shown in Figure 4.4. One can observe that for the $\varrho = 30$, the states slowly converge to zero, while for the $\varrho = 450$, states converge to zero at a comparatively faster rate. Sliding variable and control input for this system are shown in Figure 4.5. It is important to note that as we increase ϱ, both the sliding variable and control input change at a higher rate, however, for smaller ϱ, the rate of change becomes slower, yielding slow convergence of the sliding variable and moderation of the control input. Figures 4 and 5 show the system variables for perturbed DTS. Due to the presence of sinusoidal perturbation, system variables lie in an invariant set in the steady-state, depicting the similar properties of convergence as was for the unperturbed DTS. The phase portrait of the system is shown in Figure 4.8. It can be seen that, for the unperturbed case, system states converge to zero, while for the perturbed case, states stay in an invariant set. Moreover, one can see that the proposed RL drives the sliding variable to zero for unperturbed and to an invariant set for perturbed DTS in finite-time steps.



(a)



(b)

Figure 4.4: States of the unperturbed DTS for different values of ϱ

Remark 4.22 *The appropriate choice of ϱ for which the system response is not too sluggish when the $|s(k)|$ is small and the control input at the beginning is not too aggressive, depends on the constraints imposed on the closed-loop signals.*

Remark 4.23 *The proposed RL overcomes the limitations of chattering and large control effort when $|s(k)|$ is large in [25] along with achieving $|s(k)| = 0$ for unperturbed DTS. Although the issue of chattering is resolved by the non-switching type RL in [28] in which $s(k)$ need not cross and recross the sliding hyperplane at every successive control step, however, the proposed RL yields the dead-beat type response in the vicinity of the sliding*

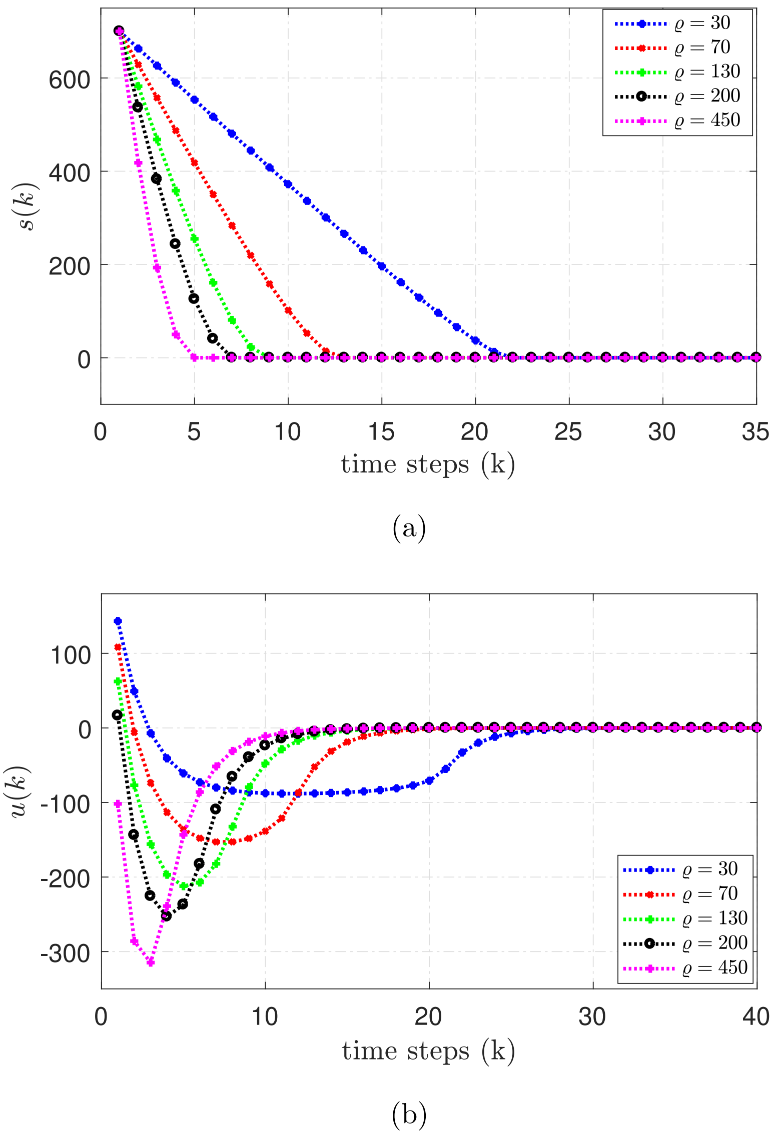


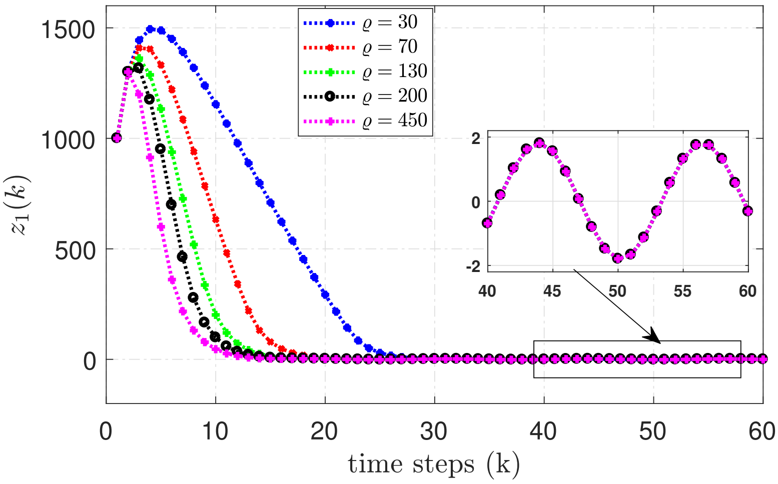
Figure 4.5: (a) Sliding variable, (b) control, of unperturbed DTS for different values of ϱ

hyperplane making $s(k)$ exactly zero.

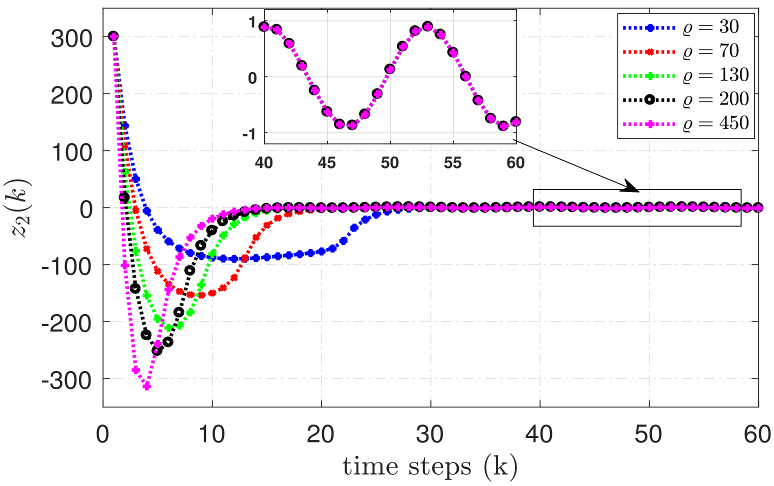
4.4.2 Example 2: A Comparative Study

Now we consider the example of a pendulum system subjected to matched type bounded perturbations. Consider the following system

$$\begin{aligned} z_1(k+1) &= z_1(k) + z_2(k) \\ z_2(k+1) &= z_2(k) - \frac{mgl}{2J}\sin(z_1(k)) - \frac{B}{J}z_2(k) + \frac{1}{J}u(k) + \frac{1}{J}\delta(k) \end{aligned} \tag{4.34}$$



(a)

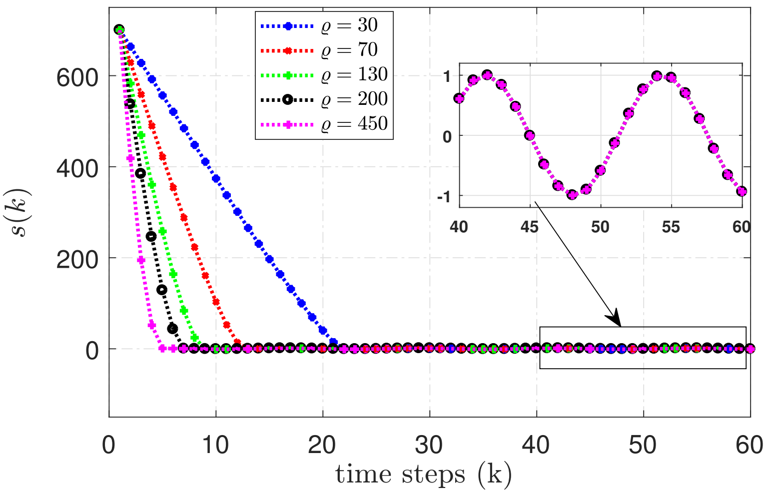


(b)

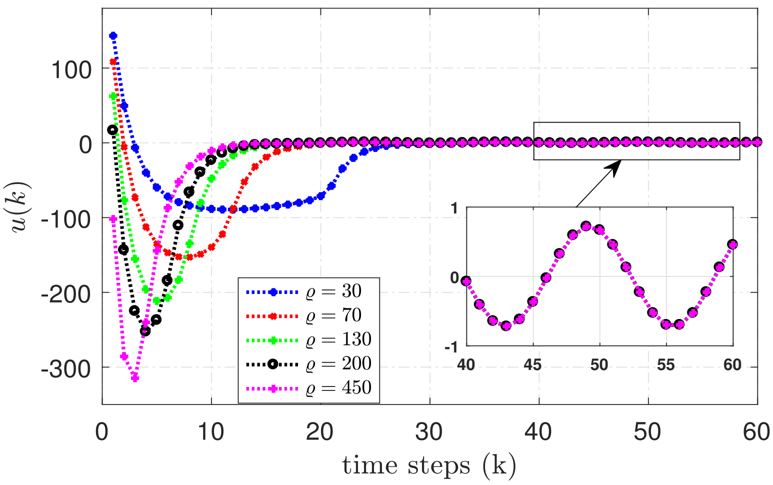
Figure 4.6: States of the perturbed DTS for different values of ϱ

where $z_1(k)$ is the pendulum angle from the mean position, $z_2(k)$ is the angular velocity, m is the mass of the pendulum, g is the acceleration due to gravity, l is the length of the pendulum, J is the inertia of the pendulum arm, B is the friction coefficient and $u(k)$ is the control input. The bounded perturbation $\delta(k)$ is considered as 0.05 sin$(0.3k)$. The control law is designed based on the proposed RL (RL2) described in equation (4.18). For the simulation purpose, the system parameters are taken as follows: $m = 1.1kg$, $l = 1m$, $g = 9.81\frac{m}{s^2}$, $B = 0.18\frac{kg \cdot m}{s^2}$. The system initial conditions are taken as $z_1(0) = 30$ and $z_2(0) = 15$.

To see the effectiveness of the proposed RL, a comparative study is conducted with



(a)



(b)

Figure 4.7: (a) Sliding variable, (b) control, of perturbed DTS for different values of ϱ

the references [25] (R_1), [27] (R_2), [60] (R_3) and the one proposed in the previous chapter (without RRF). The notations representing different parameters are taken from the corresponding references. The design parameters for [25] are taken as $q = 0.15$ and $\epsilon = 0.8$, for [27] as $k^* = 5$, for [60] as $s_0 = 25$, $\epsilon = 0.12$, for proposed (without RRF) as $\gamma = 1.8$, $\varpi = 0.4$. The selection of these parameters are done based on the conditions given in the corresponding references. For the proposed (RRF) reaching law, the design parameters were chosen as $\varrho = 10$, $\gamma = 1.8$, $\varpi = 0.4$ and the parameters for the sliding hyperplane are chosen as $c_1 = 0.3$ and $c_2 = 0.6$.

Figures 4.9, 4.10, 4.11 and 4.12 depict the state variables of the pendulum system for

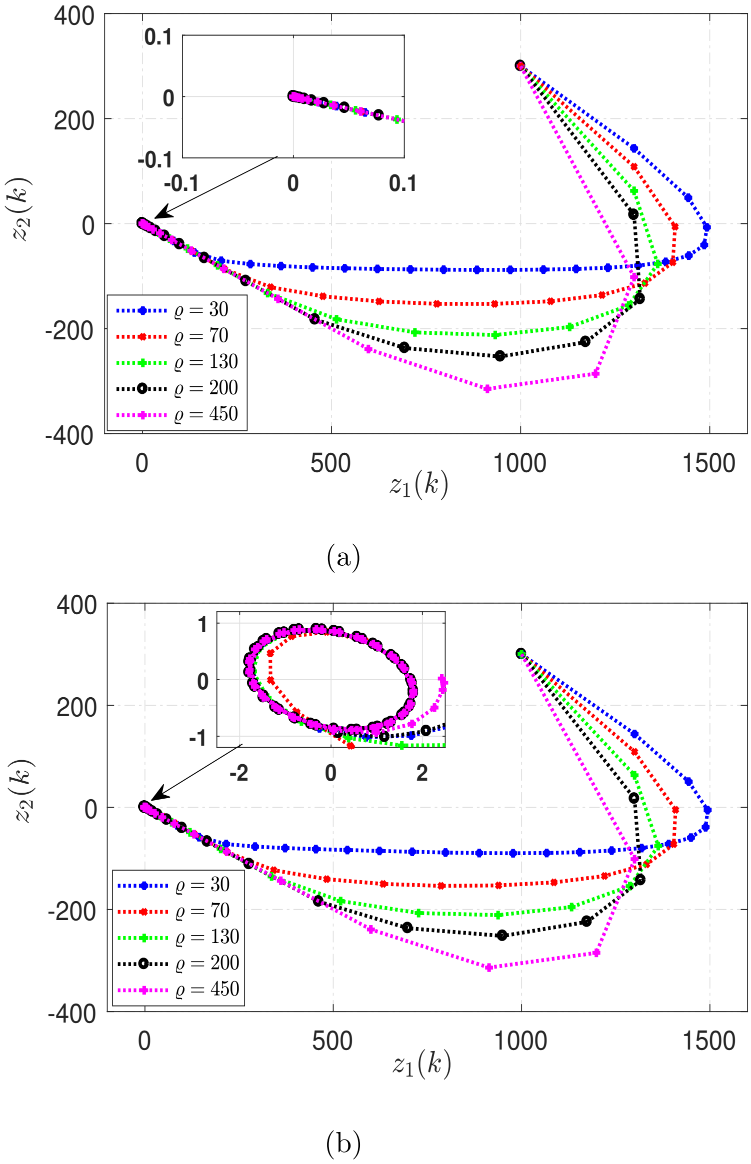


Figure 4.8: Phase portrait:(a) unperturbed, (b) perturbed, DTS for different values of ϱ

different reaching laws. In Figure 4.9 (angular position of the pendulum), one can notice that in the initial phase, the proposed reaching law has better convergence properties with least overshoot among all the methods. Moreover, system states driven by [25] and [60] have high frequency oscillatory behavior, since both the strategies adopt switching based approach, as one can see in Figure 4.10. Although, [60] adopts the reference trajectory following approach, to better compensate the effects of the disturbance and has better transient and steady state properties than [25]. Further, it is important to note that since [27], proposed (without RRF) and proposed (RRF) reaching laws are based on non-switching approaches, therefore, in steady state, state variables driven by these methods

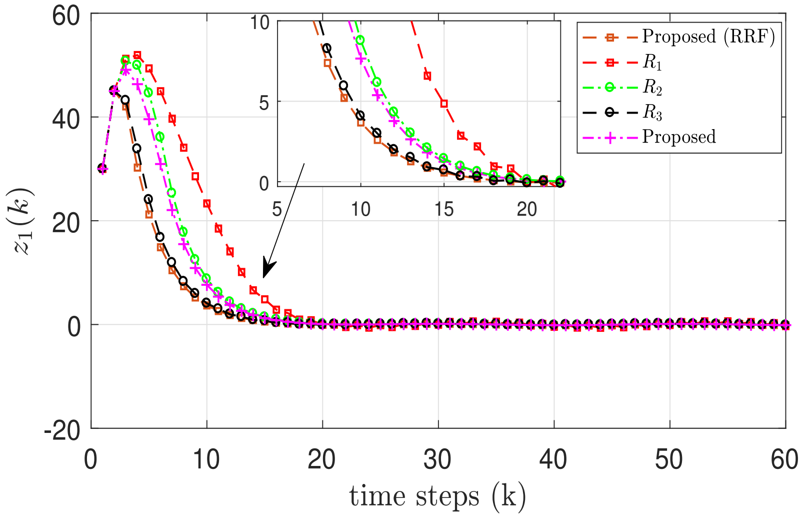


Figure 4.9: State $z_1(k)$ for the perturbed pendulum system

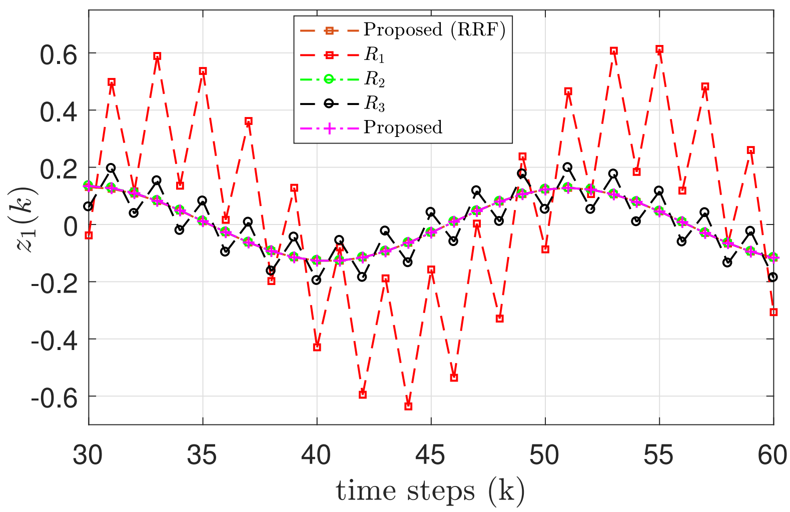


Figure 4.10: State $z_1(k)$ in sliding phase

do not show the oscillatory behavior. Moreover, similar characteristics are observed for angular velocity of the pendulum shown in Figure 4.11 and Figure 4.12. The evolution of the sliding variable can be seen from Figure 4.13 and Figure 4.14. The evolution of sliding variable with proposed law converge faster to the ultimate band without generating unnecessary oscillations as compared to the other methods. Since, $|c^\top b| = 0.027$, which is less than 1, the ultimate band width is $|s(k)| \leq 0.027$. Further, it is to note that keeping the values of γ and ϖ identical for both proposed (without RRF) and proposed (RRF) method, the latter performs better because of the parameter ϱ, as discussed with

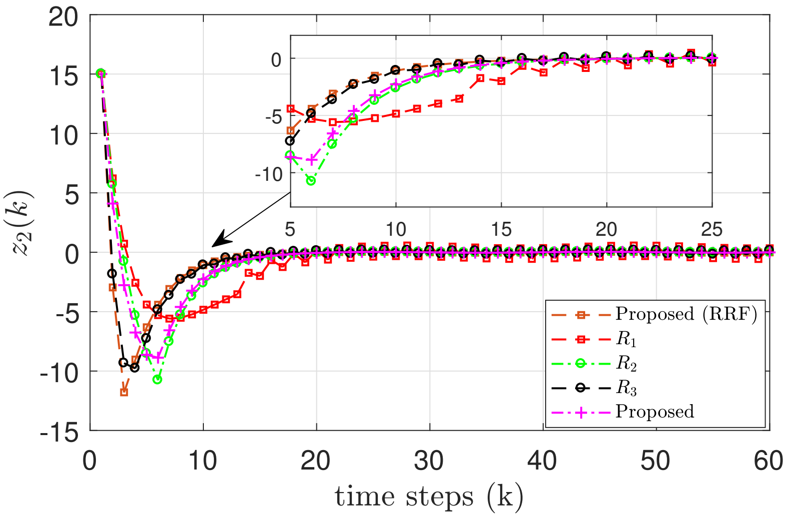


Figure 4.11: State $z_2(k)$ for the perturbed pendulum system

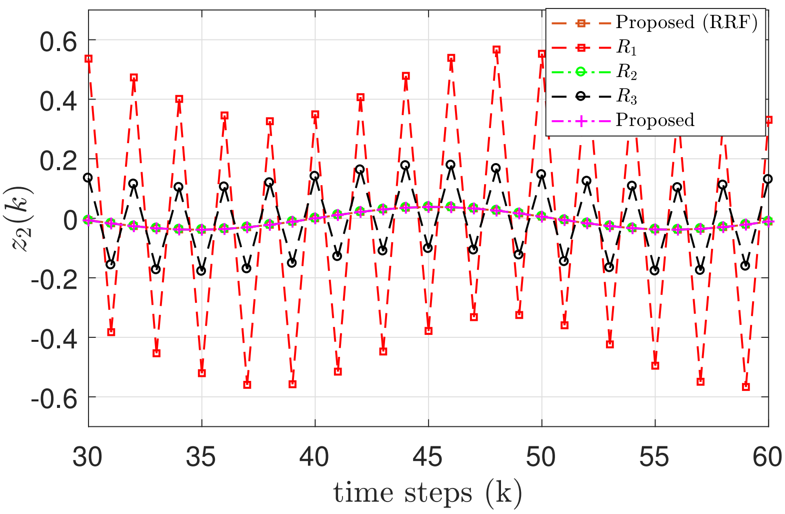


Figure 4.12: State $z_2(k)$ in sliding phase

graphical illustration. Additionally, the band size is comparatively large in case of [25] and [60] with large oscillations. For [27], proposed (without RRF) and proposed (RRF) reaching laws, no such oscillations are observed and the band width is also less. Finally, the control input can be seen from Figure 4.15 and Figure 4.16, and similar observation can be made for the control signal. Although, it is difficult to clearly see the control signals from different reaching laws due to congestion of data points, however, one can gaze at Figure 4.16 to take note of the chattering, which is significant in [25] and [60]. Since other three methods are non-switching based, we do not observe any chattering

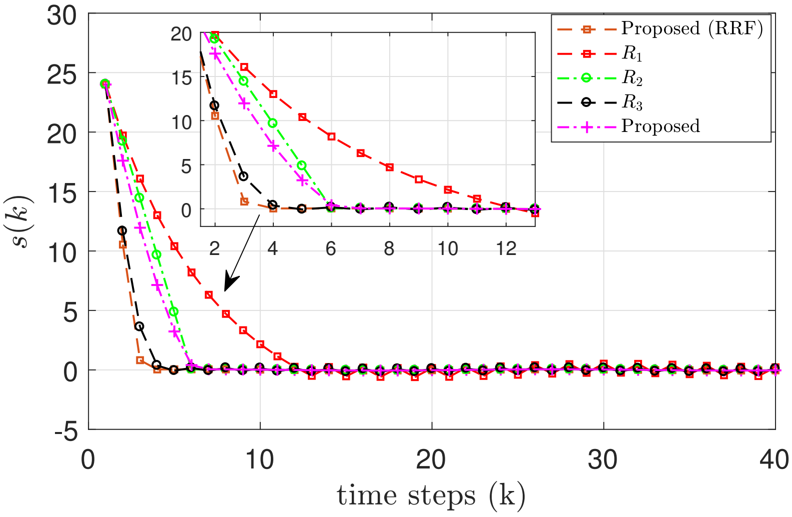


Figure 4.13: Evolution of sliding variable for the perturbed pendulum system

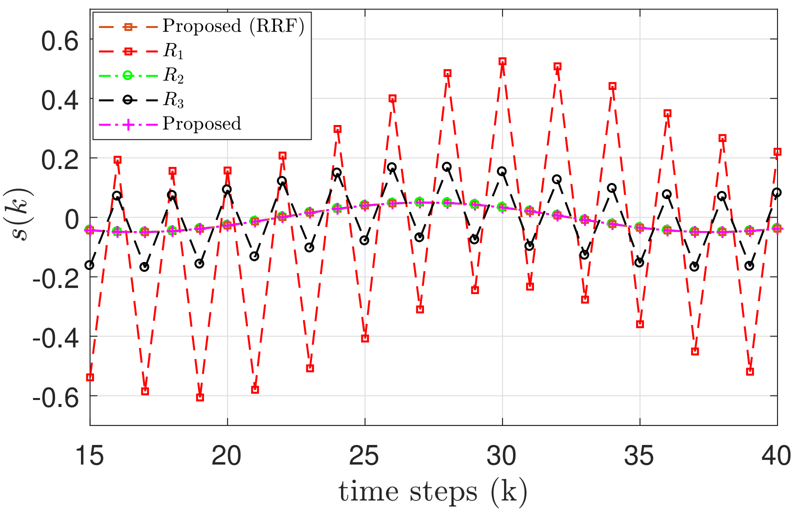


Figure 4.14: Ultimate bands of sliding variable for different reaching laws

there.

4.5 Conclusion

In this chapter, a reaching law based on a difference equation with minima, incorporating a rate-regulatory function, was introduced for discrete sliding mode control. This modified approach addresses the limitations of both Gao's reaching law and Utkin's equivalent control-based approach by steering the evolution of sliding variable efficiently. The pro-

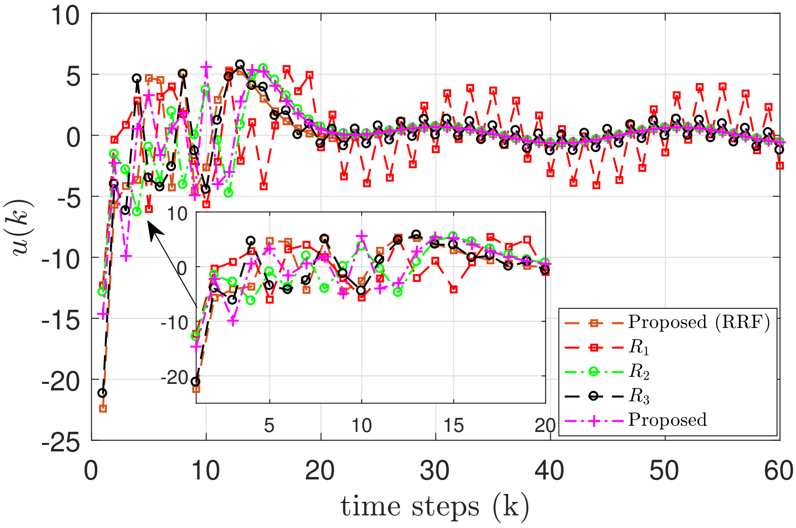


Figure 4.15: Evolution of control signal for the perturbed pendulum system

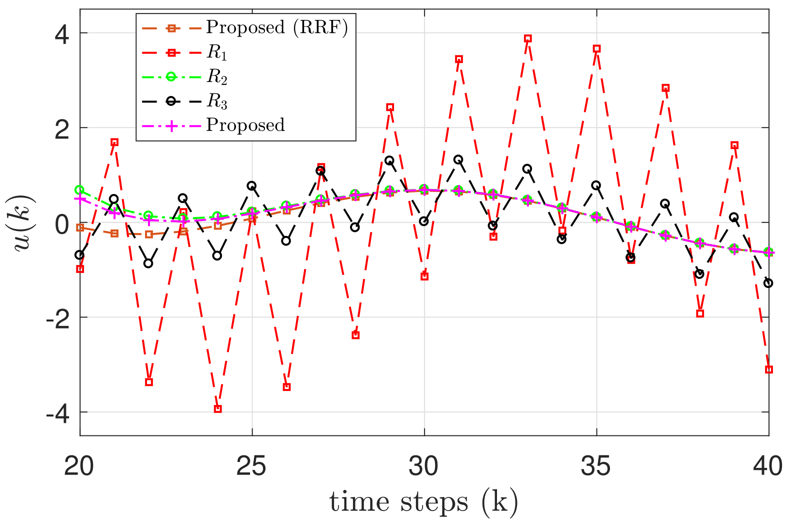


Figure 4.16: Control signal in steady state

posed reaching law effectively eliminates chattering and minimizes the need for extensive control action by introducing a design parameter, denoted as ϱ. Notably, when the system states are far from the origin, the reaching law guides the sliding variable towards the origin while ensuring that the rate of change does not exceed a reasonable threshold. This introduces a trade-off between control magnitude and system response speed, with the ability to tailor this trade-off by selecting an appropriate value for ϱ. Simulation results are presented to illustrate the effectiveness of the proposed reaching law. Additionally, a comparative study is conducted through simulations using a pendulum system. The results highlight that the proposed method achieves rapid convergence without excessive sliding variable rate of change, and it leads to a narrower bandwidth in the steady state without introducing undesirable oscillations.

In the forthcoming chapter, our attention will be directed towards the Lyapunov characterization of the discrete sliding mode control approach presented in this study.