Chapter 2

Normal and Tangent Cones for Set of Intervals and Their Application in Optimization with Functions of Interval Variables

2.1 Introduction

In classical optimization, the concept of normal and tangent cones play a crucial role in characterizing optimum solutions [157]. From a geometric viewpoint, a tangent cone comprises of all possible directions of movement inside the set, whereas normal cones comprises of directions perpendicularly away from the set [34]. It is well-known that tangent and normal cone are dual of each other [59]. For the optimization of a smooth function f under the constraint set S:

$$x^* \in \operatorname{argmin}\{f(x) : x \in S\},$$

the necessary first-order condition for optimality with respect to normal cone $\mathcal{N}_S(x^*)$ on S at x^* is given by

$$-\nabla f(x^*) \in \mathcal{N}_S(x^*).$$

With respect to tangent cone, the necessary condition for x^* being an optimum solution is given by

$$\nabla f(x^*)^\top d \geq 0 \text{ for all } d \in \mathcal{T}_S(x^*),$$

where $\mathcal{T}_S(x^*)$ is the tangent cone on S at x^*.

The concepts of normal and tangent cones are used in various fields of mathematics. For any constrained smooth optimization problem, the Karush-Kuhn-Tucker

(KKT) optimality condition requires assumptions of regularity, and almost all of these assumptions incorporate the concepts of normal and tangents cones [59]. In case of multiobjective optimization problems, [82,139] have demonstrated the utility of Pareto directions of tangent cone to calculate trade-offs in such problems. Further, the significance of normal and tangent cones in variational and nonsmooth analysis have been shown in [31,34]. Finally, applications of normal and tangent cones in highly structured mathematical programming, such as semidefinite programming, second-order programming etc., are presented in [131,191].

Besides the mathematical applications, normal and tangent cones find use in many practical applications as well. In [19], normal cones have been employed to sublevel sets to analyse the asymptotic characteristics of gradient dynamical systems. For studying nonsmooth mechanical systems, appropriate set-valued laws of forces can be constructed using normal cone [169]. The field of stability analysis involves the calculation of limiting normal cone as an application [83]. In theory of elastoplasticity [72], for a network made up of elastoplastic springs having time-varying displacement-controlled loading, the stresses are characterized based on a normal cone. Finally, in data envelopment analysis (DEA) [75], the calculation of value efficiency score of every decision-making unit involves the concept of tangent cone.

2.2 Motivation

Clearly, the concepts of normal and tangent cones are critical for characterizing the optimum solutions of problems not only in optimization, but also variational analysis, non-smooth analysis etc. It also has applications in many practical problems. Particularly, for the subject of constrained interval optimization problems (IOPs), which has become very important in recent years due to their capacity of modelling real-world optimization problems that exhibit inherent variability and uncertainty in their parameters, characterization of optimum solution has become a topic of prime importance. Development of necessary optimality conditions for these constrained IOPs may require some regularity conditions or constraint qualifications as well. In this regard, normal and tangent cone become very crucial for interval optimization. Therefore, through this thesis, we establish and study the notions of normal and tangent cones for a set of interval variables and their importance in Lagrange model for IOP.

2.3 Contributions

In this chapter, we define the concept of normal and tangent cone for set of intervals. We study their algebraic properties in terms of product rule, intersection rule etc. We employ the distance function and its property of Lipschitz continuity to characterize tangent cone for a set of intervals. We also analyze the relationship of duality between normal and tangent cones. Further, we explore the concepts of gH-gradient, gH-partial derivative, gH-directional derivative, L-IVF, and gH-differentiability for IVF, where domain is a subset of $I(\mathbb{R})^n$. Finally, we demonstrate that the proposed normal and tangent cones are beneficial for characterizing efficient points of an IOP.

Overall, the major contributions of this chapter are:

(i) We explore the role of tangent and normal cone in specifying the efficient solution of a constrained IOP. This characterization gives rise to a necessary optimality condition with respect to normal cones for set of intervals. This necessary condition also becomes sufficient under the convexity assumption on IVFs.

(ii) For constrained IOPs with equality and inequality constraints, we present an efficient condition using the Lagrange multiplier rule, where the Lagrange multipliers lie in the normal cone of the constraint functions.

(iii) Lastly, we show an application of our findings regarding the efficiency conditions of the normal cone when dealing with datasets containing intervals, through an example of support vector machines. We find a solution to a one class classification or unary classification problem where the interval-valued data reside on one side of the separating hyperplane.

2.4 Normal cone for a set of intervals

In this section, we define the normal cone for a set of intervals and examine its characterization. Additionally, we aim to derive some of its calculus rules, including the weak intersection rule, algebraic preserving rule, and the rule for countable intersections.

Definition 2.1 (Normal cone in $I(\mathbb{R})^n$). *If $\boldsymbol{\mathcal{Y}}$ is a nonempty convex subset of $I(\mathbb{R})^n$ and $\widehat{\boldsymbol{Y}}' \in \boldsymbol{\mathcal{Y}}$, then, the normal cone to $\boldsymbol{\mathcal{Y}}$ at $\widehat{\boldsymbol{Y}}'$ is defined as follows:*

$$\mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\widehat{\boldsymbol{Y}}') = \left\{ \widehat{\boldsymbol{G}} \in I(\mathbb{R})^n : \widehat{\boldsymbol{G}}^\top \odot_c (\widehat{\boldsymbol{Y}} \ominus_{gH} \widehat{\boldsymbol{Y}}') \preceq \boldsymbol{0} \ \forall \ \widehat{\boldsymbol{Y}} \in \boldsymbol{\mathcal{Y}} \right\}.$$

Example 2.4.1 *For the convex set $\boldsymbol{\mathcal{Y}} = \{[y, 2y] : y \geq 0\}$, normal cones to $\boldsymbol{\mathcal{Y}}$ at $\boldsymbol{0} \in \boldsymbol{\mathcal{Y}}$ and $[3,6] \in \boldsymbol{\mathcal{Y}}$ are given by*

$$\mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\boldsymbol{0}) = \left\{ [\underline{c}, \overline{c}] \in I(\mathbb{R}) : [\underline{c}, \overline{c}] \odot [\underline{y}, \overline{y}] \preceq \boldsymbol{0} \ \forall \ [\underline{y}, \overline{y}] \in \boldsymbol{\mathcal{Y}} \right\} = I(\mathbb{R}_-),$$
$$\mathcal{N}_{\boldsymbol{\mathcal{Y}}}([3,6]) = \left\{ [\underline{c}, \overline{c}] \in I(\mathbb{R}) : [\underline{c}, \overline{c}] \odot ([\underline{y}, \overline{y}] \ominus_{gH} [3,6]) \preceq \boldsymbol{0} \ \forall \ [\underline{y}, \overline{y}] \in \boldsymbol{\mathcal{Y}} \right\} = \{\boldsymbol{0}\}.$$

Theorem 2.1 *If $\boldsymbol{\mathcal{Y}} \subset I(\mathbb{R})^n$ is a nonempty convex set, then for any $\widehat{\mathbf{Y}}' \in \boldsymbol{\mathcal{Y}}$, $\mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}')$ is closed.*

Proof: Consider $\widehat{\mathbf{G}}_k$ to be an arbitrary sequence in $\mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}')$ converging to $\widehat{\mathbf{G}} \in I(\mathbb{R})^n$. Then,

$$\begin{aligned} & \widehat{\mathbf{G}}_k^\top \odot_c (\widehat{\mathbf{Y}} \ominus_{gH} \widehat{\mathbf{Y}}') \preceq \mathbf{0} \text{ for all } \widehat{\mathbf{Y}} \in \boldsymbol{\mathcal{Y}} \\ \Longrightarrow \quad & \bigoplus_{i=1}^{n} \mathbf{G}_{ki} \odot [\min\{\underline{y}_i - \underline{y}'_i, \overline{y}_i - \overline{y}'_i\}, \max\{\underline{y}_i - \underline{y}'_i, \overline{y}_i - \overline{y}'_i\}] \preceq \mathbf{0}, \end{aligned} \tag{2.1}$$

where $\widehat{\mathbf{G}}_k = (\mathbf{G}_{k1}, \mathbf{G}_{k2}, \ldots, \mathbf{G}_{kn})^\top$ and $\widehat{\mathbf{G}} = (\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_n)^\top$. Here, there are two possibilities:

- Case 1. $[\min\{\underline{y}_i - \underline{y}'_i, \overline{y}_i - \overline{y}'_i\},\ \max\{\underline{y}_i - \underline{y}'_i, \overline{y}_i - \overline{y}'_i\}] = [\underline{y}_i - \underline{y}'_i,\ \overline{y}_i - \overline{y}'_i]$
- Case 2. $[\min\{\underline{y}_i - \underline{y}'_i, \overline{y}_i - \overline{y}'_i\},\ \max\{\underline{y}_i - \underline{y}'_i, \overline{y}_i - \overline{y}'_i\}] = [\overline{y}_i - \overline{y}'_i,\ \underline{y}_i - \underline{y}'_i]$.

Without loss of generality, consider that the first p components belong to • Case 1., and the remaining $(n-p)$ components belong to • Case 2.. Then, (2.1) implies that

$$\bigoplus_{i=1}^{p} [\underline{g}_{ki}, \overline{g}_{ki}] \odot [\underline{y}_i - \underline{y}'_i,\ \overline{y}_i - \overline{y}'_i] \bigoplus_{j=p+1}^{n} [\underline{g}_{kj}, \overline{g}_{kj}] \odot [\overline{y}_j - \overline{y}'_j,\ \underline{y}_j - \underline{y}'_j] \preceq \mathbf{0}.$$

Using, $\underline{y}_i - \underline{y}'_i = \underline{d}_i,\ \overline{y}_i - \overline{y}'_i = \overline{d}_i,\ \underline{y}_j - \underline{y}'_j = \underline{d}_j,\ \overline{y}_j - \overline{y}'_j = \overline{d}_j$, we get

$$\bigoplus_{i=1}^{p} [\underline{d}_i, \overline{d}_i] \odot [\underline{g}_{ki}, \overline{g}_{ki}] \bigoplus_{j=p+1}^{n} [\overline{d}_j, \underline{d}_j] \odot [\overline{g}_{kj}, \underline{g}_{kj}] \preceq \mathbf{0},$$

which can be divided into

$$\bigoplus_{i=1}^{p} \min\{\underline{d}_i\underline{g}_{ki}, \underline{d}_i\overline{g}_{ki}, \overline{d}_i\underline{g}_{ki}, \overline{d}_i\overline{g}_{ki}\} \bigoplus_{j=p+1}^{n} \min\{\underline{d}_j\underline{g}_{kj}, \underline{d}_j\overline{g}_{kj}, \overline{d}_j\underline{g}_{kj}, \overline{d}_j\overline{g}_{kj}\} \leq 0 \tag{2.2}$$

and
$$\bigoplus_{i=1}^{p} \max\{\underline{d}_i\underline{g}_{ki}, \underline{d}_i\overline{g}_{ki}, \overline{d}_i\underline{g}_{ki}, \overline{d}_i\overline{g}_{ki}\} \bigoplus_{j=p+1}^{n} \max\{\underline{d}_j\underline{g}_{kj}, \underline{d}_j\overline{g}_{kj}, \overline{d}_j\underline{g}_{kj}, \overline{d}_j\overline{g}_{kj}\} \leq 0. \tag{2.3}$$

Because the sequence $\{\widehat{\mathbf{G}}_k\}$ converges to $\{\widehat{\mathbf{G}}\}$, from Remark 1.1.1 the sequence $\{\underline{g}_{ki}\}$ and $\{\overline{g}_{ki}\}$ converge to $\{\underline{g}_i\}$ and $\{\overline{g}_i\}$, respectively, for all $i = 1, 2, \ldots, n$. Hence, the left hand side of inequalities (2.2) and (2.3) converge to

$$\bigoplus_{i=1}^{p} \min\{\underline{d}_i\underline{g}_i, \underline{d}_i\overline{g}_i, \overline{d}_i\underline{g}_i, \overline{d}_i\overline{g}_i\} \bigoplus_{j=p+1}^{n} \min\{\underline{d}_j\underline{g}_j, \underline{d}_j\overline{g}_j, \overline{d}_j\underline{g}_j, \overline{d}_j\overline{g}_j\} \leq 0 \tag{2.4}$$

$$\text{and} \qquad \bigoplus_{i=1}^{p} \max\{\underline{d}_i\underline{g}_i, \underline{d}_i\overline{g}_i, \overline{d}_i\underline{g}_i, \overline{d}_i\overline{g}_i\} \bigoplus_{j=p+1}^{n} \max\{\underline{d}_j\underline{g}_j, \underline{d}_j\overline{g}_j, \overline{d}_j\underline{g}_j, \overline{d}_j\overline{g}_j\} \leq 0. \quad (2.5)$$

From (2.4) and (2.5), we have

$$\begin{aligned}
& \bigoplus_{i=1}^{p} [\underline{d}_i, \overline{d}_i] \odot [\underline{g}_i, \overline{g}_i] \bigoplus_{j=p+1}^{n} [\overline{d}_j, \underline{d}_j] \odot [\overline{g}_j, \underline{g}_j] \preceq \mathbf{0} \\
\implies & \bigoplus_{i=1}^{n} [\underline{g}_i, \overline{g}_i] \odot [\min\{\underline{y}_i - \underline{y}'_i, \overline{y}_i - \overline{y}'_i\}, \max\{\underline{y}_i - \underline{y}'_i, \overline{y}_i - \overline{y}'_i\}] \preceq \mathbf{0} \\
\implies & \widehat{\mathbf{G}}^\top \odot_c (\widehat{\mathbf{Y}} \ominus_{gH} \widehat{\mathbf{Y}}') \preceq \mathbf{0}. \qquad (2.6)
\end{aligned}$$

Therefore, $\widehat{\mathbf{G}} \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}')$, and hence $\mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}')$ is closed. □

Example 2.4.2 *Through this example, we illustrate Theorem 2.1. For the set* $\boldsymbol{\mathcal{Y}} = \{[y, 2y^2] : y \geq 1\} \cup \{\boldsymbol{0}\}$*, the normal cone to* $\boldsymbol{\mathcal{Y}}$ *at* $\boldsymbol{0}$ *is given as*

$$\begin{aligned}
\mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\boldsymbol{0}) &= \{[\underline{c}, \overline{c}] \in I(\mathbb{R}) : [\underline{c}, \overline{c}] \odot ([\underline{y}, \overline{y}] \ominus_{gH} \boldsymbol{0}) \preceq \boldsymbol{0} \ \forall \ [\underline{y}, \overline{y}] \in \boldsymbol{\mathcal{Y}}\} \\
&= \{[\underline{c}, \overline{c}] \in I(\mathbb{R}) : \overline{c} \leq 0\} = I(\mathbb{R}_-).
\end{aligned}$$

Now, select an arbitrary sequence $\{\boldsymbol{P}_k\} \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\boldsymbol{0})$ *given as* $\boldsymbol{P}_k = [\underline{p}_k, \overline{p}_k]$*, with* $\underline{p}_k, \overline{p}_k \leq 0$*, and assume that* $\lim_{k\to\infty} \boldsymbol{P}_k = \boldsymbol{P} = [\underline{p}, \overline{p}]$*. We know that* $\{\boldsymbol{P}_k\}$ *satisfies*

$$\begin{aligned}
& [\underline{p}_k, \overline{p}_k] \odot [\underline{y}, \overline{y}] \preceq \boldsymbol{0} \ \forall \ [\underline{y}, \overline{y}] \in \boldsymbol{\mathcal{Y}} \\
\implies & \left[\min\{\underline{p}_k\underline{y}, \underline{p}_k\overline{y}, \overline{p}_k\underline{y}, \overline{p}_k\overline{y}\}, \max\{\underline{p}_k\underline{y}, \underline{p}_k\overline{y}, \overline{p}_k\underline{y}, \overline{p}_k\overline{y}\}\right] \preceq \boldsymbol{0} \ \forall \ [\underline{y}, \overline{y}] \in \boldsymbol{\mathcal{Y}}. \qquad (2.7)
\end{aligned}$$

Because $\underline{p}_k$ *and* $\overline{p}_k$ *converge to* $\underline{p}$ *and* $\overline{p}$*, respectively, using (2.7) we have*

$$\begin{aligned}
& \left[\min\{\underline{py}, \underline{p}\overline{y}, \overline{p}\underline{y}, \overline{py}\}, \max\{\underline{py}, \underline{p}\overline{y}, \overline{p}\underline{y}, \overline{py}\}\right] \preceq \boldsymbol{0} \ \forall \ [\underline{y}, \overline{y}] \in \boldsymbol{\mathcal{Y}} \\
\implies & [\underline{p}, \overline{p}] \odot [\underline{y}, \overline{y}] \preceq \boldsymbol{0} \ \forall \ [\underline{y}, \overline{y}] \in \boldsymbol{\mathcal{Y}} \\
\implies & \underline{p} \leq 0 \ \textit{and} \ \overline{p} \leq 0 \ \textit{since} \ \underline{y} \geq 0 \ \textit{and} \ \overline{y} \geq 0.
\end{aligned}$$

Thus, $\boldsymbol{P} \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\boldsymbol{0})$*. Since the selected* $\{\boldsymbol{P}_k\}$ *in* $\mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\boldsymbol{0})$ *is arbitrary, it proves that* $N_{\boldsymbol{\mathcal{Y}}}(\boldsymbol{0})$ *is closed.*

Theorem 2.2 *If* $\boldsymbol{\mathcal{Y}}$ *is a nonempty subset of* $I(\mathbb{R})^n$*, then, for any* $\widehat{\boldsymbol{Y}}' \in \boldsymbol{\mathcal{Y}}$ *and* $\alpha \geq 0$*, we have* $\mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\alpha \odot \widehat{\boldsymbol{Y}}') = \mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\widehat{\boldsymbol{Y}}')$*.*

Proof: Assume $\widehat{\mathbf{G}} \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\alpha \odot \widehat{\mathbf{Y}}')$, then, for any $\widehat{\mathbf{Y}} \in \boldsymbol{\mathcal{Y}}$, and $\alpha \geq 0$, we have

$$\widehat{\mathbf{G}}^\top \odot_c ((\alpha \odot \widehat{\mathbf{Y}}) \ominus_{gH} (\alpha \odot \widehat{\mathbf{Y}}')) \preceq \mathbf{0}$$

$$\begin{aligned}
&\iff \quad \alpha \odot (\widehat{\mathbf{G}}^\top \odot_c (\widehat{\mathbf{Y}} \ominus_{gH} \widehat{\mathbf{Y}}^{'})) \preceq \mathbf{0} \\
&\iff \quad \widehat{\mathbf{G}}^\top \odot_c (\widehat{\mathbf{Y}} \ominus_{gH} \widehat{\mathbf{Y}}^{'}) \preceq \mathbf{0} \\
&\iff \quad \widehat{\mathbf{G}} \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^{'}).
\end{aligned}$$

Hence, for any $\alpha \geq 0$, we get $\mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\alpha \odot \widehat{\mathbf{Y}}^{'}) = \mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^{'})$. $\square$

Note 2.1 *We see that for $\alpha < 0$, instead of $\alpha \geq 0$, Theorem 2.2 may not be true. For example, if $\boldsymbol{\mathcal{Y}} = \{[|1+y|, 2|1+y|] : y \in \mathbb{R}\}$, we have the normal cone at $\widehat{\boldsymbol{Y}}^{'} = [1,2] \in \boldsymbol{\mathcal{Y}}$ as*

$$\begin{aligned}
\mathcal{N}_{\boldsymbol{\mathcal{Y}}}([1,2]) &= \{[\underline{p}, \overline{p}] \in I(\mathbb{R}) : [\underline{p}, \overline{p}] \odot ([\underline{y}, \overline{y}] \ominus_{gH} [1,2]) \preceq \boldsymbol{0} \; \forall \; [\underline{y}, \overline{y}] \in \boldsymbol{\mathcal{Y}}\} \\
&= \{[\underline{p}, \overline{p}] \in I(\mathbb{R}) : [\underline{p}, \overline{p}] \odot ([|1+y|-1, 2|1+y|-2]) \preceq \boldsymbol{0} \; \forall \; y \geq 0 \textit{ or } y \leq -2\} \\
&\quad \cap \{[\underline{p}, \overline{p}] \in I(\mathbb{R}) : [\underline{p}, \overline{p}] \odot ([2|1+y|-2, |1+y|-1]) \preceq \boldsymbol{0} \; \forall \; -2 \leq y \leq 0\} \\
&= I(\mathbb{R}_-) \cap I(\mathbb{R}_+) = \{\boldsymbol{0}\}.
\end{aligned}$$

Note that, for any $\alpha < 0$, $\alpha \odot \widehat{\boldsymbol{Y}}^{'} = [2\alpha, \alpha] \notin \boldsymbol{\mathcal{Y}}$. Thus, the normal cone on $\boldsymbol{\mathcal{Y}}$ at $\alpha \odot \widehat{\boldsymbol{Y}}^{'}$ does not exist. Therefore, Theorem 2.2 is not true for $\alpha < 0$.

Theorem 2.3 *If $\boldsymbol{\mathcal{Y}}_1, \boldsymbol{\mathcal{Y}}_2$ are nonempty convex subsets of $I(\mathbb{R})^n$ and $\widehat{\mathbf{Y}}_i^{'} \in \boldsymbol{\mathcal{Y}}_i$, $i = 1, 2$, then, $\mathcal{N}_{\boldsymbol{\mathcal{Y}}_1 \oplus \boldsymbol{\mathcal{Y}}_2}(\widehat{\mathbf{Y}}_1^{'} \oplus \widehat{\mathbf{Y}}_2^{'}) \subseteq \mathcal{N}_{\boldsymbol{\mathcal{Y}}_1}(\widehat{\mathbf{Y}}_1^{'}) \bigcap \mathcal{N}_{\boldsymbol{\mathcal{Y}}_2}(\widehat{\mathbf{Y}}_2^{'})$.*

Proof: Assuming $\widehat{\mathbf{G}} \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}_1 \oplus \boldsymbol{\mathcal{Y}}_2}(\widehat{\mathbf{Y}}_1^{'} \oplus \widehat{\mathbf{Y}}_2^{'})$, we have

$$\widehat{\mathbf{G}}^\top \odot_c [(\widehat{\mathbf{Y}}_1 \oplus \widehat{\mathbf{Y}}_2) \ominus_{gH} (\widehat{\mathbf{Y}}_1^{'} \oplus \widehat{\mathbf{Y}}_2^{'})] \preceq \mathbf{0} \text{ for all } \widehat{\mathbf{Y}}_1 \oplus \widehat{\mathbf{Y}}_2 \in \boldsymbol{\mathcal{Y}}_1 \oplus \boldsymbol{\mathcal{Y}}_2.$$

Specifically, for $\widehat{\mathbf{Y}}_2 = \widehat{\mathbf{Y}}_2^{'}$, we get

$$\widehat{\mathbf{G}}^\top \odot_c (\widehat{\mathbf{Y}}_1 \ominus_{gH} \widehat{\mathbf{Y}}_1^{'}) \preceq \mathbf{0} \text{ for all } \widehat{\mathbf{Y}}_1 \in \boldsymbol{\mathcal{Y}}_1.$$

As a result, $\widehat{\mathbf{G}} \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}_1}(\widehat{\mathbf{Y}}_1^{'})$ for all $\widehat{\mathbf{Y}}_1 \in \boldsymbol{\mathcal{Y}}_1$. Likewise, $\widehat{\mathbf{G}} \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}_2}(\widehat{\mathbf{Y}}_2^{'})$ for all $\widehat{\mathbf{Y}}_2 \in \boldsymbol{\mathcal{Y}}_2$. Thus, $\widehat{\mathbf{G}} \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}_1}(\widehat{\mathbf{Y}}_1^{'}) \bigcap \mathcal{N}_{\boldsymbol{\mathcal{Y}}_2}(\widehat{\mathbf{Y}}_2^{'})$ and we have the result. $\square$

Note 2.2 *We show that,in Theorem 2.3, if any of $\boldsymbol{\mathcal{Y}}_1$ and $\boldsymbol{\mathcal{Y}}_2$ is a nonconvex subset of $I(\mathbb{R})^n$, then the result may not be true. For example, let*

$$\boldsymbol{\mathcal{Y}}_1 = \{[-e^u, 0] : 0 \leq u \leq 1\} \textit{ and } \boldsymbol{\mathcal{Y}}_2 = \{[e^v, e^{2v}] : 0 \leq v \leq 1\}.$$

The set of intervals $\boldsymbol{\mathcal{Y}}_1$ and $\boldsymbol{\mathcal{Y}}_2$ are shown in Figure 2.1(a) (sky blue region) and in Figure 2.1(b) (green region), respectively. The normal cone at $\widehat{\boldsymbol{Y}}_1^{'} = [-1, 0] \in \boldsymbol{\mathcal{Y}}_1$ is *described as*

$$\begin{aligned}\mathcal{N}_{\boldsymbol{\mathcal{Y}}_1}([-1,0]) &= \{[\underline{p},\overline{p}] \in I(\mathbb{R}) : [\underline{p},\overline{p}] \odot ([\underline{y},\overline{y}] \ominus_{gH} [-1,0]) \preceq \boldsymbol{0} \ \forall \ [\underline{y},\overline{y}] \in \boldsymbol{\mathcal{Y}}_1\} \\ &= \{[\underline{p},\overline{p}] \in I(\mathbb{R}) : [\underline{p},\overline{p}] \odot [-e^u+1,0] \preceq \boldsymbol{0} \ \forall \ u \in [0,1]\} = I(\mathbb{R}_+).\end{aligned}$$

$\mathcal{N}_{\boldsymbol{\mathcal{Y}}_1}([-1,0])$ *at* $[-1,0]$ *is shown in Figure 2.1(a) by orange-colored circular-sector. In Figure 2.1(a), we can observe that the directions d (brown-colored thick arrow), d′ (blue-colored thick arrow), d″ (green-colored thick arrow) are the three possible normal vectors at point* $[-1,0]$*. Next, the normal cone at* $\widehat{\mathbf{Y}}_2' = \boldsymbol{1} \in \boldsymbol{\mathcal{Y}}_2$ *is given as*

$$\begin{aligned}\mathcal{N}_{\boldsymbol{\mathcal{Y}}_2}(\boldsymbol{1}) &= \{[\underline{q},\overline{q}] \in I(\mathbb{R}) : [\underline{q},\overline{q}] \odot ([\underline{y},\overline{y}] \ominus_{gH} \boldsymbol{1}) \preceq \boldsymbol{0} \ \forall \ [\underline{y},\overline{y}] \in \boldsymbol{\mathcal{Y}}_2\} \\ &= \{[\underline{q},\overline{q}] \in I(\mathbb{R}) : [\underline{q},\overline{q}] \odot [e^v-1,e^{2v}-1] \preceq \boldsymbol{0} \ \forall \ v \in [0,1]\} = I(\mathbb{R}_-).\end{aligned}$$

$\mathcal{N}_{\boldsymbol{\mathcal{Y}}_2}(\boldsymbol{1})$ *at* $\boldsymbol{1}$ *is shown in Figure 2.1(b) by a pink-coloured circular sector region. From Figure 2.1(b), we can observe that the direction d (yellow-coloured thick arrow), d′ (blue-colored thick arrow), d″ (sky-colored thick arrow) are three possible normal vectors at point* $\boldsymbol{1}$*. Then, we have*

$$\mathcal{N}_{\boldsymbol{\mathcal{Y}}_1}([-1,0]) \cap \mathcal{N}_{\boldsymbol{\mathcal{Y}}_2}(\boldsymbol{1}) = \{\boldsymbol{0}\}.$$

Here, we see that $\boldsymbol{\mathcal{Y}}_1 \oplus \boldsymbol{\mathcal{Y}}_2 = \{[0,e^{2u}] : 0 \leq u \leq 1\}$*. The normal cone at* $[0,1] \in \boldsymbol{\mathcal{Y}}_1 \oplus \boldsymbol{\mathcal{Y}}_2$ *is given as*

$$\begin{aligned}\mathcal{N}_{\boldsymbol{\mathcal{Y}}_1 \oplus \boldsymbol{\mathcal{Y}}_2}([0,1]) &= \{[\underline{r},\overline{r}] \in I(\mathbb{R}) : [\underline{r},\overline{r}] \odot ([0,e^{2u}] \ominus_{gH} [0,1]) \preceq \boldsymbol{0} \ \forall \ u \in [0,1]\} \\ &= \{[\underline{r},\overline{r}] \in I(\mathbb{R}) : [\underline{r},\overline{r}] \odot [0,e^{2u}-1] \ \forall \ u \in [0,1]\} = I(\mathbb{R}_-).\end{aligned}$$

We can observe that

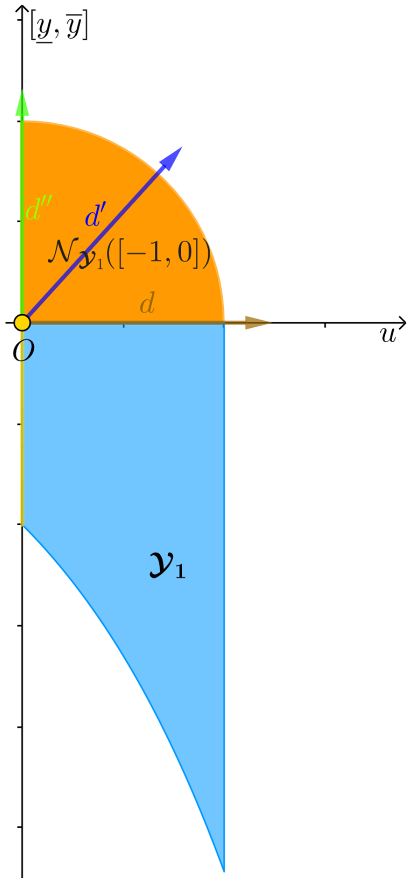
$$\mathcal{N}_{\boldsymbol{\mathcal{Y}}_1}([-1,0]) \cap \mathcal{N}_{\boldsymbol{\mathcal{Y}}_2}(\boldsymbol{1}) \subset \mathcal{N}_{\boldsymbol{\mathcal{Y}}_1 \oplus \boldsymbol{\mathcal{Y}}_2}([0,1]),$$

and this produces a strict inclusion in Theorem 2.3.

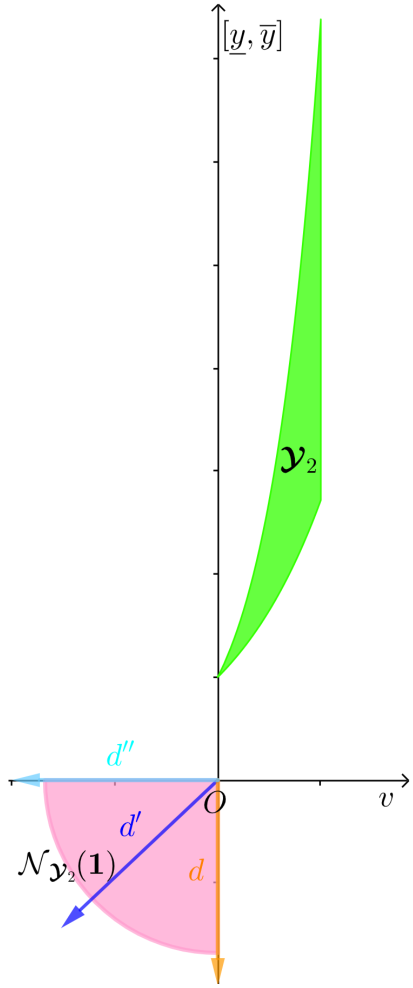
Theorem 2.4 *Let two nonempty convex sets* $\boldsymbol{\mathcal{Y}}_i$ *in* $I(\mathbb{R})^{n_i}, i = 1,2$*. Let* $\widehat{\mathbf{Y}}_i' \in \boldsymbol{\mathcal{Y}}_i, \ i = 1,2$*. Then,* $\mathcal{N}_{\boldsymbol{\mathcal{Y}}_1 \star \boldsymbol{\mathcal{Y}}_2}(\widehat{\boldsymbol{Y}}_1', \widehat{\boldsymbol{Y}}_2') = \mathcal{N}_{\boldsymbol{\mathcal{Y}}_1}(\widehat{\boldsymbol{Y}}_1') \star \mathcal{N}_{\boldsymbol{\mathcal{Y}}_2}(\widehat{\boldsymbol{Y}}_2')$*, where* $\boldsymbol{\mathcal{Y}}_1 \star \boldsymbol{\mathcal{Y}}_2 = \Big\{(\widehat{\boldsymbol{Y}}_1, \widehat{\boldsymbol{Y}}_2) \in I(\mathbb{R})^{n_1} \times I(\mathbb{R})^{n_2} : (\widehat{\boldsymbol{Y}}_1, \widehat{\boldsymbol{Y}}_2)^\top \odot_c (\widehat{\boldsymbol{Z}}_1, \widehat{\boldsymbol{Z}}_2) = \widehat{\boldsymbol{Y}}_1^\top \odot_c \widehat{\boldsymbol{Z}}_1 \oplus \widehat{\boldsymbol{Y}}_2^\top \odot_c \widehat{\boldsymbol{Z}}_2 \ \ \forall \ \ (\widehat{\boldsymbol{Z}}_1, \widehat{\boldsymbol{Z}}_2) \in I(\mathbb{R})^{n_1} \times I(\mathbb{R})^{n_2}\Big\}$*.*

Proof: Considering $\widehat{\mathbf{G}} = (\widehat{\mathbf{G}}_1, \widehat{\mathbf{G}}_2) \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}_1 \star \boldsymbol{\mathcal{Y}}_2}(\widehat{\mathbf{Y}}_1', \widehat{\mathbf{Y}}_2')$, for all $(\widehat{\mathbf{Y}}_1, \widehat{\mathbf{Y}}_2) \in \boldsymbol{\mathcal{Y}}_1 \star \boldsymbol{\mathcal{Y}}_2$, we get

$$(\widehat{\mathbf{G}}_1, \widehat{\mathbf{G}}_2)^\top \odot_c \{(\widehat{\mathbf{Y}}_1, \widehat{\mathbf{Y}}_2) \ominus_{gH} (\widehat{\mathbf{Y}}_1', \widehat{\mathbf{Y}}_2')) \preceq \mathbf{0}$$



(a) Normal cone of $\boldsymbol{\mathcal{Y}}_1$ at $\widehat{\mathbf{Y}}_1'$



(b) Normal cone of $\boldsymbol{\mathcal{Y}}_2$ at $\widehat{\mathbf{Y}}_2'$

Figure 2.1: Illustration of normal cones on $\boldsymbol{\mathcal{Y}}_1$ and $\boldsymbol{\mathcal{Y}}_2$ at $[-1, 0]$ and $\mathbf{1}$, respectively, of Note 2.2

.

$$\implies \quad (\widehat{\mathbf{G}}_1, \widehat{\mathbf{G}}_2)^\top \odot_c (\widehat{\mathbf{Y}}_1 \ominus_{gH} \widehat{\mathbf{Y}}_1', \widehat{\mathbf{Y}}_2 \ominus_{gH} \widehat{\mathbf{Y}}_2') \preceq \mathbf{0}$$
$$\implies \quad \widehat{\mathbf{G}}_1^\top \odot_c (\widehat{\mathbf{Y}}_1 \ominus_{gH} \widehat{\mathbf{Y}}_1') \oplus \widehat{\mathbf{G}}_2^\top \odot_c (\widehat{\mathbf{Y}}_2 \ominus_{gH} \widehat{\mathbf{Y}}_2') \preceq \mathbf{0}.$$

For $\widehat{\mathbf{Y}}_2 = \widehat{\mathbf{Y}}_2'$, from the above inequality, we have

$$\widehat{\mathbf{G}}_1^\top \odot_c (\widehat{\mathbf{Y}}_1 \ominus_{gH} \widehat{\mathbf{Y}}_1') \preceq \mathbf{0} \; \forall \; \widehat{\mathbf{Y}}_1 \in \boldsymbol{\mathcal{Y}}_1.$$

Hence, $\widehat{\mathbf{G}}_1 \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}_1}(\widehat{\mathbf{Y}}_1')$. Following similar argument, we have $\widehat{\mathbf{G}}_2 \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}_2}(\widehat{\mathbf{Y}}_2')$.
Because $(\widehat{\mathbf{G}}_1, \widehat{\mathbf{G}}_2)$ was considered to be arbitrary, we get

$$\mathcal{N}_{\boldsymbol{\mathcal{Y}}_1 \star \boldsymbol{\mathcal{Y}}_2}(\widehat{\mathbf{Y}}_1' \star \widehat{\mathbf{Y}}_2') \subseteq \mathcal{N}_{\boldsymbol{\mathcal{Y}}_1}(\widehat{\mathbf{Y}}_1') \star \mathcal{N}_{\boldsymbol{\mathcal{Y}}_2}(\widehat{\mathbf{Y}}_2').$$

For proving the reverse part, consider $(\widehat{\mathbf{G}}_1, \widehat{\mathbf{G}}_2) \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}_1}(\widehat{\mathbf{Y}}_1^{'}) \star \mathcal{N}_{\boldsymbol{\mathcal{Y}}_2}(\widehat{\mathbf{Y}}_2^{'})$, where $\widehat{\mathbf{G}}_1 \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}_1}(\widehat{\mathbf{Y}}_1^{'}), \widehat{\mathbf{G}}_2 \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}_2}(\widehat{\mathbf{Y}}_2^{'})$. Then, for all $\widehat{\mathbf{Y}}_1 \in \boldsymbol{\mathcal{Y}}_1$ and $\widehat{\mathbf{Y}}_2 \in \boldsymbol{\mathcal{Y}}_2$, we can write

$$\begin{aligned}
& \widehat{\mathbf{G}}_1 \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}_1}(\widehat{\mathbf{Y}}_1^{'}) \text{ and } \widehat{\mathbf{G}}_2 \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}_2}(\widehat{\mathbf{Y}}_2^{'}) \\
\implies \ & \widehat{\mathbf{G}}_1^{\top} \odot_c (\widehat{\mathbf{Y}}_1 \ominus_{gH} \widehat{\mathbf{Y}}_1^{'}) \preceq \mathbf{0} \text{ and } \widehat{\mathbf{G}}_2^{\top} \odot_c (\widehat{\mathbf{Y}}_2 \ominus_{gH} \widehat{\mathbf{Y}}_2^{'}) \preceq \mathbf{0} \\
\implies \ & \widehat{\mathbf{G}}_1^{\top} \odot_c (\widehat{\mathbf{Y}}_1 \ominus_{gH} \widehat{\mathbf{Y}}_1^{'}) \oplus \widehat{\mathbf{G}}_2^{\top} \odot_c (\widehat{\mathbf{Y}}_2 \ominus_{gH} \widehat{\mathbf{Y}}_2^{'}) \preceq \mathbf{0} \\
\implies \ & (\widehat{\mathbf{G}}_1, \widehat{\mathbf{G}}_2)^{\top} \odot_c (\widehat{\mathbf{Y}}_1 \ominus_{gH} \widehat{\mathbf{Y}}_1^{'}, \widehat{\mathbf{Y}}_2 \ominus_{gH} \widehat{\mathbf{Y}}_2^{'}) \preceq \mathbf{0} \\
\implies \ & (\widehat{\mathbf{G}}_1, \widehat{\mathbf{G}}_2)^{\top} \odot_c \{(\widehat{\mathbf{Y}}_1, \widehat{\mathbf{Y}}_2) \ominus_{gH} (\widehat{\mathbf{Y}}_1^{'}, \widehat{\mathbf{Y}}_2^{'})) \preceq \mathbf{0}.
\end{aligned}$$

Hence, $(\widehat{\mathbf{G}}_1, \widehat{\mathbf{G}}_2) \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}_1 \star \boldsymbol{\mathcal{Y}}_2}(\widehat{\mathbf{Y}}_1^{'} \star \widehat{\mathbf{Y}}_2^{'})$ for all $(\widehat{\mathbf{Y}}_1, \widehat{\mathbf{Y}}_2) \in \boldsymbol{\mathcal{Y}}_1 \star \boldsymbol{\mathcal{Y}}_2$., and, $\mathcal{N}_{\boldsymbol{\mathcal{Y}}_1}(\widehat{\mathbf{Y}}_1^{'}) \star \mathcal{N}_{\boldsymbol{\mathcal{Y}}_2}(\widehat{\mathbf{Y}}_2^{'}) \subseteq \mathcal{N}_{\boldsymbol{\mathcal{Y}}_1 \star \boldsymbol{\mathcal{Y}}_2}(\widehat{\mathbf{Y}}_1^{'} \star \widehat{\mathbf{Y}}_2^{'})$. This proves the result. □

Theorem 2.5 *Consider $\boldsymbol{\mathcal{Y}}_1, \boldsymbol{\mathcal{Y}}_2$ to be nonempty convex sets in $I(\mathbb{R})^n$ so that $\boldsymbol{\mathcal{Y}}_1 \subset \boldsymbol{\mathcal{Y}}_2$. Then, for all $\widehat{\boldsymbol{Y}}^{'} \in cl(\boldsymbol{\mathcal{Y}}_1)$, we have*

$$\mathcal{N}_{\boldsymbol{\mathcal{Y}}_2}(\widehat{\boldsymbol{Y}}^{'}) \subseteq \mathcal{N}_{\boldsymbol{\mathcal{Y}}_1}(\widehat{\boldsymbol{Y}}^{'}).$$

Additionally, if $\boldsymbol{\mathcal{Y}}$ is a finite union i.e., $\boldsymbol{\mathcal{Y}} = \bigcup_{j=1}^{n} \boldsymbol{\mathcal{Y}}_j$, then

$$\mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\widehat{\boldsymbol{Y}}^{'}) = \bigcap\nolimits_{j=1}^{n} \mathcal{N}_{\boldsymbol{\mathcal{Y}}_j}(\widehat{\boldsymbol{Y}}^{'}).$$

Proof: Consider $\widehat{\mathbf{G}} \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}_2}(\widehat{\mathbf{Y}}^{'})$. Then, we have

$$\begin{aligned}
& \widehat{\mathbf{G}}^{\top} \odot_c (\widehat{\mathbf{Y}} \ominus_{gH} \widehat{\mathbf{Y}}^{'}) \preceq \mathbf{0}, \text{ for all } \widehat{\mathbf{Y}} \in \boldsymbol{\mathcal{Y}}_2 \\
\implies \ & \widehat{\mathbf{G}}^{\top} \odot_c (\widehat{\mathbf{Y}} \ominus_{gH} \widehat{\mathbf{Y}}^{'}) \preceq \mathbf{0}, \text{ for all } \widehat{\mathbf{Y}} \in \boldsymbol{\mathcal{Y}}_1 \text{ since } \boldsymbol{\mathcal{Y}}_1 \subset \boldsymbol{\mathcal{Y}}_2.
\end{aligned}$$

Hence, $\widehat{\mathbf{G}} \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}_1}(\widehat{\mathbf{Y}}^{'})$. Thus, $\mathcal{N}_{\boldsymbol{\mathcal{Y}}_2}(\widehat{\mathbf{Y}}^{'}) \subseteq \mathcal{N}_{\boldsymbol{\mathcal{Y}}_1}(\widehat{\mathbf{Y}}^{'})$. Further,

$$\begin{aligned}
\boldsymbol{\mathcal{Y}} = \bigcup_{j=1}^{n} \boldsymbol{\mathcal{Y}}_j \quad & \implies \quad \boldsymbol{\mathcal{Y}}_j \subset \boldsymbol{\mathcal{Y}} \text{ for all } j = 1, 2, \ldots, n \\
& \implies \quad \mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^{'}) \subseteq \mathcal{N}_{\boldsymbol{\mathcal{Y}}_j}(\widehat{\mathbf{Y}}^{'}) \text{ for all } j = 1, 2, \ldots, n \\
& \implies \quad \mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^{'}) \subseteq \bigcap_{j=1}^{n} \mathcal{N}_{\boldsymbol{\mathcal{Y}}_j}(\widehat{\mathbf{Y}}^{'}).
\end{aligned}$$

Conversely, consider $\widehat{\mathbf{G}} \in \bigcap_{j=1}^{n} \mathcal{N}_{\boldsymbol{\mathcal{Y}}_j}(\widehat{\mathbf{Y}}^{'})$. So, for any $\widehat{\mathbf{Y}} \in \boldsymbol{\mathcal{Y}}_j$, $j = 1, 2, \ldots, n$, we get

$$\widehat{\mathbf{G}} \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}_j}(\widehat{\mathbf{Y}}^{'})$$

$$\implies \quad \widehat{\mathbf{G}}^{\top} \odot_c (\widehat{\mathbf{Y}} \ominus_{gH} \widehat{\mathbf{Y}}') \preceq \mathbf{0},$$

which holds for any $\widehat{\mathbf{Y}} \in \bigcup_{j=1}^{n} \boldsymbol{\mathcal{Y}}_j$. Therefore, $\widehat{\mathbf{G}} \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}')$. □

Example 2.4.3 *Through this example, we present a validation of Theorem 2.5. Let the sets* $\boldsymbol{\mathcal{Y}}_1 = \{\boldsymbol{0}\}$ *and* $\boldsymbol{\mathcal{Y}}_2 = \{[|y,|, 2|y|] : y \in \mathbb{R}\}$*. Note that* $\boldsymbol{\mathcal{Y}}_1 \subset \boldsymbol{\mathcal{Y}}_2$ *and* $\boldsymbol{0} \in Cl(\boldsymbol{\mathcal{Y}}_1)$*. Normal cones to* $\boldsymbol{\mathcal{Y}}_1$ *and* $\boldsymbol{\mathcal{Y}}_2$ *at* $\boldsymbol{0}$ *are given as*

$$\begin{aligned} \mathcal{N}_{\boldsymbol{\mathcal{Y}}_1}(\boldsymbol{0}) =& \{[\underline{c}, \overline{c}] \in I(\mathbb{R}) : [\underline{c}, \overline{c}] \odot (\boldsymbol{0} \ominus_{gH} \boldsymbol{0}) \preceq \boldsymbol{0}\} = I(\mathbb{R}) \\ \textit{and } \mathcal{N}_{\boldsymbol{\mathcal{Y}}_2}(\boldsymbol{0}) =& \{[\underline{c}, \overline{c}] \in I(\mathbb{R}) : [\underline{c}, \overline{c}] \odot ([\underline{y}, \overline{y}] \ominus_{gH} \boldsymbol{0}) \preceq \boldsymbol{0} \ \forall \ [\underline{y}, \overline{y}] \in \boldsymbol{\mathcal{Y}}_2\} \\ =& \{[\underline{c}, \overline{c}] \in I(\mathbb{R}) : [\underline{c}, \overline{c}] \odot ([|y|, 2|y|] \ominus_{gH} \boldsymbol{0}) \preceq \boldsymbol{0} \ \forall \ y \in \mathbb{R}\} = I(\mathbb{R}_-), \textit{respectively.} \end{aligned}$$

We can see that $\mathcal{N}_{\boldsymbol{\mathcal{Y}}_2}(\boldsymbol{0}) \subseteq \mathcal{N}_{\boldsymbol{\mathcal{Y}}_1}(\boldsymbol{0})$*. To validate the second half of Theorem 2.5, consider* $\boldsymbol{\mathcal{Y}}_1 \cup \boldsymbol{\mathcal{Y}}_2 = \boldsymbol{\mathcal{Y}}_2$*. Since* $\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{Y}}_2$*, we have*

$$\mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\boldsymbol{0}) = \mathcal{N}_{\boldsymbol{\mathcal{Y}}_2}(\boldsymbol{0}) = I(\mathbb{R}_-) = I(\mathbb{R}) \cap I(\mathbb{R}_-) = \mathcal{N}_{\boldsymbol{\mathcal{Y}}_1}(\boldsymbol{0}) \cap \mathcal{N}_{\boldsymbol{\mathcal{Y}}_2}(\boldsymbol{0}).$$

Remark 2.4.1 *Here, we present an example that if either* $\boldsymbol{\mathcal{Y}}_1$ *or* $\boldsymbol{\mathcal{Y}}_2$ *is a nonconvex subset of* $I(\mathbb{R})^n$*, then the first half of Theorem 2.5 may not be true. For this, let the sets* $\boldsymbol{\mathcal{Y}}_1 = \{[|y|, 2|y|] : y \geq 0\}$ *and* $\boldsymbol{\mathcal{Y}}_2 = \{[y, 2y] : y \geq 0\} \cup \{[2y, y] : y \leq 0\}$*. We can see that* $\boldsymbol{\mathcal{Y}}_1 \subset \boldsymbol{\mathcal{Y}}_2$ *and* $\boldsymbol{0} \in cl(\boldsymbol{\mathcal{Y}}_1)$*. Normal cones to* $\boldsymbol{\mathcal{Y}}_1$ *and* $\boldsymbol{\mathcal{Y}}_2$ *at* $\boldsymbol{0}$ *are given as*

$$\begin{aligned} \mathcal{N}_{\boldsymbol{\mathcal{Y}}_1}(\boldsymbol{0}) =& \{[\underline{c}, \overline{c}] \in I(\mathbb{R}) : [\underline{c}, \overline{c}] \odot ([\underline{y}, \overline{y}] \ominus_{gH} \boldsymbol{0}) \preceq \boldsymbol{0} \ \forall \ [\underline{y}, \overline{y}] \in \boldsymbol{\mathcal{Y}}_1\} = I(\mathbb{R}_-) \\ \textit{and } \mathcal{N}_{\boldsymbol{\mathcal{Y}}_2}(\boldsymbol{0}) =& \{[\underline{c}, \overline{c}] \in I(\mathbb{R}) : [\underline{c}, \overline{c}] \odot ([y, 2y] \ominus_{gH} \boldsymbol{0}) \preceq \boldsymbol{0} \ \forall \ y \geq 0\} \\ & \cap \{[\underline{c}, \overline{c}] \in I(\mathbb{R}) : [\underline{c}, \overline{c}] \odot ([2y, y] \ominus_{gH} \boldsymbol{0}) \preceq \boldsymbol{0} \ \forall \ y \leq 0\} \\ =& \ I(\mathbb{R}_-) \cap I(\mathbb{R}_+) = \{\boldsymbol{0}\}, \ \textit{respectively.} \end{aligned}$$

Thus, observe that $\mathcal{N}_{\boldsymbol{\mathcal{Y}}_1}(\boldsymbol{0}) \subset \mathcal{N}_{\boldsymbol{\mathcal{Y}}_2}(\boldsymbol{0})$*, i.e.,* $\mathcal{N}_{\boldsymbol{\mathcal{Y}}_2}(\boldsymbol{0}) \subseteq N_{\boldsymbol{\mathcal{Y}}_1}(\boldsymbol{0})$ *does not hold.*

2.5 Tangent cone for a set of intervals

In this section, we study the tangent cone for a set of intervals, along with its different characterizations. We also study some of its calculus rules. Notably, the tangent cone is defined as the closure of all feasible directions.

Definition 2.2 (Tangent cone via feasible direction in $I(\mathbb{R})^n$). *If* $\boldsymbol{\mathcal{Y}} \subseteq I(\mathbb{R})^n$*, then, the* tangent cone *to* $\boldsymbol{\mathcal{Y}}$ *at* $\widehat{\boldsymbol{Y}}^* \in \boldsymbol{\mathcal{Y}}$ *is defined by*

$$\mathcal{T}^f_{\boldsymbol{\mathcal{Y}}}(\widehat{\boldsymbol{Y}}^*) = \{\widehat{\boldsymbol{D}} \in I(\mathbb{R})^n : \ \exists \ \alpha_k \downarrow 0, \ \exists \ \widehat{\boldsymbol{D}}_k \to \widehat{\boldsymbol{D}}, \ (\widehat{\boldsymbol{Y}}^* \oplus \alpha_k \odot \widehat{\boldsymbol{D}}_k) \in \boldsymbol{\mathcal{Y}} \ \forall \ k \in \mathbb{N}\}.$$

Example 2.5.1 *Let a nonempty set* $\boldsymbol{\mathcal{Y}} = \{([y_1, y_1], [\underline{y}_2, \overline{y}_2]) \in I(\mathbb{R})^2 : \underline{y}_2 \geq y_1^5, \overline{y}_2 \leq y_1^3, 0 \leq \underline{y}_2, \overline{y}_2 \leq 1\}$*, shown by pink-shaded region in Figure 2.2. Consider* $\{\widehat{\boldsymbol{D}}_k\} = \{(\boldsymbol{D}_{1k}, \boldsymbol{D}_{2k})\} = \{([\underline{d}_{1k}, \overline{d}_{1k}], [\underline{d}_{2k}, \overline{d}_{2k}])\}$ *to be a sequence of directions of* $\boldsymbol{\mathcal{Y}}$ *at* $\widehat{\boldsymbol{Y}}^* = (\boldsymbol{0}, \boldsymbol{0})$ *and* $\{\widehat{\boldsymbol{D}}_k\}$ *converges to a tangent direction* $\widehat{\boldsymbol{D}} = (\boldsymbol{D}_1, \boldsymbol{D}_2) = ([\underline{d}_1, \overline{d}_1], [\underline{d}_2, \overline{d}_2])$*. In Figure 2.2, clearly for any* k*,* $\boldsymbol{0} \preceq \boldsymbol{D}_{1k}$ *and* $\boldsymbol{0} \preceq \boldsymbol{D}_{2k}$*. Here, we have two cases to consider:*

- *Case 1. For any* k*,* $\boldsymbol{0} \preceq \boldsymbol{D}_{1k}, 0 \notin \boldsymbol{D}_{2k}$*. Because* $\{(\boldsymbol{D}_{1k}, \boldsymbol{D}_{2k})\}$ *is a sequence of directions of* $\boldsymbol{\mathcal{Y}}$*, there exists* $\delta > 0$ *so that*

  $$(\boldsymbol{0}, \boldsymbol{0}) \oplus \alpha_k \odot (\boldsymbol{D}_{1k}, \boldsymbol{D}_{2k}) \in \boldsymbol{\mathcal{Y}} \text{ and } \alpha_k \in [0, \delta] \text{ for all } k.$$

  From the definition of $\boldsymbol{\mathcal{Y}}$*, we can get*

  $$\alpha_k \overline{d}_{2k} \leq (\alpha_k d_{1k})^3 \text{ and } (\alpha_k d_{1k})^5 \leq \alpha_k \underline{d}_{2k} \text{ and } \alpha_k \in [0, \delta] \text{ for any } k \tag{2.8}$$

  Then, select $\alpha_k = \min\left\{\frac{1}{2d_{1k}}\sqrt{\frac{\overline{d}_{2k}}{d_{1k}}}, \frac{\delta}{2}\right\}$ *for any* k*. It is clear that* $\alpha_k \in [0, \delta]$ *and* $\alpha_k < \frac{1}{d_{1k}}\sqrt{\frac{\overline{d}_{2k}}{d_{1k}}}$ *for all* k*. This means*

  $$(d_{1k}\alpha_k)^2 < \frac{\overline{d}_{2k}}{d_{1k}} \implies (d_{1k}\alpha_k)^3 < \frac{d_{2k}}{d_{1k}} d_{1k}\alpha_k = \alpha_k \overline{d}_{2k} \text{ for all } k,$$

  that is, $(\alpha_k d_{1k})^3 < \alpha_k \overline{d}_{2k}$ *with* $\alpha_k \in [0, \delta]$*, which contradicts the first inequality in the relation (2.8). Thus, for Case 1,* $\{(\boldsymbol{D}_{1k}, \boldsymbol{D}_{2k})\}$ *can not be a proper selection for a sequence of directions so that Definition 2.2 gets satisfied.*

- *Case 2. For any* k*,* $\boldsymbol{D}_{1k} = \boldsymbol{0}$*. We select, for any* $\delta \geq 0$*,* $\alpha_k = \frac{\delta}{2} \in [0, \delta]$ *for all* k*. Here,*

  $$(\boldsymbol{0}, \boldsymbol{0}) \oplus \alpha_k \odot (\boldsymbol{D}_{1k}, \boldsymbol{D}_{2k}) = (\boldsymbol{0}, \alpha \odot \boldsymbol{D}_{2k}),$$

  which lies in $\boldsymbol{\mathcal{Y}}$ *only if* $\alpha_k \odot \boldsymbol{D}_{2k} = \boldsymbol{0}$ *for all* k*. Thus* $\{(\boldsymbol{D}_{1k}, \boldsymbol{D}_{2k})\} = \{(\boldsymbol{0}, \boldsymbol{0})\}$ *is the only tangent direction at* $(\boldsymbol{0}, \boldsymbol{0})$*.*

Thus, $\{(\boldsymbol{D}_{1k}, \boldsymbol{D}_{2k})\}$ *with* $\boldsymbol{0} \preceq \boldsymbol{D}_{1k}$ *and* $\boldsymbol{D}_{2k} = \boldsymbol{0}$ *for all* k *be an appropriate choice of the sequence of directions of* $\boldsymbol{\mathcal{Y}}$ *at* $(\boldsymbol{0}, \boldsymbol{0})$*. Because* $\{(\boldsymbol{D}_{1k}, \boldsymbol{D}_{2k})\}$ *converges to the tangent direction* $\widehat{\boldsymbol{D}} = (\boldsymbol{D}_1, \boldsymbol{D}_2)$*, then*

$$\mathcal{T}^f_{\boldsymbol{\mathcal{Y}}}(\widehat{\boldsymbol{Y}}^*) = \{(\boldsymbol{D}_1, \boldsymbol{D}_2) \in I(\mathbb{R})^2 : \boldsymbol{0} \preceq \boldsymbol{D}_1, \boldsymbol{D}_2 = \boldsymbol{0}\}.$$

The tangent cone $\mathcal{T}^f_{\boldsymbol{\mathcal{Y}}}(\widehat{\boldsymbol{Y}}^*)$ *is shown as blue arrow in Figure 2.2. Hence, the above set*

$\mathcal{T}^f_{\mathcal{Y}}(\widehat{\boldsymbol{Y}}^*)$ *simply reduces to the curves* $\underline{y}_2 = y_1^3$ *and* $\overline{y}_2 = y_1^5$, *separately as they have tangents* $\underline{d}_2 = 0$ *and* $\overline{d}_2 = 0$ *with* $d_1 \geq 0$ *from the perspective of* $(y_1, \underline{y}_2)$ *and* $(y_1, \overline{y}_2)$, *respectively. This also verifies that tangent cone via feasible directions* $\mathcal{T}^f_{\mathcal{Y}}(\widehat{\boldsymbol{Y}}^*)$ *includes all the points on* $\boldsymbol{D}_2 = \boldsymbol{0}$ *with* $\boldsymbol{0} \preceq \boldsymbol{D}_1$.

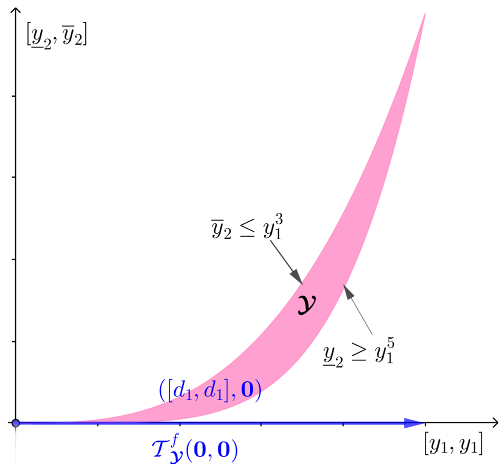


Figure 2.2: Illustration of tangent direction of $\mathcal{Y}$ at $(\mathbf{0}, \mathbf{0})$, in Example 2.5.1

Definition 2.3 (Tangent cone via sequence in $I(\mathbb{R})^n$). *Let* $\mathcal{Y} \subseteq I(\mathbb{R})^n$. *Then, the* tangent cone *to* $\mathcal{Y}$ *at* $\widehat{\boldsymbol{Y}}^* \in \mathcal{Y}$ *is given by*

$$\mathcal{T}^s_{\mathcal{Y}}(\widehat{\boldsymbol{Y}}^*) = \left\{ \widehat{\boldsymbol{D}} \in I(\mathbb{R})^n : \exists \, \{\widehat{\boldsymbol{Y}}_k\} \in \mathcal{Y},\ \widehat{\boldsymbol{Y}}_k \to \widehat{\boldsymbol{Y}}^* \text{ such that } \lim_{k\to\infty} \frac{1}{\alpha_k} \odot (\widehat{\boldsymbol{Y}}_k \ominus_{gH} \widehat{\boldsymbol{Y}}^*) = \frac{1}{\|\widehat{\boldsymbol{D}}\|} \odot \widehat{\boldsymbol{D}} \right\},$$

where $\alpha_k = \|\widehat{\boldsymbol{Y}}_k \ominus_{gH} \widehat{\boldsymbol{Y}}^*\|$.

Example 2.5.2 *Let us assume a nonempty set* $\mathcal{Y} = \{\boldsymbol{Y} \in I(\mathbb{R}) | \|\boldsymbol{Y}\| \leq 2\}$. *We begin by calculating the tangent vectors using a sequential method, as described in Definition 2.3, by considering various sequences in* $\mathcal{Y}$. *We will demonstrate that the resulting tangent vector also meets the feasibility criterion as specified in Definition 2.2.*
For obtaining $\mathcal{T}^s_{\mathcal{Y}}(\boldsymbol{Y}^*)$, *let sequence* $\boldsymbol{Y}_k^{(1)} = \left[\frac{1}{\sqrt{2k}}, 1 - \frac{1}{\sqrt{k}}\right], k > 2$ *which converges to* $\boldsymbol{Y}^* = [0, 1]$ *and*

$$\lim_{k\to\infty} \frac{1}{\|\boldsymbol{Y}_k^{(1)} \ominus_{gH} \boldsymbol{Y}^*\|} \odot (\boldsymbol{Y}_k^{(1)} \ominus_{gH} \boldsymbol{Y}^*) = \lim_{k\to\infty} \sqrt{k} \odot \left[\frac{-1}{\sqrt{k}}, \frac{1}{\sqrt{2k}}\right] = \left[-1, \frac{1}{\sqrt{2}}\right].$$

Hence, $\left[-1, \frac{1}{\sqrt{2}}\right]$ *is a tangent via sequence in* $I(\mathbb{R})$.
For satisfying the feasibility criterion, if we take $\alpha_k = \frac{1}{k}$, *we have*

$$\widehat{\boldsymbol{Y}}^* \oplus \alpha_k \odot \widehat{\boldsymbol{Y}}_k^{(1)} = [0, 1] \oplus \tfrac{1}{k} \odot \left[\tfrac{1}{\sqrt{2k}}, 1 - \tfrac{1}{\sqrt{k}}\right] \in \mathcal{Y} \text{ as } \left\|[0, 1] \oplus \tfrac{1}{k} \odot \left[\tfrac{1}{\sqrt{2k}}, 1 - \tfrac{1}{\sqrt{k}}\right]\right\| \leq 2.$$

Likewise, for second sequence $\boldsymbol{Y}_k^{(2)} = \left[\frac{1}{\sqrt{3k}}, 1 - \frac{1}{\sqrt{k}}\right], k > 3$ *which converges to* $\boldsymbol{Y}^* = [0,1]$*, we have*

$$\lim_{k\to\infty} \frac{1}{\|\,\boldsymbol{Y}_k^{(2)} \ominus_{gH} \boldsymbol{Y}^*\|} \odot (\,\boldsymbol{Y}_k^{(2)} \ominus_{gH} \boldsymbol{Y}^*) = \lim_{k\to\infty} \sqrt{k} \odot \left[\frac{-1}{\sqrt{k}}, \frac{1}{\sqrt{3k}}\right] = \left[-1, \frac{1}{\sqrt{3}}\right].$$

Hence, $\left[-1, \frac{1}{\sqrt{3}}\right]$ *is a tangent via sequence in* $I(\mathbb{R})$*.*
Again, for meeting the feasibility criterion, take $\alpha_k = \frac{1}{k}$*. Then, we get*

$$\widehat{\boldsymbol{Y}}^* \oplus \alpha_k \odot \widehat{\boldsymbol{Y}}_k^{(2)} = [0,1] \oplus \tfrac{1}{k} \odot \left[\tfrac{1}{\sqrt{3k}}, 1 - \tfrac{1}{\sqrt{k}}\right] \in \boldsymbol{\mathcal{Y}} \text{ i.e } \left\|[0,1] \oplus \tfrac{1}{k} \odot \left[\tfrac{1}{\sqrt{3k}}, 1 - \tfrac{1}{\sqrt{k}}\right]\right\| \leq 2.$$

Using induction, the sequence of tangent vectors can be obtained by sequence $\boldsymbol{Y}_k^{(n)} = \left[\frac{1}{\sqrt{(n+1)k}}, 1 - \frac{1}{\sqrt{k}}\right], k > n$ *which converges to* $\boldsymbol{Y}^* = [0,1]$*. Thus, for all n, set of tangents is given as*

$$\lim_{k\to\infty} \frac{1}{\|\,\boldsymbol{Y}_k^{(n)} \ominus_{gH} \boldsymbol{Y}^*\|} \odot (\,\boldsymbol{Y}_k^{(n)} \ominus_{gH} \boldsymbol{Y}^*) = \lim_{k\to\infty} \sqrt{k} \odot \left[\frac{-1}{\sqrt{k}}, \frac{1}{\sqrt{(n+1)k}}\right] = \left[-1, \frac{1}{\sqrt{n+1}}\right].$$

For satisfying feasibility criterion, take $\alpha_k = \frac{1}{k}$*, and we get*

$$[0,1] \oplus \frac{1}{k} \odot \left[\frac{1}{\sqrt{(n+1)k}}, 1 - \frac{1}{\sqrt{k}}\right] \in \boldsymbol{\mathcal{Y}} \text{ i.e } \left\|[0,1] \oplus \frac{1}{k} \odot \left[\frac{1}{\sqrt{(n+1)k}}, 1 - \frac{1}{\sqrt{k}}\right]\right\| \leq 2.$$

Therefore, $\left\{\left[-1, \frac{1}{\sqrt{2}}\right], \left[-1, \frac{1}{\sqrt{3}}\right], \ldots, \left[-1, \frac{1}{\sqrt{n+1}}\right], \ldots\right\}$ *results in tangent cone via both feasible direction and sequential approach in* $I(\mathbb{R})$ *as given in Definition 2.2 and Definition 2.3, respectively.*

Theorem 2.6 *Consider* $\boldsymbol{\mathcal{Y}} \subseteq I(\mathbb{R})^n$*. Then, for any* $\widehat{\boldsymbol{Y}}^* \in \boldsymbol{\mathcal{Y}}$*, the tangent cone via sequence* $\mathcal{T}^s_{\boldsymbol{\mathcal{Y}}}(\widehat{\boldsymbol{Y}}^*)$ *is closed.*

Proof: Assume that sequence $\{\widehat{\mathbf{D}}_k\}$ in $\mathcal{T}^s_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*)$ converges to some $\widehat{\mathbf{D}} \in I(\mathbb{R})^n$. So, for each k, there exists a sequence $\{\widehat{\mathbf{Y}}_k^{(i)}\} \in \boldsymbol{\mathcal{Y}}$ with $\widehat{\mathbf{Y}}_k^{(i)} \neq \widehat{\mathbf{Y}}^*$ satisfying

$$\lim_{i\to\infty} \widehat{\mathbf{Y}}_k^{(i)} = \widehat{\mathbf{Y}}^* \text{ and } \lim_{i\to\infty} \frac{1}{\alpha_k} \odot (\widehat{\mathbf{Y}}_k^{(i)} \ominus_{gH} \widehat{\mathbf{Y}}^*) = \frac{1}{\|\widehat{\mathbf{D}}_k\|} \odot \widehat{\mathbf{D}}_k,$$

where $\alpha_k^{(i)} = \|\widehat{\mathbf{Y}}_k^{(i)} \ominus_{gH} \widehat{\mathbf{Y}}^*\|$. Thus, for each k, select an index i_k so that $i_1 < i_2 < i_3 <$

$\cdots < i_k$. Here, let's define $\alpha_k^{(i_k)} = \|\widehat{\mathbf{Y}}_k^{(i_k)} \ominus_{gH} \widehat{\mathbf{Y}}^*\|$. Then, for each k,

$$\lim_{k\to\infty} \widehat{\mathbf{Y}}_k^{(i_k)} = \widehat{\mathbf{Y}}^*, \qquad \lim_{k\to\infty} \left\| \left(\frac{1}{\alpha_k^{(i_k)}} \odot (\widehat{\mathbf{Y}}_k^{(i_k)} \ominus_{gH} \widehat{\mathbf{Y}}^*) \right) \ominus_{gH} \left(\frac{1}{\|\widehat{\mathbf{D}}_k\|} \odot \widehat{\mathbf{D}}_k \right) \right\| = 0. \quad (2.9)$$

Using Lemma 2.3, p. 6 in [64] and Lemma 2.3 (i), p. 4 in [123] and for all k, we get

$$\begin{aligned}
&\lim_{k\to\infty} \left\| \left(\frac{1}{\alpha_k^{(i_k)}} \odot (\widehat{\mathbf{Y}}_k^{(i_k)} \ominus_{gH} \widehat{\mathbf{Y}}^*) \right) \ominus_{gH} \left(\frac{1}{\|\widehat{\mathbf{D}}\|} \odot \widehat{\mathbf{D}} \right) \right\| \\
&\leq \lim_{k\to\infty} \left\| \left(\frac{1}{\alpha_k^{(i_k)}} \odot (\widehat{\mathbf{Y}}_k^{(i_k)} \ominus_{gH} \widehat{\mathbf{Y}}^*) \right) \ominus_{gH} \left(\frac{1}{\|\widehat{\mathbf{D}}_k\|} \odot \widehat{\mathbf{D}}_k \right) \oplus \left(\frac{1}{\|\widehat{\mathbf{D}}_k\|} \odot \widehat{\mathbf{D}}_k \right) \ominus_{gH} \left(\frac{1}{\|\widehat{\mathbf{D}}\|} \odot \widehat{\mathbf{D}} \right) \right\| \\
&\leq \lim_{k\to\infty} \left\| \left(\frac{1}{\alpha_k^{(i_k)}} \odot (\widehat{\mathbf{Y}}_k^{(i_k)} \ominus_{gH} \widehat{\mathbf{Y}}^*) \right) \ominus_{gH} \left(\frac{1}{\|\widehat{\mathbf{D}}_k\|} \odot \widehat{\mathbf{D}}_k \right) \right\| \oplus \left\| \left(\frac{1}{\|\widehat{\mathbf{D}}_k\|} \odot \widehat{\mathbf{D}}_k \right) \ominus_{gH} \left(\frac{1}{\|\widehat{\mathbf{D}}\|} \odot \widehat{\mathbf{D}} \right) \right\|.
\end{aligned}$$

Because $\widehat{\mathbf{D}}_k \to \widehat{\mathbf{D}}$, $\|\widehat{\mathbf{D}}_k\| \to \|\widehat{\mathbf{D}}\|$, using in (2.9), we get

$$\lim_{k\to\infty} \widehat{\mathbf{Y}}_k^{(i_k)} = \widehat{\mathbf{Y}}^* \text{ and } \lim_{k\to\infty} \left\| \left(\frac{1}{\alpha_k^{(i_k)}} \odot (\widehat{\mathbf{Y}}_k^{(i_k)} \ominus_{gH} \widehat{\mathbf{Y}}^*) \right) \ominus_{gH} \left(\frac{1}{\|\widehat{\mathbf{D}}\|} \odot \widehat{\mathbf{D}} \right) \right\| = 0.$$

This means that $\widehat{\mathbf{D}} \in \mathcal{T}^s_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*)$. Thus, $\mathcal{T}^s_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*)$ is closed. $\square$

Example 2.5.3 *Through this example, we illustrate Theorem 2.6. Let*

$$\boldsymbol{\mathcal{Y}} \;\;=\;\; \{([\underline{y}_1, \overline{y}_1], [\underline{y}_2, \overline{y}_2]) \in I(\mathbb{R})^2 : \underline{y}_1 \leq 0, \overline{y}_1 \leq 0, \underline{y}_2 \geq 0, \overline{y}_2 \geq 0\}.$$

Since $\boldsymbol{\mathcal{Y}}$ is a cone in itself, we have the tangent cone via sequence in $I(\mathbb{R})^n$ as

$$\begin{aligned}
\mathcal{T}^s_{\boldsymbol{\mathcal{Y}}}(\boldsymbol{0}, \boldsymbol{0}) &= \{([\underline{d}_1, \overline{d}_1], [\underline{d}_2, \overline{d}_2]) : \underline{d}_1 \leq 0, \overline{d}_1 \leq 0, \underline{d}_2 \geq 0, \overline{d}_2 \geq 0\} \\
&= I(\mathbb{R}_-) \times I(\mathbb{R}_+),
\end{aligned}$$

which is definitely a closed set in $I(\mathbb{R})^2$.

Theorem 2.7 *Consider $\boldsymbol{\mathcal{Y}}$ to be a convex subset of $I(\mathbb{R})^n$. Then, for any $\widehat{\boldsymbol{Y}}^* \in \boldsymbol{\mathcal{Y}}$, the tangent cone $\mathcal{T}^f_{\boldsymbol{\mathcal{Y}}}(\widehat{\boldsymbol{Y}}^*)$ is a convex set in $I(\mathbb{R})^n$.*

Proof: Consider $\widehat{\mathbf{D}}_1,\ \widehat{\mathbf{D}}_2 \in \mathcal{T}^f_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*)$. Our task is to show that

$$\widehat{\mathbf{V}} = (\lambda \odot \widehat{\mathbf{D}}_1 \oplus (1-\lambda) \odot \widehat{\mathbf{D}}_2) \in \mathcal{T}^f_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*) \text{ for all } \lambda \in [0,1].$$

Because $\widehat{\mathbf{D}}_1 \in \mathcal{T}^f_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*)$, there exist sequences $\widehat{\mathbf{D}}_k^1 \to \widehat{\mathbf{D}}_1$ and $\alpha_k \downarrow 0$ such that $\widehat{\mathbf{Y}}^* \oplus \alpha_k \odot \widehat{\mathbf{D}}_k^1 \in \boldsymbol{\mathcal{Y}}$. Likewise, for $\widehat{\mathbf{D}}_2 \in \mathcal{T}^f_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*)$, there exist sequences $\widehat{\mathbf{D}}_k^2 \to \widehat{\mathbf{D}}_2$ and $\beta_k \downarrow 0$ such that $\widehat{\mathbf{Y}}^* \oplus \beta_k \odot \widehat{\mathbf{D}}_k^2 \in \boldsymbol{\mathcal{Y}}$.

Then, we take a sequence $\widehat{\mathbf{V}}_k = \lambda \odot \widehat{\mathbf{D}}_k^1 \oplus (1-\lambda) \odot \widehat{\mathbf{D}}_k^2 \to \widehat{\mathbf{V}}$. Selecting $\gamma_k = \min\{\alpha_k, \beta_k\}$ for any k we see that $\gamma_k \downarrow 0$. We now consider the following scenarios for each $k \in \mathbb{N}$:

(i) For $\gamma_k = \alpha_k$, we have

$$\begin{aligned}
\widehat{\mathbf{Y}}^* \oplus \gamma_k \odot \widehat{\mathbf{V}}_k &= \widehat{\mathbf{Y}}^* \oplus \alpha_k \odot (\lambda \odot \widehat{\mathbf{D}}_k^1 \oplus (1-\lambda) \odot \widehat{\mathbf{D}}_k^2) \\
&= \lambda \odot \widehat{\mathbf{Y}}^* \oplus (1-\lambda) \odot \widehat{\mathbf{Y}}^* \oplus \alpha_k \odot (\lambda \odot \widehat{\mathbf{D}}_k^1 \oplus (1-\lambda) \odot \widehat{\mathbf{D}}_k^2) \\
&= \lambda \odot \widehat{\mathbf{Y}}^* \oplus (\alpha_k \lambda) \odot \widehat{\mathbf{D}}_k^1 \oplus (1-\lambda) \odot \widehat{\mathbf{Y}}^* \oplus (\alpha_k(1-\lambda)) \odot \widehat{\mathbf{D}}_k^2 \\
&= \lambda \odot \{\widehat{\mathbf{Y}}^* \oplus \alpha_k \odot \widehat{\mathbf{D}}_k^1\} \oplus (1-\lambda) \odot \{\widehat{\mathbf{Y}}^* \oplus \alpha_k \odot \widehat{\mathbf{D}}_k^2\}.
\end{aligned}$$

Since $\widehat{\mathbf{Y}}^* \oplus \alpha_k \odot \widehat{\mathbf{D}}_k^1$ and $\widehat{\mathbf{Y}}^* \oplus \alpha_k \odot \widehat{\mathbf{D}}_k^2$ belong to $\boldsymbol{\mathcal{Y}}$, and $\boldsymbol{\mathcal{Y}}$ is a convex subset of $I(\mathbb{R})^n$, therefore $\widehat{\mathbf{Y}}^* \oplus \gamma_k \odot \widehat{\mathbf{V}}_k \in \boldsymbol{\mathcal{Y}}$.

(ii) For $\gamma_k = \beta_k$, we can use the step similar to that in Case ((i)).

This proves that for the sequence $\widehat{\mathbf{V}}_k = \lambda \odot \widehat{\mathbf{D}}_k^1 \oplus (1-\lambda) \odot \widehat{\mathbf{D}}_k^2 \to \widehat{\mathbf{V}}$, there exists $\gamma_k \downarrow 0$ such that $\widehat{\mathbf{Y}}^* \oplus \gamma_k \odot \widehat{\mathbf{V}}_k \in \boldsymbol{\mathcal{Y}}$. Hence, $\widehat{\mathbf{V}} = (\lambda \odot \widehat{\mathbf{D}}_1 \oplus (1-\lambda) \odot \widehat{\mathbf{D}}_2) \in \mathcal{T}^f_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*)$ for all $\lambda \in [0,1]$. $\square$

Theorem 2.8 *With $\boldsymbol{\mathcal{Y}}$ as a convex subset of $I(\mathbb{R})^n$, $\mathcal{T}^s_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*)$ is the closure of the cone generated by $\boldsymbol{\mathcal{Y}} \ominus_{gH} \widehat{\mathbf{Y}}^*$, i.e., $\mathcal{T}^s_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*) = cl(cone(\boldsymbol{\mathcal{Y}} \ominus_{gH} \widehat{\mathbf{Y}}^*))$.*

Proof: Assume $\widehat{\mathbf{D}} \in \mathcal{T}^s_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*)$. Then, a sequence $\widehat{\mathbf{Y}}_k \in \boldsymbol{\mathcal{Y}}$ exists that converges to $\widehat{\mathbf{Y}}^*$ satisfying

$$\lim_{k\to\infty} \frac{1}{\|\widehat{\mathbf{Y}}_k \ominus_{gH} \widehat{\mathbf{Y}}^*\|} \odot (\widehat{\mathbf{Y}}_k \ominus_{gH} \widehat{\mathbf{Y}}^*) = \frac{1}{\|\widehat{\mathbf{D}}\|} \odot \widehat{\mathbf{D}}.$$

Let $\alpha_k = \|\widehat{\mathbf{Y}}_k \ominus_{gH} \widehat{\mathbf{Y}}\| > 0$. Then, we get $\frac{1}{\alpha_k} > 0$. Because $\widehat{\mathbf{Y}}_k \ominus_{gH} \widehat{\mathbf{Y}}^* \in \boldsymbol{\mathcal{Y}} \ominus_{gH} \widehat{\mathbf{Y}}^*$,

$$\begin{aligned}
& \frac{1}{\alpha_k} \odot (\widehat{\mathbf{Y}}_k \ominus_{gH} \widehat{\mathbf{Y}}^*) \in \text{cone}(\boldsymbol{\mathcal{Y}} \ominus_{gH} \widehat{\mathbf{Y}}^*) \\
\implies & \frac{\|\widehat{\mathbf{D}}\|}{\alpha_k} \odot (\widehat{\mathbf{Y}}_k \ominus_{gH} \widehat{\mathbf{Y}}^*) \in \text{cone}(\boldsymbol{\mathcal{Y}} \ominus_{gH} \widehat{\mathbf{Y}}^*),\ \|\widehat{\mathbf{D}}\| \geq 0 \\
\implies & \|\widehat{\mathbf{D}}\| \odot \left(\lim_{k\to\infty} \frac{1}{\alpha_k} \odot (\widehat{\mathbf{Y}}_k \ominus_{gH} \widehat{\mathbf{Y}}^*) \right) \in \text{cl}(\text{cone}(\boldsymbol{\mathcal{Y}} \ominus_{gH} \widehat{\mathbf{Y}}^*).
\end{aligned}$$

As $\lim_{k\to\infty} \frac{1}{\alpha_k} \odot (\widehat{\mathbf{Y}}_k \ominus_{gH} \widehat{\mathbf{Y}}^*) \to \frac{1}{\|\widehat{\mathbf{D}}\|} \odot \widehat{\mathbf{D}}$, $\widehat{\mathbf{D}} \in \text{cl}(\text{cone}(\boldsymbol{\mathcal{Y}} \ominus_{gH} \widehat{\mathbf{Y}}^*))$. This means that $\mathcal{T}^s_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*) \subseteq \text{cl}(\text{cone}(\boldsymbol{\mathcal{Y}} \ominus_{gH} \widehat{\mathbf{Y}}^*))$.

For showing the reverse inclusion, let an arbitrary element $\widehat{\mathbf{Y}}$ in $\boldsymbol{\mathcal{Y}}$. Take a sequence

$$\widehat{\mathbf{Y}}_k = \frac{1}{k} \odot \widehat{\mathbf{Y}} \oplus \left(1 - \frac{1}{k}\right) \odot \widehat{\mathbf{Y}}^*, k \in \mathbb{N}.$$

Hence, by (ii) of Lemma (1.1), $k \odot (\widehat{\mathbf{Y}}_k \ominus_{gH} \widehat{\mathbf{Y}}^*) = \widehat{\mathbf{Y}} \ominus_{gH} \widehat{\mathbf{Y}}^*$.

As $\boldsymbol{\mathcal{Y}}$ is convex, $\{\widehat{\mathbf{Y}}_k\} \in \boldsymbol{\mathcal{Y}}$ and $\widehat{\mathbf{Y}}_k \to \widehat{\mathbf{Y}}^*$ as $k \to \infty$. Let $\alpha_k = \frac{1}{k} > 0, \alpha_k \to 0$ so that

$$\lim_{k\to\infty} \frac{1}{\alpha_k} \odot (\widehat{\mathbf{Y}}_k \ominus_{gH} \widehat{\mathbf{Y}}^*) = \widehat{\mathbf{Y}} \ominus_{gH} \widehat{\mathbf{Y}}^*$$

which means $\widehat{\mathbf{Y}} \ominus_{gH} \widehat{\mathbf{Y}}^* \in \mathcal{T}^s_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*)$. As $\widehat{\mathbf{Y}}$ is arbitrary, thus $\boldsymbol{\mathcal{Y}} \ominus_{gH} \widehat{\mathbf{Y}}^* \subseteq \mathcal{T}^s_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*)$. Since $\mathcal{T}^s_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*)$ is a cone, $\text{cone}(\widehat{\mathbf{Y}} \ominus_{gH} \widehat{\mathbf{Y}}^*) \subseteq \mathcal{T}^s_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*)$. Using Theorem 2.6, we can show that $\text{cl}(\text{cone}(\boldsymbol{\mathcal{Y}} \ominus_{gH} \widehat{\mathbf{Y}}^*) \subseteq \mathcal{T}^s_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*)$. Thus, $\mathcal{T}^s_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*) = \text{cl}(\text{cone}(\boldsymbol{\mathcal{Y}} \ominus_{gH} \widehat{\mathbf{Y}}^*))$. $\square$

Note 2.3 *The rule of intersection* $\mathcal{T}^s_{\boldsymbol{\mathcal{Y}}_1}(\widehat{\boldsymbol{Y}}^*) \bigcap \mathcal{T}^s_{\boldsymbol{\mathcal{Y}}_2}(\widehat{\boldsymbol{Y}}^*) = \mathcal{T}^s_{\boldsymbol{\mathcal{Y}}_1 \cap \boldsymbol{\mathcal{Y}}_2}(\widehat{\boldsymbol{Y}}^*)$ *does not hold for set of intervals. For example, assume the sets*

$$\boldsymbol{\mathcal{Y}}_1 = \left\{([x,x],[y_1,y_2]) \in I(\mathbb{R})^2 : [x^2,x^2] \preceq [y_1,y_2]\right\} \text{ and}$$
$$\boldsymbol{\mathcal{Y}}_2 = \left\{([x,x],[z_1,z_2]) \in I(\mathbb{R})^2 : [z_1,z_2] \preceq [-x^2,-x^2]\right\}.$$

The set of intervals $\boldsymbol{\mathcal{Y}}_1$ *and* $\boldsymbol{\mathcal{Y}}_2$ *are shown in Figure 3(a) (green colour) and Figure 3(b) (cyan colour), respectively. Consider* $([x,x],\ [y_1,y_2]) = ([x,x],[1,k_1] \odot x^2) \in \boldsymbol{\mathcal{Y}}_1,\ k_1 \geq 1$ *and a sequence* $\{\widehat{\boldsymbol{Y}}^{(1)}_n\} = \left\{\left[\frac{1}{n},\frac{1}{n}\right],[1,k_1] \odot \frac{1}{n^2}\right\} \in \mathcal{Y}_1$ *so that* $\{\widehat{\boldsymbol{Y}}^{(1)}_n\} \to (\boldsymbol{0},\boldsymbol{0})$. *The tangent to* $\boldsymbol{\mathcal{Y}}_1$ *at* $\widehat{\boldsymbol{Y}}^* = (\boldsymbol{0},\boldsymbol{0})$ *is*

$$\lim_{n\to\infty} \frac{1}{\|\widehat{\boldsymbol{Y}}^{(1)}_n \ominus_{gH} \widehat{\boldsymbol{Y}}^*\|} \odot (\widehat{\boldsymbol{Y}}^{(1)}_n \ominus_{gH} \widehat{\boldsymbol{Y}}^*) = \lim_{n\to\infty} \frac{n^2}{n+k_1} \odot \left(\left[\frac{1}{n},\frac{1}{n}\right],[1,k_1] \odot \frac{1}{n^2}\right) = (\boldsymbol{1},\boldsymbol{0}).$$

Now, select $\{\widehat{\boldsymbol{Z}}^{(1)}_n\} = \left\{\left[-\frac{1}{n},-\frac{1}{n}\right],[1,k_1] \odot \frac{1}{n^2}\right\} \in \boldsymbol{\mathcal{Y}}_1,\ k_1 \geq 1$, *then the tangent to* $\boldsymbol{\mathcal{Y}}_1$ *at* $\widehat{\boldsymbol{Y}}^* = (\boldsymbol{0},\boldsymbol{0})$ *is* $(-\boldsymbol{1},\boldsymbol{0})$.
Because $\mathcal{T}^s_{\boldsymbol{\mathcal{Y}}_1}(\widehat{\boldsymbol{Y}}^*)$ *is a cone, then* $d \odot (\boldsymbol{1},\boldsymbol{0}) \in \mathcal{T}^s_{\boldsymbol{\mathcal{Y}}_1}(\widehat{\boldsymbol{Y}}^*)$ *for any* $0 \neq d \in \mathbb{R}$. *From Figure 2.3, we observe that the directions* d' *(red arrow) and* d'' *(purple arrow) are the two possible tangent directions of* $\boldsymbol{\mathcal{Y}}_1$ *at* $(\boldsymbol{0},\boldsymbol{0})$.
Similarly, consider $([x,x],\ [z_1,z_2]) = ([x,x],[-k_2,-1] \odot x^2) \in \boldsymbol{\mathcal{Y}}_2,\ k_2 \geq 1$ *and a sequence* $\{\widehat{\boldsymbol{Y}}^{(2)}_n\} = \left\{\left[\frac{1}{n},\frac{1}{n}\right],[-k_2,-1] \odot \frac{1}{n^2}\right\},\ k_2 \geq 1$. *Then, the tangent to* $\boldsymbol{\mathcal{Y}}_2$ *at* $\widehat{\boldsymbol{Y}}^* = (\boldsymbol{0},\boldsymbol{0})$ *is*

$$\lim_{n\to\infty} \frac{1}{\|\widehat{\boldsymbol{Y}}^{(2)}_n \ominus_{gH} \widehat{\boldsymbol{Y}}^*\|} \odot (\widehat{\boldsymbol{Y}}^{(2)}_n \ominus_{gH} \widehat{\boldsymbol{Y}}^*) = \lim_{n\to\infty} \frac{n^2}{n+k_2} \odot \left(\left[\frac{1}{n},\frac{1}{n}\right],[-k_2,-1] \odot \frac{1}{n^2}\right) = (\boldsymbol{1},\boldsymbol{0}).$$

Here, if we select $\{\widehat{\boldsymbol{Z}}^{(2)}_n\} = \left\{\left[-\frac{1}{n},-\frac{1}{n}\right],[-k_2,-1] \odot \frac{1}{n^2}\right\} \in \boldsymbol{\mathcal{Y}}_2,\ k_2 \geq 1$, *the tangent to* $\boldsymbol{\mathcal{Y}}_2$ *at* $\widehat{\boldsymbol{Y}}^* = (\boldsymbol{0},\boldsymbol{0})$ *is* $(-\boldsymbol{1},\boldsymbol{0})$. *Because* $\mathcal{T}^s_{\boldsymbol{\mathcal{Y}}_2}(\widehat{\boldsymbol{Y}}^*)$ *is a cone, then* $d \odot (\boldsymbol{1},\boldsymbol{0}) \in \mathcal{T}^s_{\boldsymbol{\mathcal{Y}}_2}(\widehat{\boldsymbol{Y}}^*)$ *for any* $0 \neq d \in \mathbb{R}$. *In Figure 4.3, we have the directions* d' *(red arrow) and* d'' *(purple* *arrow) as the two possible tangent directions of* $\boldsymbol{\mathcal{Y}}_2$ *at* $(\boldsymbol{0},\boldsymbol{0})$. *Therefore, we can say that* $\mathcal{T}^s_{\boldsymbol{\mathcal{Y}}_1}(\boldsymbol{0}) \bigcap \mathcal{T}^s_{\boldsymbol{\mathcal{Y}}_2}(\boldsymbol{0}) = I(\mathbb{R}) \times \{\boldsymbol{0}\}$. *Here, see that* $\boldsymbol{\mathcal{Y}}_1 \bigcap \boldsymbol{\mathcal{Y}}_2 = \{(\boldsymbol{0},\boldsymbol{0})\}$ *and we know that infinitely many tangents can pass through a single point. Thus,* $\mathcal{T}^s_{\boldsymbol{\mathcal{Y}}_1 \bigcap \boldsymbol{\mathcal{Y}}_2}((\boldsymbol{0},\boldsymbol{0})) = I(\mathbb{R})^2$.

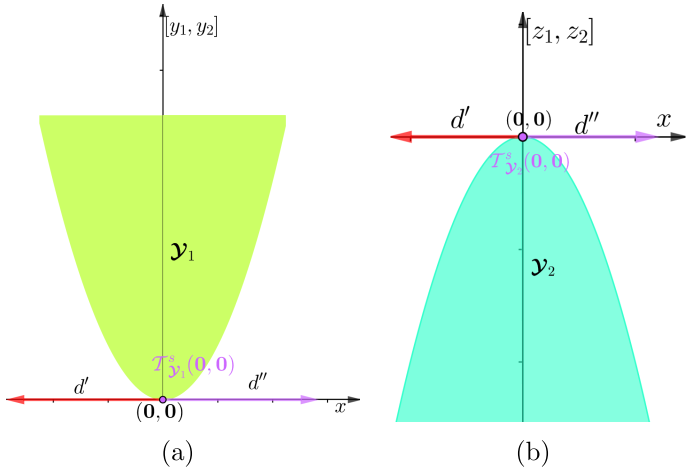


Figure 2.3: Illustration of possible tangent directions of $\boldsymbol{\mathcal{Y}}_1$ and $\boldsymbol{\mathcal{Y}}_2$ at $(\mathbf{0},\mathbf{0})$ in Note 2.3

2.5.1 Polarity between tangent cone and normal cone

Definition 2.4 (Polar cone in $I(\mathbb{R})^n$). *Let's take a set* $\boldsymbol{\mathcal{Y}} \subset I(\mathbb{R})^n$. *Then, the polar cone of* $\boldsymbol{\mathcal{Y}}$ *is given by*

$$\boldsymbol{\mathcal{Y}}^\circ = \{\widehat{\boldsymbol{Y}}^* \in I(\mathbb{R})^n : \ \widehat{\boldsymbol{Y}}^\top \odot_c \widehat{\boldsymbol{Y}}^* \preceq \boldsymbol{0} \text{ for all } \widehat{\boldsymbol{Y}} \in \boldsymbol{\mathcal{Y}}\}.$$

Note 2.4 *Let's take* $\boldsymbol{\mathcal{Y}}$ *as a cone of intervals, then the support function of* $\boldsymbol{\mathcal{Y}}$ *at* $\widehat{\boldsymbol{Y}}^* \in I(\mathbb{R})^n$ *is given as,*

$$\boldsymbol{\Psi}^*_{\boldsymbol{\mathcal{Y}}}(\widehat{\boldsymbol{Y}}^*) = \begin{cases} \sup\limits_{\widehat{\boldsymbol{Y}} \in \boldsymbol{\mathcal{Y}}} \widehat{\boldsymbol{Y}}^\top \odot_c \widehat{\boldsymbol{Y}}^* = \boldsymbol{0}, & \text{if } \widehat{\boldsymbol{Y}}^\top \odot_c \widehat{\boldsymbol{Y}}^* \preceq \boldsymbol{0},\ \forall\ \widehat{\boldsymbol{Y}} \in \boldsymbol{\mathcal{Y}} \\ \boldsymbol{\infty}, & \text{otherwise} \end{cases}$$

where, $\boldsymbol{\Psi}^*_{\boldsymbol{\mathcal{Y}}}$ *is the indicator function* $\delta_{\boldsymbol{\mathcal{Y}}^\circ}$ *on* $I(\mathbb{R})^n$ *of the polar cone of* $\boldsymbol{\mathcal{Y}}$ *as in Definition 2.4.*

Lemma 2.1 *Let a convex set* $\boldsymbol{\mathcal{Y}} \subseteq I(\mathbb{R})^n$. *Then, at any* $\widehat{\boldsymbol{Y}}^* \in \boldsymbol{\mathcal{Y}}$, $\mathcal{T}^s_{\boldsymbol{\mathcal{Y}}}(\widehat{\boldsymbol{Y}}^*)$ *and* $\mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\widehat{\boldsymbol{Y}}^*)$ *are polar to each other, i.e.,*

$$\mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\widehat{\boldsymbol{Y}}^*) = \mathcal{T}^s_{\boldsymbol{\mathcal{Y}}}(\widehat{\boldsymbol{Y}}^*)^\circ \text{ and } \mathcal{T}^s_{\boldsymbol{\mathcal{Y}}}(\widehat{\boldsymbol{Y}}^*) = \mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\widehat{\boldsymbol{Y}}^*)^\circ.$$

Proof: Consider $\widehat{\mathbf{G}} \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*)$. Then, we have

$$\widehat{\mathbf{G}}^\top \odot_c (\widehat{\mathbf{Y}} \ominus_{gH} \widehat{\mathbf{Y}}^*) \preceq \mathbf{0} \text{ for all } \widehat{\mathbf{Y}} \in \boldsymbol{\mathcal{Y}}.$$

Because $\widehat{\mathbf{Y}} \in \boldsymbol{\mathcal{Y}},\ \widehat{\mathbf{Y}} \ominus_{gH} \widehat{\mathbf{Y}}^* \in \boldsymbol{\mathcal{Y}} \ominus_{gH} \widehat{\mathbf{Y}}^*$, we have from Theorem 2.8 that

$$\widehat{\mathbf{G}} \in \text{cl}(\text{cone}(\boldsymbol{\mathcal{Y}} \ominus_{gH} \widehat{\mathbf{Y}}^*))^\circ \implies \widehat{\mathbf{G}} \in \mathcal{T}^s_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*)^\circ.$$

Hence, $\mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*) \subset \mathcal{T}^s_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*)^\circ$.
Conversely, assume that $\widehat{\mathbf{G}} \in \mathcal{T}^s_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*)^\circ$. Hence,

$$\widehat{\mathbf{G}}^\top \odot_c \widehat{\mathbf{D}} \preceq \mathbf{0} \text{ for every } \widehat{\mathbf{D}} \in \mathcal{T}^s_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*).$$

Because $\boldsymbol{\mathcal{Y}} \ominus_{gH} \widehat{\mathbf{Y}}^* \subset \mathcal{T}^s_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*)$,

$$\widehat{\mathbf{G}}^\top \odot_c (\widehat{\mathbf{Y}} \ominus_{gH} \widehat{\mathbf{Y}}^*) \preceq \mathbf{0} \text{ for all } \widehat{\mathbf{Y}} \in \boldsymbol{\mathcal{Y}}.$$

This means that $\widehat{\mathbf{G}} \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*)$. Hence, $\mathcal{T}^s_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*)^\circ \subset \mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*)$. Thus, $\mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*) = \mathcal{T}^s_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*)^\circ$. □

Lemma 2.2 *Assume $\boldsymbol{\mathcal{Y}}_1, \boldsymbol{\mathcal{Y}}_2$ are two cones in $I(\mathbb{R})^n$. Then, $(\boldsymbol{\mathcal{Y}}_1 \star \boldsymbol{\mathcal{Y}}_2)^\circ = \boldsymbol{\mathcal{Y}}_1^\circ \star \boldsymbol{\mathcal{Y}}_2^\circ$, where $\boldsymbol{\mathcal{Y}}_1 \star \boldsymbol{\mathcal{Y}}_2 = \Big\{(\widehat{\boldsymbol{Y}}_1, \widehat{\boldsymbol{Y}}_2) \in I(\mathbb{R})^{n_1} \times I(\mathbb{R})^{n_2} : (\widehat{\boldsymbol{Y}}_1, \widehat{\boldsymbol{Y}}_2)^\top \odot_c (\widehat{\boldsymbol{Z}}_1, \widehat{\boldsymbol{Z}}_2) = \widehat{\boldsymbol{Y}}_1^\top \odot_c \widehat{\boldsymbol{Z}}_1 \oplus \widehat{\boldsymbol{Y}}_2^\top \odot_c \widehat{\boldsymbol{Z}}_2 \ \ \forall \ \ (\widehat{\boldsymbol{Z}}_1, \widehat{\boldsymbol{Z}}_2) \in I(\mathbb{R})^{n_1} \times I(\mathbb{R})^{n_2}\Big\}$.*

Proof: Let $\widehat{\mathbf{G}} = (\widehat{\mathbf{G}}_1, \widehat{\mathbf{G}}_2) \in (\boldsymbol{\mathcal{Y}}_1 \star \boldsymbol{\mathcal{Y}}_2)^\circ$. Then, we have

$$(\widehat{\mathbf{G}}_1^\top, \widehat{\mathbf{G}}_2^\top) \odot_c (\widehat{\mathbf{Y}}_1, \widehat{\mathbf{Y}}_2) \preceq \mathbf{0} \text{ for all } \widehat{\mathbf{Y}} = (\widehat{\mathbf{Y}}_1, \widehat{\mathbf{Y}}_2) \in \boldsymbol{\mathcal{Y}}_1 \star \boldsymbol{\mathcal{Y}}_2. \tag{2.10}$$

Thus, for any $\widehat{\mathbf{Y}} \in \boldsymbol{\mathcal{Y}}_1 \star \boldsymbol{\mathcal{Y}}_2$, we get

$$\widehat{\mathbf{G}}_1^\top \odot_c \widehat{\mathbf{Y}}_1 \oplus \widehat{\mathbf{G}}_2^\top \odot_c \widehat{\mathbf{Y}}_2 \preceq \mathbf{0} \text{ for all } \widehat{\mathbf{Y}}_1 \in \boldsymbol{\mathcal{Y}}_1, \widehat{\mathbf{Y}}_2 \in \boldsymbol{\mathcal{Y}}_2.$$

As $\boldsymbol{\mathcal{Y}}_1$ and $\boldsymbol{\mathcal{Y}}_2$ are cones, $\mathbf{0} \in \boldsymbol{\mathcal{Y}}_i,\ i = 1, 2$. Specifically, for $\widehat{\mathbf{Y}}_2 = \mathbf{0}$,

$$\widehat{\mathbf{G}}_1^\top \odot_c \widehat{\mathbf{Y}}_1 \preceq \mathbf{0} \implies \widehat{\mathbf{G}}_1 \in \boldsymbol{\mathcal{Y}}_1^\circ.$$

Likewise, $\widehat{\mathbf{G}}_2 \in \boldsymbol{\mathcal{Y}}_2^\circ$. Hence, $(\widehat{\mathbf{G}}_1, \widehat{\mathbf{G}}_2) \in \boldsymbol{\mathcal{Y}}_1^\circ \star \boldsymbol{\mathcal{Y}}_2^\circ$. Conversely, let $\widehat{\mathbf{G}}_i \in \boldsymbol{\mathcal{Y}}_i^\circ, i = 1, 2$. Then, we have

$$\widehat{\mathbf{G}}_i^\top \odot_c \widehat{\mathbf{Y}}_i \preceq \mathbf{0} \text{ for all } \widehat{\mathbf{Y}}_i \in \boldsymbol{\mathcal{Y}}_i, i = 1, 2$$

$$\begin{aligned}
&\implies \quad \widehat{\mathbf{G}}_1^\top \odot_c \widehat{\mathbf{Y}}_1 \oplus \widehat{\mathbf{G}}_2^\top \odot_c \widehat{\mathbf{Y}}_2 \preceq \mathbf{0} \text{ for all } \widehat{\mathbf{Y}}_1 \in \boldsymbol{\mathcal{Y}}_1 \text{ and } \widehat{\mathbf{Y}}_2 \in \boldsymbol{\mathcal{Y}}_2 \\
&\implies \quad (\widehat{\mathbf{G}}_1^\top, \widehat{\mathbf{G}}_2^\top) \odot_c (\widehat{\mathbf{Y}}_1, \widehat{\mathbf{Y}}_2) \preceq \mathbf{0} \text{ for all } \widehat{\mathbf{Y}} \in \boldsymbol{\mathcal{Y}}_1 \star \boldsymbol{\mathcal{Y}}_2.
\end{aligned}$$

Hence, $(\widehat{\mathbf{G}}_1, \widehat{\mathbf{G}}_2) \in (\boldsymbol{\mathcal{Y}}_1 \star \boldsymbol{\mathcal{Y}}_2)^\circ$. Thus, $(\boldsymbol{\mathcal{Y}}_1 \star \boldsymbol{\mathcal{Y}}_2)^\circ = \boldsymbol{\mathcal{Y}}_1^\circ \star \boldsymbol{\mathcal{Y}}_2^\circ$. □

Theorem 2.9 *Consider $\boldsymbol{\mathcal{Y}}_1, \boldsymbol{\mathcal{Y}}_2$ to be two convex sets of $I(\mathbb{R})^n$. Let $\widehat{\boldsymbol{Y}}_1' \in \boldsymbol{\mathcal{Y}}_1$ and $\widehat{\boldsymbol{Y}}_2' \in \boldsymbol{\mathcal{Y}}_2$. Then, $\mathcal{T}_{\boldsymbol{\mathcal{Y}}_1 \star \boldsymbol{\mathcal{Y}}_2}(\widehat{\boldsymbol{Y}}_1', \widehat{\boldsymbol{Y}}_2') = \mathcal{T}_{\boldsymbol{\mathcal{Y}}_1}(\widehat{\boldsymbol{Y}}_1') \star \mathcal{T}_{\boldsymbol{\mathcal{Y}}_2}(\widehat{\boldsymbol{Y}}_2')$, where $\boldsymbol{\mathcal{Y}}_1 \star \boldsymbol{\mathcal{Y}}_2 = \Big\{(\widehat{\boldsymbol{Y}}_1, \widehat{\boldsymbol{Y}}_2) \in I(\mathbb{R})^{n_1} \times I(\mathbb{R})^{n_2} : (\widehat{\boldsymbol{Y}}_1, \widehat{\boldsymbol{Y}}_2)^\top \odot_c (\widehat{\boldsymbol{Z}}_1, \widehat{\boldsymbol{Z}}_2) = \widehat{\boldsymbol{Y}}_1^\top \odot_c \widehat{\boldsymbol{Z}}_1 \oplus \widehat{\boldsymbol{Y}}_2^\top \odot_c \widehat{\boldsymbol{Z}}_2 \;\; \forall \;\; (\widehat{\boldsymbol{Z}}_1, \widehat{\boldsymbol{Z}}_2) \in I(\mathbb{R})^{n_1} \times I(\mathbb{R})^{n_2}\Big\}$.*

Proof: Using Theorem 2.4, we get,

$$\mathcal{N}_{\boldsymbol{\mathcal{Y}}_1 \star \boldsymbol{\mathcal{Y}}_2}(\widehat{\mathbf{Y}}_1', \widehat{\mathbf{Y}}_2') = \mathcal{N}_{\boldsymbol{\mathcal{Y}}_1}(\widehat{\mathbf{Y}}_1') \star \mathcal{N}_{\boldsymbol{\mathcal{Y}}_2}(\widehat{\mathbf{Y}}_2').$$

Hence, using Lemma 2.2 for the above relation, we have the needed result. □

2.5.2 Tangent cone characterization with the help of continuity of distance function in $I(\mathbb{R})^n$

Definition 2.5 (Distance function on $I(\mathbb{R})^n$). *Consider $\boldsymbol{\mathcal{Y}}$ to be a nonempty subset of $I(\mathbb{R})^n$. The distance function denoted by $d_{\boldsymbol{\mathcal{Y}}} : I(\mathbb{R})^n \to \mathbb{R}_+$, at a point $\widehat{\boldsymbol{Y}}'$ is given by*

$$d_{\boldsymbol{\mathcal{Y}}}(\widehat{\boldsymbol{Y}}') = \inf_{\widehat{\mathbf{Y}} \in \boldsymbol{\mathcal{Y}}} \| \widehat{\boldsymbol{Y}} \ominus_{gH} \widehat{\boldsymbol{Y}}' \|_{I(\mathbb{R})^n}.$$

Definition 2.6 (gH-Lipschitz continuous on $I(\mathbb{R})^n$). *Consider an IVF $\boldsymbol{T}$: $\boldsymbol{\mathcal{Y}} \to I(\mathbb{R})$ on a nonempty subset $\boldsymbol{\mathcal{Y}}$ of $I(\mathbb{R})^n$. This IVF is called gH-Lipschitz continuous on $\boldsymbol{\mathcal{Y}}$ if there exists $K > 0$ satisfying*

$$\| \boldsymbol{T}(\widehat{\boldsymbol{Y}}) \ominus_{gH} \boldsymbol{T}(\widehat{\boldsymbol{Z}}) \|_{I(\mathbb{R})} \leq K \| \widehat{\boldsymbol{Y}} \ominus_{gH} \widehat{\boldsymbol{Z}} \|_{I(\mathbb{R})^n} \text{ for all } \widehat{\boldsymbol{Y}}, \widehat{\boldsymbol{Z}} \in \boldsymbol{\mathcal{Y}},$$

where K is referred to as Lipschitz constant.

Lemma 2.3 *If $\boldsymbol{\mathcal{Y}}$ is a nonempty subset of $I(\mathbb{R})^n$, then, $d_{\boldsymbol{\mathcal{Y}}}(\cdot)$ is gH-Lipschitz continuous.*

Proof: Taking $\widehat{\mathbf{X}}_1 \in I(\mathbb{R})^n$,

$$\begin{aligned}
d_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{X}}_1) = \inf_{\widehat{\mathbf{Y}} \in \boldsymbol{\mathcal{Y}}} \|\widehat{\mathbf{X}}_1 \ominus_{gH} \widehat{\mathbf{Y}}\| &\leq \inf_{\widehat{\mathbf{Y}} \in \boldsymbol{\mathcal{Y}}} \{\|\widehat{\mathbf{X}}_1 \ominus_{gH} \widehat{\mathbf{X}}_2\| + \|\widehat{\mathbf{X}}_2 \ominus_{gH} \widehat{\mathbf{Y}}\|\} \\
&\leq \|\widehat{\mathbf{X}}_1 \ominus_{gH} \widehat{\mathbf{X}}_2\| + d_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{X}}_2).
\end{aligned}$$

Likewise, $d_{\mathcal{Y}}(\widehat{\mathbf{X}}_2) = \inf_{\widehat{\mathbf{Y}} \in \mathcal{Y}} \|\widehat{\mathbf{X}}_2 \ominus_{gH} \widehat{\mathbf{Y}}\| \leq \|\widehat{\mathbf{X}}_2 \ominus_{gH} \widehat{\mathbf{X}}_1\| + d_{\mathcal{Y}}(\widehat{\mathbf{X}}_1)$.
Hence, $|d_{\mathcal{Y}}(\widehat{\mathbf{X}}_1) \ominus_{gH} d_{\mathcal{Y}}(\widehat{\mathbf{X}}_2)| \leq \|\widehat{\mathbf{X}}_1 \ominus_{gH} \widehat{\mathbf{X}}_2\|$. Therefore, $d_{\mathcal{Y}}(\cdot)$ is gH-Lipschitz continuous where the Lipschitz constant is 1. □

Theorem 2.10 *If $\mathcal{Y} \subset I(\mathbb{R})^n$, then, $\mathcal{T}^f_{\mathcal{Y}}(\widehat{\mathbf{Y}}^*) = \left\{\widehat{\boldsymbol{D}} \in I(\mathbb{R})^n : \liminf_{\alpha \to 0} \frac{d_{\mathcal{Y}}(\widehat{\mathbf{Y}}^* \oplus \alpha \odot \widehat{\boldsymbol{D}})}{\alpha} = 0\right\}$.*

Proof: Consider $\widehat{\mathbf{D}} \in \mathcal{T}^f_{\mathcal{Y}}(\widehat{\mathbf{Y}}^*)$. Then, there exists $\widehat{\mathbf{D}}_k \to \widehat{\mathbf{D}}$ and $\alpha \to 0$ satisfying $\widehat{\mathbf{Y}}^* \oplus \alpha \odot \widehat{\mathbf{D}}_k \in \mathcal{Y}$.
Thus, $d_{\mathcal{Y}}(\widehat{\mathbf{Y}}^* \oplus \alpha \odot \widehat{\mathbf{D}}_k) = 0$. Then, using Lemma 2.3, we have

$$d_{\mathcal{Y}}(\widehat{\mathbf{Y}}^* \oplus \alpha \odot \widehat{\mathbf{D}}_k) \to d_{\mathcal{Y}}(\widehat{\mathbf{Y}}^* \oplus \alpha \odot \widehat{\mathbf{D}}) \text{ as } k \to \infty.$$

This means that $d_{\mathcal{Y}}(\widehat{\mathbf{Y}}^* \oplus \alpha \odot \widehat{\mathbf{D}}) = 0$. Thus, $\liminf_{\alpha \to 0} \frac{d_{\mathcal{Y}}(\widehat{\mathbf{Y}}^* \oplus \alpha \odot \widehat{\mathbf{D}})}{\alpha} = 0$.
Conversely, consider $\liminf_{\alpha \to 0} \frac{d_{\mathcal{Y}}(\widehat{\mathbf{Y}}^* \oplus \alpha \odot \widehat{\mathbf{D}})}{\alpha} = 0$. From Definition 2.5, for each $k \in \mathbb{N}$, there exists $\widehat{\mathbf{W}}_k \in \mathcal{Y}$ satisfying

$$\|\widehat{\mathbf{W}}_k \ominus_{gH} (\widehat{\mathbf{Y}}^* \oplus \alpha \odot \widehat{\mathbf{D}})\| \leq d_{\mathcal{Y}}(\widehat{\mathbf{Y}}^* \oplus \alpha \odot \widehat{\mathbf{D}}) + \frac{\alpha}{k},$$

which is true for each k. Taking $k \to \infty$, we have

$$\frac{\|\widehat{\mathbf{W}}_k \ominus_{gH} (\widehat{\mathbf{Y}}^* \oplus \alpha \odot \widehat{\mathbf{D}})\|}{\alpha} \leq \frac{d_{\mathcal{Y}}(\widehat{\mathbf{Y}}^* \oplus \alpha \odot \widehat{\mathbf{D}})}{\alpha}.$$

Suppose $\widehat{\mathbf{W}}_k = \widehat{\mathbf{Y}}^* \oplus \alpha \odot \widehat{\mathbf{D}}_k \in \mathcal{Y}$. Then, we get

$$\begin{aligned}
\|\widehat{\mathbf{W}}_k \ominus_{gH} (\widehat{\mathbf{Y}}^* \oplus \alpha \odot \widehat{\mathbf{D}})\| &= \|(\widehat{\mathbf{Y}}^* \oplus \alpha \odot \widehat{\mathbf{D}}_k) \ominus_{gH} (\widehat{\mathbf{Y}}^* \oplus \alpha \odot \widehat{\mathbf{D}})\| \\
&= \|\alpha \odot \widehat{\mathbf{D}}_k \ominus_{gH} \alpha \odot \widehat{\mathbf{D}}\|, \text{by } ((\text{i})) \text{ of Lemma 1.1} \\
&= \alpha\|\widehat{\mathbf{D}}_k \ominus_{gH} \widehat{\mathbf{D}}\| \text{ as } \alpha \in \mathbb{R}_+.
\end{aligned}$$

Thus, for all $\alpha \to 0$, we have

$$\begin{aligned}
& \|\widehat{\mathbf{D}}_k \ominus_{gH} \widehat{\mathbf{D}}\| = \frac{\|\widehat{\mathbf{W}}_k \ominus_{gH} (\widehat{\mathbf{Y}}^* \oplus \alpha \odot \widehat{\mathbf{D}})\|}{\alpha} \leq \frac{d_{\mathcal{Y}}(\widehat{\mathbf{Y}}^* \oplus \alpha \odot \widehat{\mathbf{D}})}{\alpha} \\
\implies \quad & \|\widehat{\mathbf{D}}_k \ominus_{gH} \widehat{\mathbf{D}}\| \leq \liminf_{\alpha \to 0} \frac{d_{\mathcal{Y}}(\widehat{\mathbf{Y}}^* \oplus \alpha \odot \widehat{\mathbf{D}})}{\alpha}.
\end{aligned}$$

Therefore, there exists $\widehat{\mathbf{D}}_k \to \widehat{\mathbf{D}}$ for each k and $\alpha \to 0$ satisfying, for any $k \in \mathbb{N}$, $\widehat{\mathbf{W}}_k = \widehat{\mathbf{Y}}^* \oplus \alpha \odot \widehat{\mathbf{D}}_k \in \mathcal{Y}$. Thus, $\widehat{\mathbf{D}} \in \mathcal{T}^f_{\mathcal{Y}}(\widehat{\mathbf{Y}}^*)$. □

2.6 Efficiency conditions employing normal cone

In this section, we introduce the concepts of the gH-partial derivative, gH-gradient, and gH-directional derivative for IVF, with the domain being a subset of $I(\mathbb{R})^n$. Following this, the concept of gH-differentiation for interval variables is presented using the L-IVF (Definition 2.10). We then discuss the necessary efficiency conditions with the help of the defined calculus by proposing the Lagrangian for the IVFs and examining the roles of the normal cone and tangent cone in proposing a fresh Lagrange multiplier optimization technique.

Definition 2.7 (gH-partial derivative of IVF where the domain is subset of $I(\mathbb{R})^n$). *Consider* $\boldsymbol{\mathcal{Y}} \subseteq I(\mathbb{R})^n, n \geq 2$ *and* $\widehat{\boldsymbol{Y}} = (\boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots, \boldsymbol{Y}_{i-1}, \boldsymbol{Y}_i, \ldots, \boldsymbol{Y}_n)^\top \in \boldsymbol{\mathcal{Y}}$. *Having* $\boldsymbol{T} : \boldsymbol{\mathcal{Y}} \to I(\mathbb{R})$ *as an IVF, we take an IVF* $\boldsymbol{\Phi}_i(\boldsymbol{Y}_i) = \boldsymbol{T}(\boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots, \boldsymbol{Y}_{i-1}, \boldsymbol{Y}_i, \ldots, \boldsymbol{Y}_n)$ *on* $I(\mathbb{R})$. *If* $\boldsymbol{Y}_i$ *is displaced by small interval* $\boldsymbol{H}_i$, *at* $\widehat{\boldsymbol{Y}} \in \boldsymbol{\mathcal{Y}}$, *the i-th partial derivative of* $\boldsymbol{T}$, *written as* $\boldsymbol{T}_{\boldsymbol{Y}_i}(\widehat{\boldsymbol{Y}})$, *is given as*

$$\lim_{\boldsymbol{H}_i \to \boldsymbol{0}} (\boldsymbol{1} \oslash \boldsymbol{H}_i) \odot (\boldsymbol{\Phi}_i(\boldsymbol{Y}_i \oplus \boldsymbol{H}_i) \ominus_{gH} \boldsymbol{\Phi}_i(\boldsymbol{Y}_i)),$$

given that the limit exists finitely, where $0 \notin \boldsymbol{H}_i = [h_i, h_i], i = 1, 2, \ldots, n$.

Example 2.6.1 *Consider* $\boldsymbol{\mathcal{Y}} \subseteq I(\mathbb{R})^2$ *and an IVF* $\boldsymbol{T} : \boldsymbol{\mathcal{Y}} \to I(\mathbb{R})$

$$\boldsymbol{T}([\underline{y}_1, \overline{y}_1], [\underline{y}_2, \overline{y}_2]) = [\underline{y}_1\overline{y}_1 + \underline{y}_2\overline{y}_2, \underline{y}_1\overline{y}_1 + \underline{y}_2\overline{y}_2],$$

for all $\widehat{\boldsymbol{Y}} = (\boldsymbol{Y}_1, \boldsymbol{Y}_2) = ([\underline{y}_1, \overline{y}_1], [\underline{y}_2, \overline{y}_2]) \in I(\mathbb{R})^2$.

The gH-partial derivative of $\boldsymbol{T}$ *at point* $\widehat{\boldsymbol{Y}} = (\boldsymbol{Y}_1, \boldsymbol{Y}_2)$ *with respect to* $\boldsymbol{Y}_1$ *is given by*

$$\begin{aligned}
&\lim_{\boldsymbol{H}_1 \to 0} (\boldsymbol{1} \oslash \boldsymbol{H}_1) \odot [\boldsymbol{\Phi}_1(\boldsymbol{Y}_1 \oplus \boldsymbol{H}_1) \ominus_{gH} \boldsymbol{\Phi}_1(\boldsymbol{Y}_1)] \\
&= \lim_{h_1 \to 0} \frac{1}{h_1} \odot [\boldsymbol{T}(([\underline{y}_1 + h_1, \overline{y}_1 + h_1]), [\underline{y}_2, \overline{y}_2]) \ominus_{gH} \boldsymbol{T}([\underline{y}_1, \overline{y}_1], [\underline{y}_2, \overline{y}_2])] \\
&= \lim_{h_1 \to 0} \frac{1}{h_1} \odot \{[(\underline{y}_1 + h_1)(\overline{y}_1 + h_1) + \underline{y}_2\overline{y}_2, (\underline{y}_1 + h_1)(\overline{y}_1 + h_1) + \underline{y}_2\overline{y}_2] \\
&\qquad\qquad \ominus_{gH} [\underline{y}_1\overline{y}_1 + \underline{y}_2\overline{y}_2, \underline{y}_1\overline{y}_1 + \underline{y}_2\overline{y}_2]\} \\
&= \lim_{h_1 \to 0} \frac{1}{h_1} \odot \{[\underline{y}_1 h_1 + \overline{y}_1 h_1 + h_1^2, \underline{y}_1 h_1 + \overline{y}_1 h_1 + h_1^2]\} = [\underline{y}_1 + \overline{y}_1, \underline{y}_1 + \overline{y}_1].
\end{aligned}$$

Therefore, $\boldsymbol{T}_{\boldsymbol{Y}_1}([\underline{y}_1, \overline{y}_1], [\underline{y}_2, \overline{y}_2]) = [\underline{y}_1 + \overline{y}_1, \underline{y}_1 + \overline{y}_1]$. *Similarly,* $\boldsymbol{T}_{\boldsymbol{Y}_2}([\underline{y}_1, \overline{y}_1], [\underline{y}_2, \overline{y}_2]) = [\underline{y}_2 + \overline{y}_2, \underline{y}_2 + \overline{y}_2]$.

Note 2.5 *Considering the above mentioned* $\boldsymbol{H}_i$ *to be non-degenerate interval, say* $0 \notin \boldsymbol{H}_i = [\underline{h}_i, \overline{h}_i]$, *for* $\boldsymbol{H}_i \to 0$, *using the Example 2.6.1, we can show*

$$\boldsymbol{T}_{\boldsymbol{Y}_1}([\underline{y}_1, \overline{y}_1], [\underline{y}_2, \overline{y}_2]) = \lim_{\substack{\underline{h}_1 \to 0 \\ \overline{h}_1 \to 0}} \left[\min\left\{ \underline{y}_1 \frac{\overline{h}_1}{\underline{h}_1} + \overline{y}_1 + \overline{h}_1, \underline{y}_1 + \overline{y}_1 \frac{\underline{h}_1}{\overline{h}_1} + \underline{h}_1 \right\}, \right.$$
$$\left. \max\left\{ \underline{y}_1 \frac{\overline{h}_1}{\underline{h}_1} + \overline{y}_1 + \overline{h}_1, \underline{y}_1 + \overline{y}_1 \frac{\underline{h}_1}{\overline{h}_1} + \underline{h}_1 \right\} \right]$$

whose form is indefinite. For this scenario, there does not exist gH-partial derivative of $\boldsymbol{T}$.

Definition 2.8 (gH-gradient of IVF where domain is a subset of $I(\mathbb{R})^n$). *Consider* $\boldsymbol{\mathcal{Y}} \subseteq I(\mathbb{R})^n, n \geq 2$. *The gH-gradient of an IVF* $\boldsymbol{T} : \boldsymbol{\mathcal{Y}} \to I(\mathbb{R})$ *at a point* $\widehat{\boldsymbol{Y}} \in \boldsymbol{\mathcal{Y}}$, *is given as*

$$\nabla \boldsymbol{T}(\widehat{\boldsymbol{Y}}) = (\boldsymbol{T}_{\boldsymbol{Y}_1}(\widehat{\boldsymbol{Y}}), \boldsymbol{T}_{\boldsymbol{Y}_2}(\widehat{\boldsymbol{Y}}), \ldots, \boldsymbol{T}_{\boldsymbol{Y}_n}(\widehat{\boldsymbol{Y}}))^\top.$$

Definition 2.9 (gH-directional derivative of IVF where domain is a subset of $I(\mathbb{R})^n$). *Consider* $\boldsymbol{\mathcal{Y}} \subseteq I(\mathbb{R})^n, n \geq 2$. *Consider* $\boldsymbol{T} : \boldsymbol{\mathcal{Y}} \to I(\mathbb{R})$ *to be an IVF and let* $\widehat{\boldsymbol{\Gamma}} = (\boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2, \ldots, \boldsymbol{\Gamma}_n)^\top \in I(\mathbb{R})^n$ *be a direction at* $\widehat{\boldsymbol{Y}} \in \boldsymbol{\mathcal{Y}}$. *The gH-directional derivative of* $\boldsymbol{T}$ *in the direction* $\widehat{\boldsymbol{\Gamma}}$ *is defined by*

$$\boldsymbol{T}_{\mathscr{D}}(\widehat{\boldsymbol{Y}}; \widehat{\boldsymbol{\Gamma}}) = \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\boldsymbol{T}(\widehat{\boldsymbol{Y}} \oplus \lambda \odot \widehat{\boldsymbol{\Gamma}}) \ominus_{gH} \boldsymbol{T}(\widehat{\boldsymbol{Y}})),$$

given that the limit exists finitely.

Example 2.6.2 *Consider* $\boldsymbol{\mathcal{Y}} \subset I(\mathbb{R})^2$. *Let IVF* $\boldsymbol{T} : \boldsymbol{\mathcal{Y}} \to I(\mathbb{R})$ *be given as*

$$\boldsymbol{T}([\underline{y}_1, \overline{y}_1], [\underline{y}_2, \overline{y}_2]) = [\underline{y}_1 + \underline{y}_2 e^{\underline{y}_1}, \overline{y}_1 + \overline{y}_2 e^{\overline{y}_1}],$$

for all $\widehat{\boldsymbol{Y}} = ([\underline{y}_1, \overline{y}_1], [\underline{y}_2, \overline{y}_2]) \in I(\mathbb{R})^2$. *The gH-directional derivative of* $\boldsymbol{T}$ *at* $(\boldsymbol{0}, \boldsymbol{0})$ *in the direction* $(\boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2) = ([\underline{\gamma}_1, \overline{\gamma}_1], [\underline{\gamma}_2, \overline{\gamma}_2])$ *is then given by*

$$\begin{aligned}
\boldsymbol{T}_{\mathscr{D}}((\boldsymbol{0}, \boldsymbol{0}); (\boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2)) &= \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \{\boldsymbol{T}(\lambda \odot [\underline{\gamma}_1, \overline{\gamma}_1], \lambda \odot [\underline{\gamma}_2, \overline{\gamma}_2]) \ominus_{gH} \boldsymbol{T}(\boldsymbol{0}, \boldsymbol{0})\} \\
&= \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \{[\lambda \underline{\gamma}_1 + \lambda \underline{\gamma}_2 e^{\lambda \underline{\gamma}_1}, \lambda \overline{\gamma}_1 + \lambda \overline{\gamma}_2 e^{\lambda \overline{\gamma}_1}]\} \\
&= \lim_{\lambda \to 0+} [\underline{\gamma}_1 + \underline{\gamma}_2 e^{\lambda \underline{\gamma}_1}, \overline{\gamma}_1 + \overline{\gamma}_2 e^{\lambda \overline{\gamma}_1}] \\
&= [\underline{\gamma}_1 + \underline{\gamma}_2, \overline{\gamma}_1 + \overline{\gamma}_2].
\end{aligned}$$

Thus, $\boldsymbol{T}_{\mathscr{D}}((\boldsymbol{0}, \boldsymbol{0}); (\boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2)) = [\underline{\gamma}_1 + \underline{\gamma}_2, \overline{\gamma}_1 + \overline{\gamma}_2]$.

2.6.1 gH-differentiability of a function T: $I(\mathbb{R})^n \to I(\mathbb{R})$

Definition 2.10 (L-IVF where domain is a subset of $I(\mathbb{R})^n$). *Consider $\boldsymbol{\mathcal{Y}} \subseteq I(\mathbb{R})^n, n \geq 2$. The function $\boldsymbol{T} : \boldsymbol{\mathcal{Y}} \to I(\mathbb{R})$ is said to be L-IVF if $\boldsymbol{T}$ holds the following:*

(i) $\boldsymbol{T}(\alpha \odot \widehat{\boldsymbol{Y}}) = \alpha \odot \boldsymbol{T}(\widehat{\boldsymbol{Y}})$ *for all* $\widehat{\boldsymbol{Y}} \in \boldsymbol{\mathcal{Y}}$ *and for all* $\alpha \in \mathbb{R}$,

(ii) $\boldsymbol{T}(\widehat{\boldsymbol{Y}}) = \bigoplus_{i=1}^n \widehat{\boldsymbol{Y}}_i \odot \boldsymbol{T}(\boldsymbol{e}_i)$ *for all* $\widehat{\boldsymbol{Y}} = (\boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots, \boldsymbol{Y}_n)^\top \in \boldsymbol{\mathcal{Y}}$, *and* $\boldsymbol{e}_i = (\boldsymbol{0}, \boldsymbol{0}, \ldots, \boldsymbol{1}, \boldsymbol{0}, \ldots, \boldsymbol{0})^\top$, *i.e., i-th unit vector of* $I(\mathbb{R})^n, i = 1, 2, \ldots, n$, *and*

(iii) for all $\widehat{\boldsymbol{Y}}, \widehat{\boldsymbol{Z}} \in \boldsymbol{\mathcal{Y}}$, *either* $\boldsymbol{T}(\widehat{\boldsymbol{Y}}) \oplus \boldsymbol{T}(\widehat{\boldsymbol{Z}}) = \boldsymbol{T}(\widehat{\boldsymbol{Y}} \oplus \widehat{\boldsymbol{Z}})$ *or none of* $\boldsymbol{T}(\widehat{\boldsymbol{Y}}) \oplus \boldsymbol{T}(\widehat{\boldsymbol{Z}})$ *and* $\boldsymbol{T}(\widehat{\boldsymbol{Y}} \oplus \widehat{\boldsymbol{Z}})$ *is dominant than other,*

where '$\bigoplus_{i=1}^n$*' denotes consecutive sum of* n *intervals.*

Note 2.6 *We can easily see that for degenerate case, the Definition 2.10 becomes linear function for the real-valued functions.*

Definition 2.11 (gH-differentiability). *Consider $\boldsymbol{\mathcal{Y}}$ to be a nonempty subset of $I(\mathbb{R})^n, n \geq 2$. Then, an IVF $\boldsymbol{T} : \boldsymbol{\mathcal{Y}} \to I(\mathbb{R})$ is gH-differentiable at a point $\widehat{\boldsymbol{Y}} \in \boldsymbol{\mathcal{Y}}$ if there exists an L-IVF $\mathscr{L}_{\widehat{\boldsymbol{Y}}} : I(\mathbb{R})^n \to I(\mathbb{R})$, an error IVF $\boldsymbol{E}(\boldsymbol{T}(\widehat{\boldsymbol{Y}}); \widehat{\boldsymbol{\Gamma}})$ and $\delta > 0$ satisfying*

$$(\boldsymbol{T}(\widehat{\boldsymbol{Y}} \oplus \widehat{\boldsymbol{\Gamma}}) \ominus_{gH} \boldsymbol{T}(\widehat{\boldsymbol{Y}})) = \mathscr{L}_{\widehat{\boldsymbol{Y}}}(\widehat{\boldsymbol{\Gamma}}) \oplus \|\widehat{\boldsymbol{\Gamma}}\| \odot \boldsymbol{E}(\boldsymbol{T}(\widehat{\boldsymbol{Y}}); \widehat{\boldsymbol{\Gamma}}), \tag{2.11}$$

for all $\widehat{\boldsymbol{\Gamma}}$ such that $\|\widehat{\boldsymbol{\Gamma}}\| < \delta$ and $\boldsymbol{E}(\boldsymbol{T}(\widehat{\boldsymbol{Y}}); \widehat{\boldsymbol{\Gamma}}) \to \boldsymbol{0}$ as $\|\widehat{\boldsymbol{\Gamma}}\| \to 0$.

Example 2.6.3 *With $\boldsymbol{\mathcal{Y}} \subseteq I(\mathbb{R})^2$, let IVF $\boldsymbol{T} : \boldsymbol{\mathcal{Y}} \to I(\mathbb{R})$ be given by*

$$\boldsymbol{T}(([\underline{y}_1, \overline{y}_1], [\underline{y}_2, \overline{y}_2])) = [\underline{y}_1 + \underline{y}_1\overline{y}_1 + \underline{y}_2\overline{y}_2, \overline{y}_1 + \underline{y}_1\overline{y}_1 + \underline{y}_2\overline{y}_2],$$

for all $\widehat{\boldsymbol{Y}} = ([\underline{y}_1, \overline{y}_1], [\underline{y}_2, \overline{y}_2]) \in I(\mathbb{R})^2$. Consider $\widehat{\boldsymbol{\Gamma}} = ([\underline{\gamma}_1, \overline{\gamma}_1], [\underline{\gamma}_2, \overline{\gamma}_2])$ so that $\|\widehat{\boldsymbol{\Gamma}}\| \to 0$. Then, we have

$$\begin{aligned} \|\widehat{\boldsymbol{\Gamma}}\| = \sum_{i=1}^{2} \|[\underline{\gamma}_i, \overline{\gamma}_i]\|_{I(\mathbb{R})} &= \sum_{i=1}^{2} \max\{|\underline{\gamma}_i|, |\overline{\gamma}_i|\} \to 0 \\ \implies |\underline{\gamma}_i| \ \& \ |\overline{\gamma}_i| \to 0 \ \ \textit{for all} \ \ i = 1, 2 \ \ \textit{i.e.,} \ \underline{\gamma}_i \ \& \ \overline{\gamma}_i \to 0 \ \ \textit{for all} \ \ i = 1, 2. \end{aligned}$$

If $\boldsymbol{T}$ is gH-differentiable at $(\boldsymbol{0}, \boldsymbol{0})$, we have

$$\begin{aligned} \boldsymbol{T}([\underline{\gamma}_1, \overline{\gamma}_1], [\underline{\gamma}_2, \overline{\gamma}_2]) \ominus_{gH} \boldsymbol{T}(\boldsymbol{0}, \boldsymbol{0}) &= [\underline{\gamma}_1 + \underline{\gamma}_1\overline{\gamma}_1 + \underline{\gamma}_2\overline{\gamma}_2, \overline{\gamma}_1 + \underline{\gamma}_1\overline{\gamma}_1 + \underline{\gamma}_2\overline{\gamma}_2] \\ &= [\underline{\gamma}_1, \overline{\gamma}_1] \oplus [\underline{\gamma}_1\overline{\gamma}_1 + \underline{\gamma}_2\overline{\gamma}_2, \underline{\gamma}_1\overline{\gamma}_1 + \underline{\gamma}_2\overline{\gamma}_2]. \end{aligned} \tag{2.12}$$

Here, take $\mathscr{L}_{\widehat{\mathbf{Y}}} : I(\mathbb{R})^2 \to I(\mathbb{R})$ by $\mathscr{L}_{\widehat{\mathbf{Y}}}([\underline{\gamma}_1, \overline{\gamma}_1], [\underline{\gamma}_2, \overline{\gamma}_2]) = [\underline{\gamma}_1, \overline{\gamma}_1]$. We can easily show that $\mathscr{L}_{\widehat{\mathbf{Y}}}$ adheres to condition (i) and (iii) of Definition 2.10.

$$\begin{aligned}\mathscr{L}_{\widehat{\mathbf{Y}}}([\underline{\gamma}_1, \overline{\gamma}_1], [\underline{\gamma}_2, \overline{\gamma}_2]) = [\underline{\gamma}_1, \overline{\gamma}_1] &= [\underline{\gamma}_1, \overline{\gamma}_1] \odot \boldsymbol{1} \oplus [\underline{\gamma}_2, \overline{\gamma}_2] \odot \boldsymbol{0} \\ &= [\underline{\gamma}_1, \overline{\gamma}_1] \odot \mathscr{L}_{\widehat{\mathbf{Y}}}(\boldsymbol{1}, \boldsymbol{0}) \oplus [\underline{\gamma}_2, \overline{\gamma}_2] \odot \mathscr{L}_{\widehat{\mathbf{Y}}}(\boldsymbol{0}, \boldsymbol{1}).\end{aligned}$$

This means that $\mathscr{L}_{\widehat{\mathbf{Y}}}$ satisfies (ii) of Definition 2.10. Hence, $\mathscr{L}_{\widehat{\mathbf{Y}}}$ is L-IVF. Here, using (2.12), the error IVF is given by

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{T}(\boldsymbol{0}, \boldsymbol{0}); \widehat{\boldsymbol{\Gamma}}) &= \boldsymbol{T}([\underline{\gamma}_1, \overline{\gamma}_1], [\underline{\gamma}_2, \overline{\gamma}_2]) \ominus_{gH} \boldsymbol{T}(\boldsymbol{0}, \boldsymbol{0}) \ominus_{gH} \mathscr{L}_{\widehat{\mathbf{Y}}}([\underline{\gamma}_1, \overline{\gamma}_1], [\underline{\gamma}_2, \overline{\gamma}_2]) \\ &= [\underline{\gamma}_1 \overline{\gamma}_1 + \underline{\gamma}_2 \overline{\gamma}_2, \underline{\gamma}_1 \overline{\gamma}_1 + \underline{\gamma}_2 \overline{\gamma}_2] \to \boldsymbol{0}, \quad \textit{since } \underline{\gamma}_i \ \& \ \overline{\gamma}_i \to 0 \ \textit{ for all } \ i = 1, 2.\end{aligned}$$

As a consequence, $\boldsymbol{E}(\boldsymbol{T}(\boldsymbol{0}, \boldsymbol{0}); \widehat{\boldsymbol{\Gamma}}) \to 0$ when $\|\widehat{\boldsymbol{\Gamma}}\| \to 0$. This shows that $\boldsymbol{T}$ is gH-differentiable at $(\boldsymbol{0}, \boldsymbol{0})$.

Next, we present a lemma that shows L-IVF of $\mathbf{T}$ at $\widehat{\mathbf{H}}$ is defined using gH-directional derivative of $\mathbf{T}$ along direction $\widehat{\mathbf{H}}$.

Lemma 2.4 *Let $\boldsymbol{\mathcal{Y}} \subseteq I(\mathbb{R})^n, n \geq 2$ and an IVF $\boldsymbol{T} : \boldsymbol{\mathcal{Y}} \to I(\mathbb{R})$ be gH-differentiable at $\widehat{\boldsymbol{Y}} \in \boldsymbol{\mathcal{Y}}$. Then, there exists a nonzero $\lambda, \delta > 0$ and a L-IVF $\mathscr{L}_{\widehat{\mathbf{Y}}}$ for $\boldsymbol{T}$ in Definition 2.11, which can be rewritten as*

$$\lim_{\lambda \to 0} \frac{1}{\lambda} \odot (\boldsymbol{T}(\widehat{\boldsymbol{Y}} \oplus \lambda \odot \widehat{\boldsymbol{H}}) \ominus_{gH} \boldsymbol{T}(\widehat{\boldsymbol{Y}})) = \mathscr{L}_{\widehat{\mathbf{Y}}}(\widehat{\boldsymbol{H}}) \ \textit{for all } \widehat{\boldsymbol{H}} \in I(\mathbb{R})^n \ \textit{with } |\lambda| \|\widehat{\boldsymbol{H}}\| < \delta,$$

Proof: Consider $\mathbf{T}$ to be gH-differentiable at $\widehat{\mathbf{Y}} \in \boldsymbol{\mathcal{Y}}$. Then, there exists $\delta > 0$ satisfying

$$\mathbf{T}(\widehat{\mathbf{Y}} \oplus \widehat{\boldsymbol{\Gamma}}) \ominus_{gH} \mathbf{T}(\widehat{\mathbf{Y}}) = \mathscr{L}_{\widehat{\mathbf{Y}}}(\widehat{\boldsymbol{\Gamma}}) \oplus \|\widehat{\boldsymbol{\Gamma}}\| \odot \mathbf{E}(\mathbf{T}(\widehat{\mathbf{Y}}); \widehat{\boldsymbol{\Gamma}}) \text{ for all } \|\widehat{\boldsymbol{\Gamma}}\| < \delta,$$

where $\mathbf{E}(\mathbf{T}(\widehat{\mathbf{Y}}); \widehat{\boldsymbol{\Gamma}}) \to \mathbf{0}$. Let $\widehat{\boldsymbol{\Gamma}} = \lambda \odot \widehat{\mathbf{H}}$ for $\lambda \neq 0$ and $\widehat{\mathbf{H}} \in I(\mathbb{R})^n$ with $|\lambda| \|\widehat{\mathbf{H}}\| < \delta$. Here, observe that

$$\begin{aligned} & \frac{1}{\lambda} \odot (\mathbf{T}(\widehat{\mathbf{Y}} \oplus \lambda \odot \widehat{\mathbf{H}}) \ominus_{gH} \mathbf{T}(\widehat{\mathbf{Y}})) = \frac{1}{\lambda} \odot \left(\mathscr{L}_{\widehat{\mathbf{Y}}}(\lambda \odot \widehat{\mathbf{H}}) \oplus \|\widehat{\boldsymbol{\Gamma}}\| \odot \mathbf{E}(\mathbf{T}(\widehat{\mathbf{Y}}); \widehat{\boldsymbol{\Gamma}}) \right) \\ \implies & \frac{1}{\lambda} \odot (\mathbf{T}(\widehat{\mathbf{Y}} \oplus \lambda \odot \widehat{\mathbf{H}}) \ominus_{gH} \mathbf{T}(\widehat{\mathbf{Y}})) = \frac{1}{\lambda} \odot \left(\lambda \odot \mathscr{L}_{\widehat{\mathbf{Y}}}(\widehat{\mathbf{H}}) \oplus |\lambda| \|\widehat{\boldsymbol{\Gamma}}\| \odot \mathbf{E}(\mathbf{T}(\widehat{\mathbf{Y}}); \widehat{\mathbf{H}}) \right). \end{aligned}$$

Thus,

$$\lim_{\lambda \to 0} \frac{1}{\lambda} \odot (\mathbf{T}(\widehat{\mathbf{Y}} \oplus \lambda \odot \widehat{\mathbf{H}}) \ominus_{gH} \mathbf{T}(\widehat{\mathbf{Y}})) = \mathscr{L}_{\widehat{\mathbf{Y}}}(\widehat{\mathbf{H}}).$$

$\square$

Lemma 2.5 *Consider an IVF $\boldsymbol{T}$ on a nonempty subset $\boldsymbol{\mathcal{Y}}$ of $I(\mathbb{R})^n, n \geq 2$ to be gH-differentiable at $\widehat{\boldsymbol{Y}} \in \boldsymbol{\mathcal{Y}}$. Then, for each $\widehat{\boldsymbol{\Gamma}} = (\boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2, \boldsymbol{\Gamma}_3, \ldots, \boldsymbol{\Gamma}_n)^\top \in I(\mathbb{R})^n$, the gH-gradient of $\boldsymbol{T}$ at $\widehat{\boldsymbol{Y}}$ exists and L-IVF $\mathscr{L}_{\widehat{\mathbf{Y}}}$ for $\boldsymbol{T}$, from Definition 2.11, is given by*

$$\mathscr{L}_{\widehat{\mathbf{Y}}}(\widehat{\boldsymbol{\Gamma}}) = \widehat{\boldsymbol{\Gamma}}^\top \odot_c \nabla \boldsymbol{T}(\widehat{\boldsymbol{Y}}), \text{ where } \widehat{\boldsymbol{\Gamma}}^\top \odot_c \nabla \boldsymbol{T}(\widehat{\boldsymbol{Y}}) = \bigoplus_{i=1}^{n} \boldsymbol{\Gamma}_i \odot \boldsymbol{T}_{\mathbf{Y}_i}(\widehat{\boldsymbol{Y}}). \tag{2.13}$$

Proof: With $\widehat{\boldsymbol{\Gamma}} = \widehat{\mathbf{0}} \in I(\mathbb{R})^n$, it is trivial to show that both sides of equation (2.13) are same.
Now, consider that $\widehat{\boldsymbol{\Gamma}} \neq \mathbf{0}$. As $\mathbf{T}$ is gH-differentiable at $\widehat{\mathbf{Y}}$, we have, for $\delta > 0$,

$$\lim_{\lambda \to 0} \frac{1}{\lambda} \odot \{\mathbf{T}(\widehat{\mathbf{Y}} \oplus \lambda \odot \widehat{\mathbf{H}}) \ominus_{gH} \mathbf{T}(\widehat{\mathbf{Y}})\} = \mathscr{L}_{\widehat{\mathbf{Y}}}(\widehat{\mathbf{H}}), \tag{2.14}$$

where $\widehat{\boldsymbol{\Gamma}} = \lambda \odot \widehat{\mathbf{H}}$ with $\lambda \neq 0$, $\widehat{\mathbf{H}} \in I(\mathbb{R})^n$ and $|\lambda| \, \|\widehat{\mathbf{H}}\| < \delta$.
Now consider $\widehat{\mathbf{H}} = \mathbf{e}_i$, i-th unit vector of $I(\mathbb{R})^n, i = 1, 2, \ldots, n$. Then, using the (2.14), we have

$$\lim_{\lambda \to 0} \frac{1}{\lambda} \odot (\mathbf{T}(\widehat{\mathbf{Y}} \oplus \lambda \odot \mathbf{e}_i) \ominus_{gH} \mathbf{T}(\widehat{\mathbf{Y}})) = \mathscr{L}_{\widehat{\mathbf{Y}}}(\mathbf{e}_i) \implies \mathbf{T}_{\mathbf{Y}_i}(\widehat{\mathbf{Y}}) = \mathscr{L}_{\widehat{\mathbf{Y}}}(\mathbf{e}_i).$$

Thus, at $\widehat{\mathbf{Y}}$, all the i-th gH-partial derivatives $\mathbf{T}_{\mathbf{Y}_i}(\widehat{\mathbf{Y}})$ of $\mathbf{T}$ exist, $i = 1, 2, \ldots n$.
Hence, the gH-gradient of $\mathbf{T}$ at $\widehat{\mathbf{Y}}$ exists and we have

$$\begin{aligned}\mathscr{L}_{\widehat{\mathbf{Y}}}(\widehat{\boldsymbol{\Gamma}}) = \mathscr{L}_{\widehat{\mathbf{Y}}}(\boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2, \ldots, \boldsymbol{\Gamma}_n) &= \boldsymbol{\Gamma}_1 \odot \mathscr{L}_{\widehat{\mathbf{Y}}}(\mathbf{e}_1) \oplus \boldsymbol{\Gamma}_2 \odot \mathscr{L}_{\widehat{\mathbf{Y}}}(\mathbf{e}_2) \ \oplus \cdots \oplus \ \boldsymbol{\Gamma}_n \odot \mathscr{L}_{\widehat{\mathbf{Y}}}(\mathbf{e}_n) \\ &= \bigoplus_{i=1}^{n} \boldsymbol{\Gamma}_i \odot \mathbf{T}_{\mathbf{Y}_i}(\widehat{\mathbf{Y}}) = \boldsymbol{\Gamma}^\top \odot_c \nabla \mathbf{T}(\widehat{\mathbf{Y}}).\end{aligned}$$

$\square$

We now provide a lemma to describe Remark 2.6.1. This lemma necessitates the introduction of the following auxiliary definitions.

Definition 2.12 (Convex set in $I(\mathbb{R})^n$). *Consider $\boldsymbol{\mathcal{Y}}$ to be nonempty subset of $I(\mathbb{R})^n$. Then, $\boldsymbol{\mathcal{Y}}$ is called a convex set in $I(\mathbb{R})^n$ if for every $\widehat{\boldsymbol{Y}}, \widehat{\boldsymbol{Z}} \in \boldsymbol{\mathcal{Y}}$ and $\alpha \in [0, 1]$*

$$\alpha \odot \widehat{\boldsymbol{Y}} \oplus (1 - \alpha) \odot \widehat{\boldsymbol{Z}} \in \boldsymbol{\mathcal{Y}}.$$

Definition 2.13 (Convex IVF where domain is subset of $I(\mathbb{R})^n$). *Consider $\boldsymbol{\mathcal{Y}}$ to be a nonempty convex subset of $I(\mathbb{R})^n$. Then, an IVF $\boldsymbol{T}$: $\boldsymbol{\mathcal{Y}} \to I(\mathbb{R})$ is called convex if for* *all* $\widehat{\boldsymbol{Y}}_1, \widehat{\boldsymbol{Y}}_2 \in \boldsymbol{\mathcal{Y}}$, *and* $\alpha \in [0,1]$, *we find that*

$$\boldsymbol{T}(\alpha \odot \widehat{\boldsymbol{Y}}_1 \oplus (1-\alpha) \odot \widehat{\boldsymbol{Y}}_2) \preceq \alpha \odot \boldsymbol{T}(\widehat{\boldsymbol{Y}}_1) \oplus (1-\alpha) \odot \boldsymbol{T}(\widehat{\boldsymbol{Y}}_2).$$

Lemma 2.6 *Consider* $\boldsymbol{T}$ *to be a convex IVF on* $\mathcal{Y}$ *and gH-differentiable at* $\widehat{\boldsymbol{Y}}^* \in \boldsymbol{\mathcal{Y}}$. *Then,*

$$\boldsymbol{T}(\widehat{\boldsymbol{Y}}^*) \oplus (\widehat{\boldsymbol{Y}} \ominus_{gH} \widehat{\boldsymbol{Y}}^*)^\top \odot_c \nabla \boldsymbol{T}(\widehat{\boldsymbol{Y}}^*) \preceq \boldsymbol{T}(\widehat{\boldsymbol{Y}}).$$

Proof: Consider, for any $\widehat{\mathbf{Y}}, \widehat{\mathbf{Y}}^* \in \boldsymbol{\mathcal{Y}}$, that $\widehat{\mathbf{Y}}_\wp = \widehat{\mathbf{Y}}^* \oplus \wp \odot (\widehat{\mathbf{Y}} \ominus_{gH} \widehat{\mathbf{Y}}^*)$. Because $\mathbf{T}$ is a convex IVF, following Definition 2.13, we get

$$\begin{aligned}
&\mathbf{T}(\wp \odot \widehat{\mathbf{Y}} \oplus (1-\wp) \odot \widehat{\mathbf{Y}}^*) \preceq \wp \odot \mathbf{T}(\widehat{\mathbf{Y}}^*) \oplus (1-\wp) \odot \mathbf{T}(\widehat{\mathbf{Y}}) \\
\Longrightarrow\ &\mathbf{T}(\widehat{\mathbf{Y}}^* \oplus \wp \odot (\widehat{\mathbf{Y}} \ominus_{gH} \widehat{\mathbf{Y}}^*)) \preceq \wp \odot \mathbf{T}(\widehat{\mathbf{Y}}^*) \oplus (1-\wp) \odot \mathbf{T}(\widehat{\mathbf{Y}}) \\
\Longrightarrow\ &\mathbf{T}(\widehat{\mathbf{Y}}_\wp) \ominus_{gH} \wp \odot \mathbf{T}(\widehat{\mathbf{Y}}^*) \preceq (1-\wp) \odot \mathbf{T}(\widehat{\mathbf{Y}}) \\
\Longrightarrow\ &\frac{1}{1-\wp} \odot (\mathbf{T}(\widehat{\mathbf{Y}}_\wp) \ominus_{gH} \wp \odot \mathbf{T}(\widehat{\mathbf{Y}}^*)) \preceq \mathbf{T}(\widehat{\mathbf{Y}}) \\
\Longrightarrow\ &\frac{1}{1-\wp} \odot (\mathbf{T}(\widehat{\mathbf{Y}}_\wp) \ominus_{gH} (1-(1-\wp)) \odot \mathbf{T}(\widehat{\mathbf{Y}}^*)) \preceq \mathbf{T}(\widehat{\mathbf{Y}}) \\
\Longrightarrow\ &\mathbf{T}(\widehat{\mathbf{Y}}^*) \oplus \frac{1}{1-\wp} \odot (\mathbf{T}(\widehat{\mathbf{Y}}_\wp) \ominus_{gH} \mathbf{T}(\widehat{\mathbf{Y}}^*)) \preceq \mathbf{T}(\widehat{\mathbf{Y}}) \\
\Longrightarrow\ &\mathbf{T}(\widehat{\mathbf{Y}}^*) \oplus \frac{1}{1-\wp} \odot (\mathbf{T}(\widehat{\mathbf{Y}}^* \oplus \wp \odot (\widehat{\mathbf{Y}} \ominus_{gH} \widehat{\mathbf{Y}}^*)) \ominus_{gH} \mathbf{T}(\widehat{\mathbf{Y}}^*)) \preceq \mathbf{T}(\widehat{\mathbf{Y}}).
\end{aligned}$$

With respect to Lemma 2.5 and taking $\wp \to 0$, we get

$$\mathbf{T}(\widehat{\mathbf{Y}}^*) \oplus (\widehat{\mathbf{Y}} \ominus_{gH} \widehat{\mathbf{Y}}^*)^\top \odot_c \nabla \mathbf{T}(\widehat{\mathbf{Y}}^*) \preceq \mathbf{T}(\widehat{\mathbf{Y}}).$$

□

2.6.2 Optimality condition for constrained IOPs where domain is a subset of $I(\mathbb{R})^n$

Let the constrained IOP be given by:

$$\min_{\widehat{\mathbf{Y}} \in \boldsymbol{\mathcal{Y}}} \mathbf{T}(\widehat{\mathbf{Y}}), \tag{2.15}$$

where $\boldsymbol{\mathcal{Y}}$ is a nonempty subset of $I(\mathbb{R})^n$ and $\mathbf{T} : \boldsymbol{\mathcal{Y}} \to I(\mathbb{R})$ be an IVF.

For the constrained IOP (2.15), the concept of optimality is defined through the following definition of an efficient solution.

Definition 2.14 (Efficient solution). *Consider* $\boldsymbol{T} : \boldsymbol{\mathcal{Y}} \to I(\mathbb{R})$ *to be an IVF over* *a nonempty subset $\boldsymbol{\mathcal{Y}}$ of $I(\mathbb{R})^n$. A point $\widehat{\boldsymbol{Y}}^* \in \boldsymbol{\mathcal{Y}}$ is called an efficient solution of* $\min\limits_{\widehat{\boldsymbol{Y}} \in \boldsymbol{\mathcal{Y}}} \boldsymbol{T}(\widehat{\boldsymbol{Y}})$ *if* $\boldsymbol{T}(\widehat{\boldsymbol{Y}}) \not\prec \boldsymbol{T}(\widehat{\boldsymbol{Y}}^*)$ *for all* $\widehat{\boldsymbol{Y}} \in \boldsymbol{\mathcal{Y}}$.

Theorem 2.11 (Normal cone condition for efficiency). *Let there be an IOP of minimizing a gH-differentiable function $\boldsymbol{T}$ over a closed set $\boldsymbol{\mathcal{Y}} \subseteq I(\mathbb{R})^n$. Consider $\widehat{\boldsymbol{Y}} \in \boldsymbol{\mathcal{Y}}$. If $\widehat{\boldsymbol{Y}}^* = (\boldsymbol{Y}_1^*, \boldsymbol{Y}_2^*, \ldots, \boldsymbol{Y}_n^*)^\top \in I(\mathbb{R})^n$ is a local efficient solution of IOP (2.15) with $\boldsymbol{T}(\widehat{\boldsymbol{Y}}^*) \preceq \boldsymbol{T}(\widehat{\boldsymbol{Y}})$ for all $\widehat{\boldsymbol{Y}} \in \boldsymbol{\mathcal{Y}}$, then we have*

$$-\nabla \boldsymbol{T}(\widehat{\boldsymbol{Y}}^*) \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\widehat{\boldsymbol{Y}}^*).$$

Proof: From Definition 2.4 and Lemma 2.1, we show that

$$-\nabla \mathbf{T}(\widehat{\mathbf{Y}}^*)^\top \odot_c \widehat{\mathbf{W}} \preceq \mathbf{0} \text{ for all } \widehat{\mathbf{W}} \in \mathcal{T}^s_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*). \tag{2.16}$$

If $\widehat{\mathbf{W}} = \widehat{\mathbf{0}}$, then (2.16) holds. Take $\widehat{\mathbf{W}} \neq \widehat{\mathbf{0}}$. Then, using Definition 2.3, there exists a sequence $\widehat{\mathbf{Y}}_v = (\mathbf{Y}_1^v, \mathbf{Y}_2^v, \ldots, \mathbf{Y}_n^v)^\top \in \boldsymbol{\mathcal{Y}}$ together with a real sequence $\alpha_v \to 0$ and there exists a sequence $\widehat{\mathbf{W}}_v \to \widehat{\mathbf{W}}$ so that $\widehat{\mathbf{Y}}_v = \widehat{\mathbf{Y}}^* \oplus \alpha_v \odot \widehat{\mathbf{W}}_v \in \boldsymbol{\mathcal{Y}}$. This means that $\widehat{\mathbf{W}}_v = \frac{1}{\alpha_v} \odot (\widehat{\mathbf{Y}}_v \ominus_{gH} \widehat{\mathbf{Y}}^*) \in \boldsymbol{\mathcal{Y}}$. Now, because $\boldsymbol{\mathcal{Y}}$ is closed, we have $\widehat{\mathbf{Y}}_v \to \widehat{\mathbf{Y}}^* \in \boldsymbol{\mathcal{Y}}$ and further $\widehat{\mathbf{Y}}_v \ominus_{gH} \widehat{\mathbf{Y}}^* \in \boldsymbol{\mathcal{Y}}$. As $\mathbf{T}$ is gH-differentiable, we have

$$\mathbf{T}(\widehat{\mathbf{Y}}_v) \ominus_{gH} \mathbf{T}(\widehat{\mathbf{Y}}^*) = (\widehat{\mathbf{Y}}_v \ominus_{gH} \widehat{\mathbf{Y}}^*)^\top \odot_c \nabla \mathbf{T}(\widehat{\mathbf{Y}}^*) \oplus o(\|\widehat{\mathbf{Y}}_v \ominus_{gH} \widehat{\mathbf{Y}}^*\|) \odot \mathbf{1},$$

where $o : [0, \infty) \to \mathbb{R}$ is a function satisfying $\lim\limits_{t \to 0} \frac{o(t)}{t} = 0$.
Furthermore, because $\widehat{\mathbf{Y}}^*$ is a local efficient solution with $\mathbf{T}(\widehat{\mathbf{Y}}^*) \preceq \mathbf{T}(\widehat{\mathbf{Y}}_v)$ for v that is large enough, we get

$$\mathbf{0} \preceq (\widehat{\mathbf{Y}}_v \ominus_{gH} \widehat{\mathbf{Y}}^*)^\top \odot_c \nabla \mathbf{T}(\widehat{\mathbf{Y}}^*) \oplus o(\|\widehat{\mathbf{Y}}_v \ominus_{gH} \widehat{\mathbf{Y}}^*\|) \odot \mathbf{1}.$$

Based on the fact that $\alpha_v \geq 0$ and $\widehat{\mathbf{Y}}_v \neq \widehat{\mathbf{Y}}^*$ for all v, we have

$$\mathbf{0} \preceq \frac{1}{\alpha_v} \odot (\widehat{\mathbf{Y}}_v \ominus_{gH} \widehat{\mathbf{Y}}^*)^\top \odot_c \nabla \mathbf{T}(\widehat{\mathbf{Y}}^*) \oplus \frac{\|\widehat{\mathbf{Y}}_v \ominus_{gH} \widehat{\mathbf{Y}}^*\|}{\alpha_v} \cdot \frac{o(\|\widehat{\mathbf{Y}}_v \ominus_{gH} \widehat{\mathbf{Y}}^*\|)}{\|\widehat{\mathbf{Y}}_v \ominus_{gH} \widehat{\mathbf{Y}}^*\|} \odot \mathbf{1} \tag{2.17}$$

with large enough v. From inequality (2.17), we get

$$\mathbf{0} \preceq \widehat{\mathbf{W}}_v^\top \odot_c \nabla \mathbf{T}(\widehat{\mathbf{Y}}^*) \oplus \|\widehat{\mathbf{W}}_v\| \odot \mathbf{0} \implies -\widehat{\mathbf{W}}_v^\top \odot_c \nabla \mathbf{T}(\widehat{\mathbf{Y}}^*) \preceq \mathbf{0}.$$

Letting $v \to \infty$,

$$-\widehat{\mathbf{W}}^\top \odot_c \nabla \mathbf{T}(\widehat{\mathbf{Y}}^*) \preceq \mathbf{0} \implies -\nabla \mathbf{T}(\widehat{\mathbf{Y}}^*)^\top \odot_c \widehat{\mathbf{W}} \preceq \mathbf{0}.$$

Thus, $-\nabla \mathbf{T}(\widehat{\mathbf{Y}}^*) \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*)$. $\square$

Note 2.7 *The converse of above theorem does not hold. Take the IOP*

$$\min_{\widehat{\boldsymbol{Y}} \in \boldsymbol{\mathcal{Y}}} \boldsymbol{T}(\widehat{\boldsymbol{Y}}), \tag{2.18}$$

where $\boldsymbol{T} : \boldsymbol{\mathcal{Y}} \to I(\mathbb{R})$ *be an IVF described as*

$$\boldsymbol{T}(([\underline{y}_1, \overline{y}_1], [\underline{y}_2, \overline{y}_2])) = [\underline{y}_1^3 + \underline{y}_2^3, \overline{y}_1^3 + \overline{y}_2^3] \text{ for all } \widehat{\boldsymbol{Y}} = ([\underline{y}_1, \overline{y}_1], [\underline{y}_2, \overline{y}_2]) \in I(\mathbb{R})^2,$$

on a subset $\boldsymbol{\mathcal{Y}} = \left\{([\underline{y}_1, \overline{y}_1], [\underline{y}_2, \overline{y}_2]) | -3 \leq \underline{y}_1, \underline{y}_2 \leq 3, -3 \leq \overline{y}_1, \overline{y}_2 \leq 3\right\} \subseteq I(\mathbb{R})^2$.

Because $\underline{y}_1^3 + \underline{y}_2^3$, $\overline{y}_1^3 + \overline{y}_2^3$ *are non-convex on* $\mathbb{R}^2$, *we get* $\boldsymbol{T}$ *to be non-convex IVF as well. For ease, let us consider* $\boldsymbol{Y}_1 = [\underline{y}_1, \overline{y}_1]$, *and* $\boldsymbol{Y}_2 = [\underline{y}_2, \overline{y}_2]$. *The gH-partial derivative of* $\boldsymbol{T}_{\boldsymbol{Y}_1}([\underline{y}_1, \overline{y}_1], [\underline{y}_2, \overline{y}_2])$ *is expressed as*

$$\begin{aligned}
\boldsymbol{T}_{\boldsymbol{Y}_1}([\underline{y}_1, \overline{y}_1], [\underline{y}_2, \overline{y}_2]) &= \lim_{h \to 0} \frac{1}{h} \odot \left([(\underline{y}_1 + h)^3 + \underline{y}_2^3, (\overline{y}_1 + h)^3 + \overline{y}_2^3] \ominus_{gH} [\underline{y}_1^3 + \underline{y}_2^3, \overline{y}_1^3 + \overline{y}_2^3]\right) \\
&= \lim_{h \to 0} \frac{1}{h} \odot ([\underline{y}_1^3 + 3\underline{y}_1^2 h + 3\underline{y}_1 h^2 + h^3 + \underline{y}_2^3, \overline{y}_1^3 + 3\overline{y}_1^2 h + 3\overline{y}_1 h^2 + h^3 \\
&\qquad + \overline{y}_2^3] \ominus_{gH} [\underline{y}_1^3 + \underline{y}_2^3, \overline{y}_1^3 + \overline{y}_2^3]) \\
&= \lim_{h \to 0} [3\underline{y}_1^2 + 3\underline{y}_1 h + h^2, 3\overline{y}_1^2 + 3\overline{y}_1 h + h^2] \\
&= [3\underline{y}_1^2, 3\overline{y}_1^2].
\end{aligned}$$

Likewise, $\boldsymbol{T}_{\boldsymbol{Y}_2}\left([\underline{y}_1, \overline{y}_1], [\underline{y}_2, \overline{y}_2]\right) = [3\underline{y}_2^2, 3\overline{y}_2^2]$. *Thus,* $\boldsymbol{T}$ *is gH-differentiable. Hence,* $\nabla \boldsymbol{T}([\underline{y}_1, \overline{y}_1], [\underline{y}_2, \overline{y}_2]) = ([3\underline{y}_1^2, 3\overline{y}_1^2], [3\underline{y}_2^2, 3\overline{y}_2^2])$ *and* $\nabla \boldsymbol{T}((\boldsymbol{0}, \boldsymbol{0})) = (\boldsymbol{0}, \boldsymbol{0})$. *Clearly,* $\nabla \boldsymbol{T}((\boldsymbol{0}, \boldsymbol{0})) = (\boldsymbol{0}, \boldsymbol{0}) \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}}((\boldsymbol{0}, \boldsymbol{0}))$. *But, at the point* $([-3, -1], [-2, 0])$

$$[-35, -1] = \boldsymbol{T}(([-3, -1], [-2, 0])) \preceq \boldsymbol{T}((\boldsymbol{0}, \boldsymbol{0})) = \boldsymbol{0}.$$

Therefore, for IOP (2.18), $\boldsymbol{0}$ *is not an efficient point.*

Remark 2.6.1 *In Note 2.7, we observed that the converse of Theorem 2.11 does not hold. However, the converse can be shown to be true if we consider* $\boldsymbol{\mathcal{Y}}$ *to be a convex subset of* $I(\mathbb{R})^n$.

For this, consider $\boldsymbol{\mathcal{Y}}$ *to be a closed and convex subset of* $I(\mathbb{R})^n$ *and* $-\nabla \boldsymbol{T}(\widehat{\boldsymbol{Y}}^*) \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\widehat{\boldsymbol{Y}}^*)$. *Using Definition 2.1 and for all* $\widehat{\boldsymbol{Y}} \in \boldsymbol{\mathcal{Y}}$, *we get*

$$\begin{aligned}
&(\widehat{\boldsymbol{Y}} \ominus_{gH} \widehat{\boldsymbol{Y}}^*)^\top \odot_c ((-1) \odot \nabla \boldsymbol{T}(\widehat{\boldsymbol{Y}}^*)) \preceq \boldsymbol{0} \\
\implies &\boldsymbol{0} \preceq (\widehat{\boldsymbol{Y}} \ominus_{gH} \widehat{\boldsymbol{Y}}^*)^\top \odot_c \nabla \boldsymbol{T}(\widehat{\boldsymbol{Y}}^*)
\end{aligned}$$

$$\Longrightarrow \boldsymbol{T}(\widehat{\boldsymbol{Y}}^{*}) \preceq \boldsymbol{T}(\widehat{\boldsymbol{Y}}^{*}) \oplus (\widehat{\boldsymbol{Y}} \ominus_{gH} \widehat{\boldsymbol{Y}}^{*})^{\top} \odot_c \nabla \boldsymbol{T}(\widehat{\boldsymbol{Y}}^{*}).$$

Because $\boldsymbol{T}$ is convex and gH-differentiable at $\widehat{\boldsymbol{Y}}^{*} \in \boldsymbol{\mathcal{Y}}$, using Lemma 2.6, we have

$$\boldsymbol{T}(\widehat{\boldsymbol{Y}}^{*}) \preceq \boldsymbol{T}(\widehat{\boldsymbol{Y}}^{*}) \oplus (\widehat{\boldsymbol{Y}} \ominus_{gH} \widehat{\boldsymbol{Y}}^{*})^{\top} \odot_c \nabla \boldsymbol{T}(\widehat{\boldsymbol{Y}}^{*}) \preceq \boldsymbol{T}(\widehat{\boldsymbol{Y}}).$$

Thus, $\widehat{\boldsymbol{Y}}^{*}$ is a local efficient solution of IOP (2.15) with $\boldsymbol{T}(\widehat{\boldsymbol{Y}}^{*}) \preceq \boldsymbol{T}(\widehat{\boldsymbol{Y}})$ for all $\widehat{\boldsymbol{Y}} \in \boldsymbol{\mathcal{Y}}$.

Now, we will determine an efficiency condition using the Lagrange multiplier rule for constrained IOP that include inequality and equality constraints. The respective Lagrange coefficient (i.e., Lagrange multiplier) $\mathbf{Z}_j$ must lie on the normal cone for the range set of the jth constraint IVF (i.e., $\mathbf{T}_j(\widehat{\mathbf{Y}})$). Consider

$$\boldsymbol{\mathcal{Z}} = I(\mathbb{R})^s \times I(\mathbb{R})^{n-s} = \left\{ \widehat{\mathbf{Z}} = (\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n)^{\top} : \begin{cases} \mathbf{Z}_j \succeq \mathbf{0} & \text{for } j = 1, 2, 3, \ldots, s \\ \mathbf{Z}_j = \mathbf{0} & \text{for } j = s+1, \ldots, n \end{cases} \right\},$$

$$\text{and} \quad \boldsymbol{\mathcal{U}} = \left\{ \widehat{\mathbf{U}} = (\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_n)^{\top} : \begin{cases} \mathbf{U}_j \preceq \mathbf{0} & \text{for } j = 1, 2, \ldots, s \\ \mathbf{U}_j = \mathbf{0} & \text{for } j = s+1, \ldots, n \end{cases} \right\}.$$

Then, let the constrained IOP

$$\min_{\widehat{\mathbf{Y}} \in \boldsymbol{\mathcal{Y}}} \mathbf{T}_0(\widehat{\mathbf{Y}}) \text{ such that } \mathbf{T}_j(\widehat{\mathbf{Y}}) = \begin{cases} \preceq \mathbf{0} & \text{for } j = 1, 2, \ldots, s. \\ = \mathbf{0} & \text{for } j = s+1, s+2, \ldots, n. \end{cases} \tag{2.19}$$

The Lagrange problem on $I(\mathbb{R})^n \times I(\mathbb{R})^n$ is expressed as

$$\mathbf{L}(\widehat{\mathbf{Y}}, \widehat{\mathbf{Z}}) = \mathbf{T}_0(\widehat{\mathbf{Y}}) \oplus \widehat{\mathbf{Z}}_1 \odot \mathbf{T}_1(\widehat{\mathbf{Y}}) \oplus \widehat{\mathbf{Z}}_2 \odot \mathbf{T}_2(\widehat{\mathbf{Y}}) \oplus \cdots \oplus \widehat{\mathbf{Z}}_n \odot \mathbf{T}_n(\widehat{\mathbf{Y}}), \tag{2.20}$$

for all $\widehat{\mathbf{Z}} \in \mathcal{N}_{\boldsymbol{\mathcal{U}}}(\mathbf{T}_1(\widehat{\mathbf{Y}}), \mathbf{T}_2(\widehat{\mathbf{Y}}), \ldots, \mathbf{T}_n(\widehat{\mathbf{Y}}))$ and $\mathbf{C} = \{\widehat{\mathbf{Y}} \in \boldsymbol{\mathcal{Y}} : \mathbf{T}(\widehat{\mathbf{Y}}) \in \boldsymbol{\mathcal{U}}\}$,
where $\mathbf{T}(\widehat{\mathbf{Y}}) = (\mathbf{T}_1(\widehat{\mathbf{Y}}), \mathbf{T}_2(\widehat{\mathbf{Y}}), \ldots, \mathbf{T}_n(\widehat{\mathbf{Y}}))$ and $\boldsymbol{\mathcal{D}} = \{\mathbf{T}(\widehat{\mathbf{Y}}) : \widehat{\mathbf{Y}} \in \boldsymbol{\mathcal{Y}}\}$.

Definition 2.15 (Linear independence in $I(\mathbb{R})^n$). *A set $\boldsymbol{\mathcal{Y}} = \{\boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots, \boldsymbol{Y}_n\}$ consisting of nonzero vectors form $I(\mathbb{R})^n$ is linearly independent, if for the n real numbers $\beta_1, \beta_2, \ldots, \beta_n$, we have:*

$$\boldsymbol{0} \in \beta_1 \odot \boldsymbol{Y}_1 \oplus \beta_2 \odot \boldsymbol{Y}_2 \oplus \cdots \oplus \beta_n \odot \boldsymbol{Y}_n \text{ if and only if } \beta_1 = 0, \beta_2 = 0, \ldots, \beta_n = 0.$$

Theorem 2.12 *Consider $\boldsymbol{\mathcal{Y}}$ to be a nonempty closed subset of $I(\mathbb{R})^n$ and $\boldsymbol{T} : \boldsymbol{\mathcal{Y}} \to I(\mathbb{R})$ as a gH-differentiable function. If $\widehat{\boldsymbol{Y}}^{*} = (\boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots, \boldsymbol{Y}_n)^{\top} \in I(\mathbb{R})^n$ is a locally efficient solution for IOP (2.19) with $\boldsymbol{T}_0(\widehat{\boldsymbol{Y}}^{*}) \preceq \boldsymbol{T}_0(\widehat{\boldsymbol{Y}})$ for all $\widehat{\boldsymbol{Y}} \in \boldsymbol{\mathcal{Y}}$, at which the gradient*

$\nabla \boldsymbol{T}_j(\widehat{\boldsymbol{Y}}^*)$, $j = 1, 2, \ldots, n$ are linearly independent, there exists a vector $\widehat{\boldsymbol{Z}}$ in $\boldsymbol{\mathcal{Z}}$ with $\widehat{\boldsymbol{Z}} \in \mathcal{N}_{\boldsymbol{\mathcal{U}}}(\boldsymbol{T}(\widehat{\boldsymbol{Y}}^*))$ satisfying $\nabla_{\widehat{\boldsymbol{Y}}}\boldsymbol{L}(\widehat{\boldsymbol{Y}}^*, \widehat{\boldsymbol{Z}}) = \boldsymbol{0}$ and $-\nabla_{\widehat{\boldsymbol{Y}}}\boldsymbol{L}(\widehat{\boldsymbol{Y}}^*, \widehat{\boldsymbol{Z}}) \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\widehat{\boldsymbol{Y}}^*)$.

Proof: Let a vector $\widehat{\mathbf{V}} = \widehat{\mathbf{Z}}^\top \odot_c \nabla \mathbf{T}(\widehat{\mathbf{Y}}^*)$. Since $\nabla \mathbf{T}_j(\widehat{\mathbf{Y}}^*)$ are linearly independent and at least one $\mathbf{Z}_j$ is not zero, we have $\widehat{\mathbf{V}} \neq \mathbf{0}$. As $\widehat{\mathbf{Z}}$ is arbitrary, we have $\widehat{\mathbf{V}}$ as arbitrary as well. Here, we will prove that $\widehat{\mathbf{V}} \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*)$. This is same as showing that $\widehat{\mathbf{V}}^\top \odot_c \widehat{\mathbf{W}} \preceq \mathbf{0}$ for any arbitrary vector $\widehat{\mathbf{W}} \in \mathcal{T}^s_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*)$. This kind of vector $\widehat{\mathbf{W}}$ adheres to Definition 2.3. Assume a sequence of vectors $\widehat{\mathbf{W}}_v = \frac{1}{\alpha_v} \odot (\widehat{\mathbf{Y}}_v \ominus_{gH} \widehat{\mathbf{Y}}^*)$ with $\widehat{\mathbf{Y}}_v \in \boldsymbol{\mathcal{Y}}$ and $\alpha_v \to 0$. The gH-differentiability of $\mathbf{T}$ at $\widehat{\mathbf{Y}}^*$ is given as

$$\mathbf{T}(\widehat{\mathbf{Y}}_v) \ominus_{gH} \mathbf{T}(\widehat{\mathbf{Y}}^*) = \nabla \mathbf{T}(\widehat{\mathbf{Y}}^*)^\top \odot_c (\widehat{\mathbf{Y}}_v \ominus_{gH} \widehat{\mathbf{Y}}^*) \oplus o(\widehat{\mathbf{Y}}_v \ominus_{gH} \widehat{\mathbf{Y}}^*) \odot \mathbf{1}. \tag{2.21}$$

As a consequence,

$$\frac{1}{\alpha_v} \odot (\mathbf{T}(\widehat{\mathbf{Y}}_v) \ominus_{gH} \mathbf{T}(\widehat{\mathbf{Y}}^*)) = \nabla \mathbf{T}(\widehat{\mathbf{Y}}^*)^\top \odot_c \widehat{\mathbf{W}}_v \oplus \frac{o(\widehat{\mathbf{Y}}_v \ominus_{gH} \widehat{\mathbf{Y}}^*)}{\|\widehat{\mathbf{Y}}_v \ominus_{gH} \widehat{\mathbf{Y}}^*\|} \cdot \|\widehat{\mathbf{W}}_v\| \odot \mathbf{1}$$
$$\to \ \widehat{\mathbf{W}}^\top \odot_c \nabla \mathbf{T}(\widehat{\mathbf{Y}}^*) \text{ as } \alpha_v \to 0.$$

Because $\mathbf{T}_0(\widehat{\mathbf{Y}}^*) \preceq \mathbf{T}_0(\widehat{\mathbf{Y}})$ for all $\widehat{\mathbf{Y}} \in \boldsymbol{\mathcal{Y}}$, using Definition 2.3, we get

$$\widehat{\mathbf{W}}^\top \odot_c \nabla \mathbf{T}(\widehat{\mathbf{Y}}^*) \in \mathcal{T}^s_{\boldsymbol{\mathcal{U}}}(\mathbf{T}(\widehat{\mathbf{Y}}^*)).$$

Because $\widehat{\mathbf{Z}} \in \mathcal{N}_{\boldsymbol{\mathcal{U}}}(\mathbf{T}(\widehat{\mathbf{Y}}^*))$, thus we have

$$\widehat{\mathbf{Z}}^\top \odot_c (\widehat{\mathbf{W}}^\top \odot_c \nabla \mathbf{T}(\widehat{\mathbf{Y}}^*)) \preceq \mathbf{0} \implies (\widehat{\mathbf{Z}}^\top \odot_c \nabla \mathbf{T}(\widehat{\mathbf{Y}}^*)) \odot_c \widehat{\mathbf{W}} \preceq \mathbf{0} \implies \widehat{\mathbf{V}} \odot_c \widehat{\mathbf{W}} \preceq \mathbf{0}.$$

Hence, $\widehat{\mathbf{V}} \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*)$. Because $\widehat{\mathbf{Y}}^*$ is a locally efficient solution of the constrained IOP (2.19) with $\mathbf{T}_0(\widehat{\mathbf{Y}}^*) \preceq \mathbf{T}_0(\widehat{\mathbf{Y}})$, using Theorem 2.11, we have $-\nabla \mathbf{T}_0(\widehat{\mathbf{Y}}^*) \in \mathcal{N}_{\boldsymbol{\mathcal{Y}}}(\widehat{\mathbf{Y}}^*)$. Because of arbitrariness of $\widehat{\mathbf{V}}$, we get $\widehat{\mathbf{V}} = -\nabla \mathbf{T}_0(\widehat{\mathbf{Y}}^*)$, and this means

$$\nabla \mathbf{T}_0(\widehat{\mathbf{Y}}^*) \oplus \widehat{\mathbf{Z}}_1 \odot \nabla \mathbf{T}_1(\widehat{\mathbf{Y}}^*) \oplus \widehat{\mathbf{Z}}_2 \odot \nabla \mathbf{T}_2(\widehat{\mathbf{Y}}^*) \oplus \cdots \oplus \widehat{\mathbf{Z}}_n \odot \nabla \mathbf{T}_n(\widehat{\mathbf{Y}}^*) = \mathbf{0}.$$

This immediately shows that $\nabla_{\widehat{\mathbf{Y}}} \mathbf{L}(\widehat{\mathbf{Y}}^*, \widehat{\mathbf{Z}}) = \mathbf{0}$. Thus, $-\nabla_{\widehat{\mathbf{Y}}} \mathbf{L}(\widehat{\mathbf{Y}}^*, \widehat{\mathbf{Z}})^\top \odot (\widehat{\mathbf{Y}} \ominus_{gH} \widehat{\mathbf{Y}}^*) = \mathbf{0}$, which is the criterion for normal cone. This leads to the desired result. □

2.6.3 An application

In this subsection, we apply the proposed normal cone to a set of intervals within the context of an interval-valued support vector machine (SVM) problem. In many classification problems, input data can be imprecise and contain uncertainty, often due to measurement errors, implementation issues, and other factors. SVMs are a common approach to address such problems. For this application, we will primarily use the following auxiliary definitions in the computation of the normal cone.

Definition 2.16 (Support function on $I(\mathbb{R})^n$). *If $\boldsymbol{\mathcal{Y}}$ is a nonempty subset of $I(\mathbb{R})^n$. The support function of $\boldsymbol{\mathcal{Y}}$, denoted by $\boldsymbol{\Psi}^*_{\boldsymbol{\mathcal{Y}}} : \boldsymbol{\mathcal{Y}} \to I(\mathbb{R})$, is expressed as*

$$\boldsymbol{\Psi}^*_{\boldsymbol{\mathcal{Y}}}(\widehat{\boldsymbol{Y}}) = \sup_{\widehat{\boldsymbol{A}} \in \boldsymbol{\mathcal{Y}}} \widehat{\boldsymbol{Y}}^\top \odot_c \widehat{\boldsymbol{A}}.$$

Here, we use Definition 2.12 for the definition of supremum of a set of intervals and Remark 3 of [65].

Definition 2.17 (Indicator function on $I(\mathbb{R})^n$). *For a nonempty set $\boldsymbol{\mathcal{Y}} \subseteq I(\mathbb{R})^n$, the indicator function is given as*

$$\delta_{\boldsymbol{\mathcal{Y}}}(\widehat{\boldsymbol{Y}}) = \begin{cases} \boldsymbol{0}, & \text{if } \widehat{\boldsymbol{Y}} \in \boldsymbol{\mathcal{Y}} \\ \infty, & \text{if } \widehat{\boldsymbol{Y}} \notin \boldsymbol{\mathcal{Y}}. \end{cases}$$

Definition 2.18 (Conjugate function on $I(\mathbb{R})^n$). *If $\boldsymbol{\mathcal{Y}}$ is a nonempty subset of $I(\mathbb{R})^n$, then the conjugate of an IVF $\boldsymbol{T} : \boldsymbol{\mathcal{Y}} \to I(\mathbb{R})$ is expressed as*

$$\boldsymbol{T}^*(\widehat{\boldsymbol{Y}}) = \sup_{\widehat{\boldsymbol{A}} \in \boldsymbol{\mathcal{Y}}} \{(\widehat{\boldsymbol{Y}}^\top \odot_c \widehat{\boldsymbol{A}}) \ominus_{gH} \boldsymbol{T}(\widehat{\boldsymbol{A}})\}.$$

Note 2.8 *We can readily see that the conjugate of the indicator function $\delta_{\boldsymbol{\mathcal{Y}}}$ on $I(\mathbb{R})^n$ is the support function $\boldsymbol{\Psi}^*_{\boldsymbol{\mathcal{Y}}}$ on $I(\mathbb{R})^n$.*

Utilizing the above definitions, we address a unary classification problem. Therefore, we consider an SVM-type IOP for an interval-valued data set given by

$$\{(\mathbf{Y}, c) : \mathbf{Y} \in I(\mathbb{R}_+), c = -1\}.$$

The formulation of the SVM-type IOP for the above data set is given by

$$\left.\begin{array}{ll} \min & \mathbf{T}(\mathbf{W}) = \frac{1}{2} \odot [\|\underline{w} - p\|^2, \|\overline{w} - p\|^2] \\ \text{subject to} & \mathbf{1} \preceq c \odot \left(\mathbf{W} \odot \mathbf{Y} \oplus \left(\frac{1}{c} \ominus_{gH} \mathbf{B}\right)\right), \ \mathbf{W}, \mathbf{B} \in I(\mathbb{R}_+), \end{array}\right\} \tag{2.22}$$

where $\mathbf{W} \in I(\mathbb{R}_+)$ is the weight vector for interval, $\mathbf{B} \in I(\mathbb{R}_+)$ is the interval bias, and $\mathbf{0} \preceq \mathbf{p} = [p, p]$. The constraint is applied to determine the condition on the normal cone, ensuring that the interval-valued data reside on one side of the classifying hyperplane

$\mathbf{W} \odot \mathbf{Y} \oplus (\frac{1}{c} \ominus_{gH} \mathbf{B}) = \mathbf{-1}$, i.e., $\mathbf{W} \odot \mathbf{Y} \ominus_{gH} \mathbf{B} = \mathbf{0}$. Thus, an equivalent IOP to (2.22) can be defined as

$$\left.\begin{array}{ll} \min & \mathbf{T}(\mathbf{W}) = \frac{1}{2} \odot [\|\underline{w} - p\|^2, \|\overline{w} - p\|^2] \\ \text{subject to} & \mathbf{\Phi}(\mathbf{W}, \mathbf{B}) = \mathbf{W} \odot \mathbf{Y} \ominus_{gH} \mathbf{B} \preceq \mathbf{0},\ \ \mathbf{W}, \mathbf{B} \in I(\mathbb{R}_+). \end{array}\right\} \tag{2.23}$$

We take $\mathbf{C} = \{\mathbf{W} \in I(\mathbb{R}_+) : \mathbf{\Phi}(\mathbf{W}, \mathbf{B}) \preceq \mathbf{0}\}$, which is a closed set. Here, if we consider $\mathbf{W}^* = [\underline{w}^*, \overline{w}^*]$ to be a local efficient solution to the IOP (2.23) with $\mathbf{T}(\mathbf{W}^*) \preceq \mathbf{T}(\mathbf{W})$, then using Theorem 2.11, we get

$$-\nabla \mathbf{T}(\mathbf{W}^*) \in \mathcal{N}_{\mathbf{C}}(\mathbf{W}^*). \tag{2.24}$$

We can see that $\mathbf{T}$ is gH-differentiable since $\underline{t}(\underline{w}) = \|\underline{w} - p\|$ and $\overline{t}(\overline{w}) = \|\overline{w} - p\|$ are differentiable. The gH-gradient of $\mathbf{T}$ at $\mathbf{W}^*$ is expressed as

$$\nabla \mathbf{T}(\mathbf{W}^*) = \mathbf{W}^* \ominus_{gH} \mathbf{p}.$$

For validating the inclusion (2.24), we require to compute $\mathcal{N}_{\mathbf{C}}(\mathbf{W}^*)$. To address this, we consider two cases, where in the first case, we take $\mathbf{W}^* \in \text{int}(\mathbf{C})$. To get the normal cone $\mathcal{N}_{\mathbf{C}}(\mathbf{W}^*)$, we select an arbitrary $\mathbf{G} \in \mathcal{N}_{\mathbf{C}}(\mathbf{W}^*)$. Then, there exists some $r > 0$ satisfying $\mathbf{W}^* \oplus r \odot \mathbf{G} \in \mathbf{C}$. Thus, we get

$$\begin{aligned} & \mathbf{G} \odot ((\mathbf{W} \oplus r \odot \mathbf{G}) \ominus_{gH} \mathbf{W}) \preceq \mathbf{0} \\ \Longrightarrow\ & \mathbf{G} \odot (r \odot \mathbf{G}) \preceq \mathbf{0} \\ \Longrightarrow\ & r \odot \|\mathbf{G}\|^2 \in \mathbf{G} \odot (r \odot \mathbf{G}) \preceq \mathbf{0} \\ \Longrightarrow\ & \mathbf{G} = \mathbf{0}. \end{aligned}$$

Hence, if $\mathbf{W}^* \in \text{int}(\mathbf{C})$, then $\mathcal{N}_{\mathbf{C}}(\mathbf{W}^*) = \{\mathbf{0}\}$. Thus, using the relation (2.24) we get

$$-\nabla \mathbf{T}(\mathbf{W}^*) \in \mathcal{N}_{\mathbf{C}}(\mathbf{W}^*) = \{\mathbf{0}\} \implies \mathbf{W}^* = \mathbf{p}. \tag{2.25}$$

For the second case, we consider $\mathbf{W}^*$ to reside on the boundary of $\mathbf{C}$. Then, we have $\mathbf{W}^* \odot \mathbf{Y} = \mathbf{B}$. To get the normal cone $\mathcal{N}_{\mathbf{C}}(\mathbf{W}^*)$, we utilize the polarity of tangent cone. To compute the tangent cone via sequence in $I(\mathbb{R})^n$, denoted by $\mathcal{T}^s_{\mathbf{C}}(\mathbf{W}^*)$, it is necessary to compute the set $\frac{1}{\alpha} \odot (\mathbf{C} \ominus_{gH} \mathbf{W}^*)$. For this, consider $\mathbf{W} \in \mathbf{C}$ and we see that

$$(\mathbf{W} \ominus_{gH} \mathbf{W}^*) \odot \mathbf{Y} \subseteq \mathbf{W} \odot \mathbf{Y} \ominus_{gH} \mathbf{W}^* \odot \mathbf{Y} \preceq \mathbf{B} \ominus_{gH} \mathbf{B} = \mathbf{0} \text{ and}$$

$$\frac{1}{\alpha} \odot (\mathbf{C} \ominus_{gH} \mathbf{W}^*) \subseteq \left\{ \frac{1}{\alpha} \odot (\mathbf{W} \ominus_{gH} \mathbf{W}^*) \in I(\mathbb{R}) : \frac{1}{\alpha} \odot (\mathbf{W} \ominus_{gH} \mathbf{W}^*) \odot \mathbf{Y} \preceq \mathbf{0} \right\}. \tag{2.26}$$

For proving " $\supseteq$ " of (2.26), take $\frac{1}{\alpha} \odot (\mathbf{W} \ominus_{gH} \mathbf{W}^*) \in I(\mathbb{R})$ such that $\frac{1}{\alpha} \odot (\mathbf{W} \ominus_{gH} \mathbf{W}^*) \odot \mathbf{Y} \preceq \mathbf{0}$ with $\mathbf{W} \in I(\mathbb{R}_+)$. This means that $\mathbf{W} \odot \mathbf{Y} \preceq \mathbf{W}^* \odot \mathbf{Y} = \mathbf{B}$ as $(\mathbf{W} \ominus_{gH} \mathbf{W}^*) \odot \mathbf{Y} \subseteq \mathbf{W} \odot \mathbf{Y} \ominus_{gH} \mathbf{W}^* \odot \mathbf{Y}$. Hence, we get $\mathbf{W} \in I(\mathbb{R}_+)$ satisfying $\mathbf{W} \odot \mathbf{Y} \preceq \mathbf{B}$. This shows that $\mathbf{W} \in \mathbf{C}$ and $\mathbf{W} \ominus_{gH} \mathbf{W}^* \in \mathbf{C} \ominus_{gH} \mathbf{W}^*$. Hence,

$$\left\{ \frac{1}{\alpha} \odot (\mathbf{W} \ominus_{gH} \mathbf{W}^*) \in I(\mathbb{R}) : \frac{1}{\alpha} \odot (\mathbf{W} \ominus_{gH} \mathbf{W}^*) \odot \mathbf{Y} \preceq \mathbf{0} \right\} \subseteq \frac{1}{\alpha} \odot (\mathbf{C} \ominus_{gH} \mathbf{W}^*). \tag{2.27}$$

With both the inclusions (2.26) and (2.27) together, we get

$$\frac{1}{\alpha} \odot (\mathbf{C} \ominus_{gH} \mathbf{W}^*) = \left\{ \frac{1}{\alpha} \odot (\mathbf{W} \ominus_{gH} \mathbf{W}^*) \in I(\mathbb{R}) : \frac{1}{\alpha} \odot (\mathbf{W} \ominus_{gH} \mathbf{W}^*) \odot \mathbf{Y} \preceq \mathbf{0} \right\}.$$

Thus, with respect to Theorem 2.8, we get

$$\begin{aligned} \mathcal{T}^s_{\mathbf{C}}(\mathbf{W}^*) &= \left\{ \frac{1}{\alpha} \odot (\mathbf{W} \ominus_{gH} \mathbf{W}^*) \in I(\mathbb{R}) : \frac{1}{\alpha} \odot (\mathbf{W} \ominus_{gH} \mathbf{W}^*) \odot \mathbf{Y} \preceq \mathbf{0} \right\} \\ &= \left\{ \mathbf{W}' \in I(\mathbb{R}) : \mathbf{W}' \odot \mathbf{Y} \preceq \mathbf{0} \right\} \text{ by taking } \mathbf{W}' = \frac{1}{\alpha} \odot (\mathbf{W} \ominus_{gH} \mathbf{W}^*). \end{aligned}$$

Because $\mathcal{N}_{\mathbf{C}}(\mathbf{W}^*) = \mathcal{T}^s_{\mathbf{C}}(\mathbf{W}^*)^\circ$, we then have

$$\begin{aligned} \mathcal{N}_{\mathbf{C}}(\mathbf{W}^*) =& \{\mathbf{Z} \in I(\mathbb{R}) : \mathbf{Z} \odot \mathbf{W}' \preceq \mathbf{0},\ \forall\ \mathbf{W}' : \mathbf{W}' \odot \mathbf{Y} \preceq \mathbf{0}\} \\ =& \{\mathbf{Z} \in I(\mathbb{R}) : \sup_{\mathbf{W}' \odot \mathbf{Y} \preceq \mathbf{0}} \mathbf{Z} \odot \mathbf{W}' \preceq \mathbf{0}\} \\ =& \{\mathbf{Z} \in I(\mathbb{R}) : \boldsymbol{\Psi}^*_{\{\mathbf{W}' : \mathbf{W}' \odot \mathbf{Y} \preceq \mathbf{0}\}}(\mathbf{Z}) \preceq \mathbf{0}\} \text{ by Definition 2.16} \\ =& \{\mathbf{Z} \in I(\mathbb{R}) : \boldsymbol{\delta}_{\{\lambda \odot \mathbf{Y} : \lambda \geq 0\}}(\mathbf{Z}) \preceq \mathbf{0}\} \text{ by Note 2.4} \\ =& \{\lambda \odot \mathbf{Y} : \lambda \geq 0\} = I(\mathbb{R}_+). \end{aligned}$$

Here, using (2.24), we get

$$\begin{aligned} & -(\mathbf{W}^* \ominus_{gH} \mathbf{p}) \in I(\mathbb{R}_+) \\ \Longrightarrow\ & \mathbf{0} \preceq -1 \odot [\underline{w}^* - p, \overline{w}^* - p] \\ \Longrightarrow\ & \mathbf{W}^* \preceq \mathbf{p}. \end{aligned} \tag{2.28}$$

Then, using (2.25) and (2.28), we get the common value of $\mathbf{W}^*$ as

$$\bigcap_{\mathbf{W}^* \in \mathbf{C}} \{\mathbf{W}^* : -\nabla \mathbf{T}(\mathbf{W}^*) \in \mathcal{N}_{\mathbf{C}}(\mathbf{W}^*)\} = \mathbf{p}.$$

Related to any $\mathbf{W}$ that satisfies the inclusion (2.24), we can write the set of possible values of the bias as

$$\{\mathbf{B} \in I(\mathbb{R}) : \mathbf{\Phi}(\mathbf{W}, \mathbf{B}) \preceq \mathbf{0}\} \cup \{\mathbf{B} \in I(\mathbb{R}) : \mathbf{\Phi}(\mathbf{W}, \mathbf{B}) = \mathbf{0}\}. \tag{2.29}$$

Therefore, for any solution $\mathbf{W}$ and $\mathbf{B}$ of (2.24) and (2.29), the classifying hyperplane is obtained as

$$\mathbf{W} \odot \mathbf{Y} \ominus_{gH} \mathbf{B} = \mathbf{0}.$$

Example 2.6.4 *Using this example, we demonstrate the SVM-type application of IOPs. Consider the interval-valued data set given by* $\{(\boldsymbol{Y}, c) : \boldsymbol{Y} = [3, 4],\ c = -1\}$*. For this data set, we identify a classifying hyperplane by solving the SVM-type IOP given by*

$$\left.\begin{array}{ll} \min & \boldsymbol{T}(\boldsymbol{W}) = \frac{1}{2} \odot [\|\underline{w} - 1\|^2, \|\overline{w} - 1\|^2] \\ \textit{subject to} & \mathbf{\Phi}(\boldsymbol{W}, \boldsymbol{B}) = \boldsymbol{W} \odot [3, 4] \ominus_{gH} \boldsymbol{B} \preceq \boldsymbol{0}, \quad \boldsymbol{W}, \boldsymbol{B} \in I(\mathbb{R}_+). \end{array}\right\} \tag{2.30}$$

Let $\boldsymbol{W}^* = [\underline{w}^*, \overline{w}^*]$ *be a local efficient solution to the IOP (2.30) such that* $\boldsymbol{T}(\boldsymbol{W}^*) \preceq \boldsymbol{T}(\boldsymbol{W})$ *for all* $\boldsymbol{W} \in I(\mathbb{R}_+)$*. We have* $\boldsymbol{C} = \{[\underline{w}, \overline{w}] : 3\underline{w} \leq \underline{b}, 4\overline{w} \leq \overline{b}\}$*.*
For finding the classifying hyperplane, we need to determine a possible solution $(\boldsymbol{W}^*, b)$ *by applying Theorem 2.11. Thus, we have*

$$-\nabla \boldsymbol{T}(\boldsymbol{W}^*) = -[\underline{w}^* - 1, \overline{w}^* - 1] \in \mathcal{N}_{\boldsymbol{C}}(\boldsymbol{W}^*). \tag{2.31}$$

Here, if $\boldsymbol{W}^* \in int(\boldsymbol{C})$*, following the same steps as in the above application, we get* $\mathcal{N}_{\boldsymbol{C}}(\boldsymbol{W}^*) = \{\boldsymbol{0}\}$*. Next, from the inclusion (2.31), we can see that* $\boldsymbol{W}^* = \boldsymbol{1}$ *and the corresponding set of bias* $\boldsymbol{B}$ *is* $\{[\underline{b}, \overline{b}] : \underline{b} > 3, \overline{b} > 4\}$ *obtained using the definition of* $\boldsymbol{C}$*. In the other case, if* $\boldsymbol{W}^*$ *lies on the boundary of* $\boldsymbol{C}$*, then proceeding as given earlier, we get* $\mathcal{N}_{\boldsymbol{C}}(\boldsymbol{W}^*) = \{\lambda \odot [3, 4] : \lambda \geq 0\} = I(\mathbb{R}_+)$*. Further, following the inclusion (2.31), we observe that the possible values of* $\boldsymbol{W}^*$ *is* $\{[\underline{w}^*, \overline{w}^*] : 0 < \underline{w}^* \leq 1, 0 < \overline{w}^* \leq 1\}$*. Then,* $\boldsymbol{W}^*$ *is*

$$\bigcap_{\boldsymbol{W}^* \in \boldsymbol{C}} \{\boldsymbol{W}^* : -\nabla \boldsymbol{T}(\boldsymbol{W}^*) \in \mathcal{N}_{\boldsymbol{C}}(\boldsymbol{W}^*)\} = \boldsymbol{1}$$

and the set of values for bias $\boldsymbol{B}$ *given by* $\{[\underline{b}, \overline{b}] : \underline{b} \geq 3, \overline{b} \geq 4\}$*. Therefore, with respect to* $\boldsymbol{W}^* = \boldsymbol{1}$*, the set of classifying hyperplane is* $\boldsymbol{Y} \ominus_{gH} [\underline{b}, \overline{b}] = \boldsymbol{0}$*, where* $\underline{b} \geq 3, \overline{b} \geq 4$*.*

For any $\boldsymbol{B}$ *in* $\{[\underline{b}, \overline{b}] : \underline{b} \geq 3, \overline{b} \geq 4\}$, *the value of the objective IVF* $\boldsymbol{T}$ *is the same i.e.* $\boldsymbol{0}$.

2.7 Conclusion

In this chapter, we have presented the concepts of normal cone (Definition 2.1) and tangent cone (Definitions 2.2 and 2.3) for sets of intervals. We have shown that the normal cone adheres to set inclusion 2.5, complies with the countable intersection rule (Theorem 2.5), and that the tangent and normal cones are polar of each other (Theorem 2.1). We have also introduced the concept of gH-differentiability (Definition 2.11) for IVF involving interval variables. To establish this concept, we study the definitions of the partial derivative (Definition 2.7), direction derivative (Definition 2.9), gH-gradient (Definition 2.8), and L-IVF (Definition 2.10) where the domain is a subset of $I(\mathbb{R})^n$. Furthermore, these concepts are applied to identify the optimality conditions for two constrained IOPs (Theorem 2.11 and 2.12). Among them, latter is shown by using Lagrangian technique (Theorem 2.12). At the end, we have presented an example demonstrating the applicability of the proposed results on the normal cone for finding the efficiency conditions in a interval-valued support vector machine problem.

Finally, We note that the sum-intersection rule of the normal cone incorporates the distribution law, specifically $(\mathbf{X} \oplus \mathbf{Y}) \odot \mathbf{Z} = \mathbf{X} \odot \mathbf{Z} \oplus \mathbf{Y} \odot \mathbf{Z}$, which are not true for nondegenerate intervals. Hence, the equality of sum-intersection rule for normal cone does not hold for IVFs. For example, consider two nonempty convex sets $\mathbf{C}_1 = \{[1, 2] \odot y : y \geq 0\}$ and $\mathbf{C}_2 = \{[-2, -1] \odot y : y \leq 0\}$. Then, at $\mathbf{0}$,

$$\mathcal{N}_{\mathbf{C}_1}(\mathbf{0}) \oplus \mathcal{N}_{\mathbf{C}_2}(\mathbf{0}) = I(\mathbb{R}_-) \oplus I(\mathbb{R}_-) = I(\mathbb{R}_-) \subset I(\mathbb{R}) = \mathcal{N}_{\mathbf{C}_1 \bigcap \mathbf{C}_2}(\mathbf{0}).$$

In such scenario, if the normal cone resides within symmetric interval space, degenerate interval space, or particularly on $I(\mathbb{R}_+)$, $I(\mathbb{R}_-)$, then the sum-intersection rule applies. Exploring this problem further without imposing restrictions on the normal cone could be an interesting pursuit in future.

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