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Szegö-Weinberger type inequalities for symmetric domains in simply connected space forms

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ABSTRACT

We consider the Neumann eigenvalue problem for Laplacian on a bounded multi-connected domain contained in simply connected space forms. Under certain symmetry assumptions on the domain, we prove Szegö-Weinberger type inequalities for the first n positive Neumann eigenvalues.

1. Introduction

Let $\Omega \subset M$ be a bounded domain with smooth boundary $\partial\Omega$ in a complete connected Riemannian manifold M of dimension $n \geq 2$. Consider the Neumann eigenvalue problem on Ω

$$\begin{array}{rcll} \Delta\phi & = & \mu\phi & \text{in } \Omega, \\ \frac{\partial\phi}{\partial\nu} & = & 0 & \text{on } \partial\Omega, \end{array} \tag{1.1}$$

where ν represents the outward unit normal to $\partial\Omega$. The set of all Neumann eigenvalues form a discrete sequence $0 = \mu_1(\Omega) < \mu_2(\Omega) \leq \mu_3(\Omega) \leq \cdots \nearrow \infty$. This problem (1.1) models the vibrations of a homogeneous free membrane.

In 1952, Kornhauser and Stakgold [16] using a perturbation method, showed that the disk is a local maximizer of $\mu_2(\Omega)$ on the class of all simply connected planar domains of a given area. Later, Szegö [18] with the help of conformal mapping techniques (the 'method of conformal transplantation'), proved that the ball is indeed a global maximum on the same class of domains. In 1956, Weinberger [23] generalized this result for the class of bounded Lipschitz domains in $\mathbb{R}^n$ with given volume, without the simply connectedness

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assumption. Let Ω^* be the ball centered at the origin in $\mathbb{R}^n$ of the same volume as Ω. Then Weinberger's result can be stated as the following inequality (Szegö-Weinberger inequality):

$$\mu_2(\Omega) \leq \mu_2(\Omega^*). \tag{1.2}$$

Later, Szegö and Weinberger observed that (see page 634 of [23]) Szegö's proof for (1.2) for simply connected planar domains can be extended to the following inequality:

$$\frac{1}{\mu_2(\Omega)} + \frac{1}{\mu_3(\Omega)} \geq \frac{1}{\mu_2(\Omega^*)} + \frac{1}{\mu_3(\Omega^*)}. \tag{1.3}$$

Since $\mu_2(\Omega^*) = \mu_3(\Omega^*)$, and $\mu_3(\Omega) \geq \mu_2(\Omega)$, the above inequality clearly yields (1.2).

In [5], Bandle extended the (1.3) (and hence (1.2)) to the case of an inhomogeneous free membrane using conformal mapping techniques, see also [6, Theorem 3.12]. In [10, page 94], Chavel remarked that using Weinberger's approach, one can easily prove the Szegö-Weinberger inequality for domains in space forms of constant negative sectional curvature. However, in the case of space forms of constant positive sectional curvature, Weinberger's approach will work only for the domains that are contained in sufficiently small geodesic balls. For example, for dimension 2, Ω must be contained in a geodesic ball of radius $\pi/(4\sqrt{\kappa})$, where κ is the constant positive sectional curvature of the space form, see [9, page 80]). In [4], Ashbaugh and Benguria improved this result for the domains contained in a geodesic ball of radius $\pi/(2\sqrt{\kappa})$ for all dimensions. For extension of (1.2) to domains in more general manifolds, see [1,19,21,24]. It is known that

$$\mu_3(\Omega) < 2^{\frac{2}{n}} \mu_3(\Omega^*) \tag{1.4}$$

due to Girouard, Nadirashvili, & Polterovich for bounded Jordan domains in $\mathbb{R}^2$ [14] and due to Bucur & Henrot [8] for general domains in $\mathbb{R}^n$ with $n \geq 2$. However, to establish the Szegö-Weinberger type inequality for the higher Neumann eigenvalues one needs to make certain symmetry restriction on Ω.

For $k \in \mathbb{N}$, and $i, j \in \{1, 2, \dots, n\}$, let $R_{i,j}^{\frac{2\pi}{k}}$ be the rotation (in the anti-clockwise direction with respect to the origin) by an angle $\frac{2\pi}{k}$ in the coordinate plane (x_i, x_j). A domain $\Omega \subset \mathbb{R}^n$ is said to be *symmetric of order k* with respect to the origin, if there exists a rotation R on $\mathbb{R}^n$ such that $R_{i,j}^{\frac{2\pi}{k}}(R(\Omega)) = R(\Omega), \forall\, i, j \in \{1, 2, \dots, n\}$. A domain $\Omega \subset \mathbb{R}^n$ is said to be *centrally symmetric* with respect to the origin, if $x \in \Omega$ if and only if $-x \in \Omega$.

In [15, Section 5], Hersch proved that for any Jordan domain $\Omega \subset \mathbb{R}^2$, symmetric of order $k \geq 3$,

$$\mu_3(\Omega) \leq \mu_3(\Omega^*). \tag{1.5}$$

For a smooth simply connected domain Ω with such symmetries, Ashbaugh and Benguria [3, Lemma 4.1] proved that, $\mu_2(\Omega) = \mu_3(\Omega)$. In particular, this yields (1.5) from (1.2). If Ω is symmetric of order 4, then they obtained (1.5), without the simply connectedness assumption on Ω, see [3, Theorem 4.3]. However, for a Jordan domain Ω with symmetry of order 4, Hersch [15, Section 5] obtained

$$\mu_4(\Omega) \leq \mu_4(\Omega^*). \tag{1.6}$$

For further inequalities involving $\mu_k(\Omega)$ for domains with symmetry of order $k \geq 2$, see [11,12].

Recently, the inequalities for higher eigenvalues have been extended for multiply connected domains by considering the eigenvalues of an appropriate concentric annular region instead of the ball Ω^*. In [2], authors compared the Neumann eigenvalues of Laplacian on a Lipschitz domain $\Omega = \Omega_{out} \setminus \overline{\Omega}_{in}$ and a concentric annular region $B_\beta \setminus \overline{B}_\alpha$, where

(i) Ω_{in} is compactly contained in Ω_{out},
(ii) $0 \le \alpha < \beta$ is such that $B_\alpha \subset \Omega_{in}$ and $|\Omega| = |B_\beta \setminus \overline{B}_\alpha|$.

They proved that, if Ω is either symmetric of order 2 or centrally symmetric, then

$$\mu_2(\Omega) \le \mu_2(B_\beta \setminus \overline{B}_\alpha),$$

and if domain Ω is symmetric of order 4, then

$$\mu_i(\Omega) \le \mu_i(B_\beta \setminus \overline{B}_\alpha) \text{ for } i = 2, 3, \ldots, n+2. \tag{1.7}$$

In [20], a similar result has been proved for the first nonzero Neumann eigenvalue for centrally symmetric doubly connected domains contained in noncompact rank-1 symmetric spaces with the additional assumption that inner domain Ω_{in} is a geodesic ball.

In this article, we partially extend (1.7) for the domains with holes contained in a simply connected space form. It is well known that a smooth, connected, and simply-connected complete Riemannian manifold with constant curvature is isometric to either of $\mathbb{R}^n$ (for zero curvature), $\mathbb{S}^n$ (for positive curvature) and $\mathbb{H}^n$ (for negative curvature). Due to this fact, we prove our main result for $M = \mathbb{S}^n$ and $\mathbb{H}^n$.

Now we provide the notations used in this article and state the notion of k-symmetry for the domains in space forms.

Notations. Throughout this article, M represents either $\mathbb{S}^n$ or $\mathbb{H}^n$, $n \ge 2$. Let $p = (1, 0, 0, \ldots, 0) \in M$ be a fixed point and B_r denotes the geodesic ball in M of radius r. Let $\exp_p : T_p(M) \to M$ be the exponential map and $X = (X_1, X_2, \ldots, X_n)$ be the geodesic normal coordinates centered at p. We identify any domain Ω in M with $\exp_p^{-1}(\Omega)$.

Definition 1.1. Let a domain $\Omega \subset M$ be represented as $\Omega = \exp_p(U)$ for $p \in M$ and $U \subset T_p(M)$. Then Ω is said to be *symmetric of order k* with respect to p, if U is symmetric of order k with respect to the origin. The domain Ω is said to be *geodesically symmetric* or *centrally symmetric* with respect to the point p, if U is centrally symmetric about the origin.

Remark 1.2. For $n = 2$, a domain Ω in M is centrally symmetric if and only if it is symmetric of order 2. For $n \ge 3$, the central symmetry does not imply the symmetry of order 2 [2, Lemma 5.2]. If Ω is a domain having symmetry of order $k \in \mathbb{N}$ with $k \ne 2, 4$, then Ω is either a geodesic ball or a concentric annular domain. For a proof, see [2, Proposition 5.1].

We make the following assumptions on the domain Ω:

($\mathbf{A_1}$) Let $\Omega = \Omega_{out} \setminus \overline{\Omega}_{in}$ be a domain in M, where $\overline{\Omega}_{in} \subset \Omega_{out}$. Here domain Ω might possess other holes except Ω_{in} or might possess no holes at all i.e., $\Omega_{in} = \emptyset$. If Ω_{in} is non empty, then we assume that $p \in \Omega_{in}$. In the case of $M = \mathbb{S}^n$, we additionally assume that Ω is contained in a geodesic ball of radius $\pi/2$ with center at p.

($\mathbf{A_2}$) Let B_{R_1} and B_{R_2} be the geodesic balls in M, centered at p, of radius R_1 and R_2, respectively such that $B_{R_1} \subset \Omega_{in}$ and $\mathrm{Vol}(\Omega) = \mathrm{Vol}(B_{R_2} \setminus \overline{B}_{R_1})$.

Remark 1.3. It follows from assumption ($\mathbf{A_1}$) that for our choice of Ω, there exists a neighborhood U of the origin in T_pM such that $\exp_p : U \to \Omega$ is a diffeomorphism.

The main result of this article is as follows.

Theorem 1.4. *Let $\Omega \subset M$, R_1 and R_2 be as given in* $(\mathbf{A_1})$ *and* $(\mathbf{A_2})$. *In addition,*

(1) if Ω is either symmetric of order 2 or centrally symmetric with respect to the point p, then

$$\mu_2(\Omega) \le \mu_2(B_{R_2} \setminus \overline{B}_{R_1}), \tag{1.8}$$

(2) if Ω is symmetric of order 4 with respect to the point p, then for $i = 2, \dots, n+1$,

$$\mu_i(\Omega) \le \mu_i(B_{R_2} \setminus \overline{B}_{R_1}) = \mu_2(B_{R_2} \setminus \overline{B}_{R_1}). \tag{1.9}$$

Remark 1.5. If $\Omega_{in} = \emptyset$, then (1.9) extend [3, Theorem 4.3] to $\mathbb{S}^n$ and $\mathbb{H}^n$, $n \ge 2$. To the best of our knowledge, this is the first result that gives the Szegö-Weinberger type inequality for the higher Neumann eigenvalues on space forms. In [13], authors proved that the union of two disjoint geodesic balls of the same volume maximize the third Neumann eigenvalue among the regions of given volume in hyperbolic spaces.

Remark 1.6. As an easy consequence of the above theorem, we also obtain an isoperimetric inequality for the harmonic mean of the first n nonzero Neumann eigenvalues of Laplacian on a domain Ω which is symmetric of order 4 as given below:

$$\frac{1}{\mu_2(\Omega)} + \frac{1}{\mu_3(\Omega)} + \dots + \frac{1}{\mu_{n+1}(\Omega)} \ge \frac{n}{\mu_2(B_{R_2} \setminus \overline{B}_{R_1})}.$$

Whereas in [7,22], authors proved a similar bound for the harmonic mean of the first $(n-1)$ nonzero Neumann eigenvalues in terms of the first nonzero Neumann eigenvalue of the geodesic ball, without any symmetry assumption.

The remaining part of this article is organized as follows. Section 2 is devoted to the study of Neumann eigenvalues and eigenfunctions on an annular domain contained in M. This section also includes some relations satisfied by these eigenvalues and eigenfunctions, which are important to prove our main result. We give detailed computation of geodesic normal coordinates on $M (= \mathbb{S}^n$ and $\mathbb{H}^n)$ and derive some of their properties in Section 3. In this section, we also obtain some integral identities, which helps us to find test functions for the variational characterization of Neumann eigenvalues In Section 4, we prove the main result and provide some concluding remarks.

2. The Neumann problem on an annular domain

This section describes the Neumann eigenvalues on an annular domain as the eigenvalues of certain Sturm–Liouville eigenvalue problems. These Sturm–Liouville problems are determined by the eigenvalues of $\Delta_{\mathbb{S}^{n-1}}$. We also prove some inequalities between the eigenvalues of distinct Sturm–Liouville problems which are essential for proving the main result.

2.1. The spherical harmonics

A spherical harmonic Y on $\mathbb{S}^{n-1}$ of degree $k \ge 0$ is the restriction of $\tilde{Y}$, a harmonic homogeneous polynomial of degree k on $\mathbb{R}^n$, to $\mathbb{S}^{n-1}$. Let $\mathcal{H}_k, k \ge 0$ represent the space of harmonic homogeneous polynomials of degree k on $\mathbb{R}^n$. Then dim $\mathcal{H}_k = \binom{k+n-1}{n-1} - \binom{k+n-3}{n-1}$. Notice that,

$$\mathcal{H}_0 = \text{ span } \{1\}$$
$$\mathcal{H}_1 = \text{ span } \{x_i : i \in \{1, \dots, n\}$$

$$\mathcal{H}_2 = \text{ span } \{x_i x_j, x_1^2 - x_k^2 : i, j \in \{1, \ldots, n\} \text{ and } k \in \{2, \ldots, n\}\}.$$

The following proposition describe the set of eigenvalues and the eigenfunctions of $\Delta_{\mathbb{S}^{n-1}}$ [17, Sections 22.3, 22.4].

Proposition 2.1. *The set of all eigenvalues of $\Delta_{\mathbb{S}^{n-1}}$ is $\{k(k+n-2) : k \in \mathbb{N} \cup \{0\}\}$. The eigenfunctions corresponding to each eigenvalue $k(k+n-2)$ are the spherical harmonics of degree k and thus the multiplicity of $k(k+n-2)$ is equal to dim $\mathcal{H}_k$.*

2.2. The Neumann eigenvalues and eigenfunctions on annular domain

Fix $p \in M$, we describe the Neumann eigenvalues and corresponding eigenfunctions on annular domain $(B_{R_2} \setminus \overline{B}_{R_1}) \subset M$ centered at the point p.

Recall that, the Riemannian metric g_M on M in terms of geodesic polar coordinates is of the form $g_M(r, \Theta) = dr^2 + \sin_M^2(r) g_0(\Theta)$. Here $(r, \Theta) \in [0, L] \times \mathbb{S}^{n-1}$, g_0 is the canonical metric on the $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}$ and function $\sin_M(r)$ is defined as

$$\sin_M(r) := \begin{cases} \sin r, & M = \mathbb{S}^n, \\ \sinh r, & M = \mathbb{H}^n. \end{cases}$$

Let $S(r)$ be the geodesic sphere of radius r centered at p and $\mathrm{tr}(A(r))$ be the trace of the second fundamental form $A(r)$. Then Δ on M can be decomposed in terms of $\Delta_{S(r)}$ as given below:

$$\Delta = -\frac{d^2}{dr^2} - \ \mathrm{tr}(A(r)) \frac{d}{dr} + \Delta_{S(r)}.$$

For $M = \mathbb{S}^n$ and $\mathbb{H}^n$, one can verify that

$$\mathrm{tr(A(r))} = \frac{1}{\sin_M^{n-1}(r)} \frac{d}{dr} \left(\sin_M^{n-1}(r) \right) \text{ and } \Delta_{S(r)} = \frac{1}{\sin_M^2(r)} \Delta_{\mathbb{S}^{n-1}}.$$

Thus for a function $f(r, \Theta) = u(r)v(\Theta)$ defined on $(B_{R_2} \setminus B_{R_1})$,

$$\begin{aligned} \Delta(u(r)v(\Theta)) &= \left(-\frac{d^2 u}{dr^2} - \frac{1}{\sin_M^{n-1}(r)} \frac{d}{dr} \left(\sin_M^{n-1}(r) \right) \frac{du}{dr} \right) v(\Theta) + \Delta_{S(r)}(u(r)v(\Theta)) \\ &= -\frac{1}{\sin_M^{n-1}(r)} \frac{d}{dr} \left(\sin_M^{n-1}(r) \frac{d}{dr} u \right) v(\Theta) + \frac{u}{\sin_M^2(r)} \Delta_{\mathbb{S}^{n-1}} v(\Theta). \end{aligned}$$

If v is spherical harmonic of degree k, then we obtain

$$\Delta(u(r)v(\Theta)) = -\frac{1}{\sin_M^{n-1}(r)} \frac{d}{dr} \left(\sin_M^{n-1}(r) \frac{d}{dr} u \right) v(\Theta) + \frac{u}{\sin_M^2(r)} k(k+n-2) v(\Theta).$$

Thus by separation of variable technique, for an eigenpair (μ, f) of the Neumann problem on $B_{R_2} \setminus B_{R_1}$, f is of the form $f(r, \Theta) = u(r)v(\Theta)$, where v is a spherical harmonic corresponding to the eigenvalue $k(k+n-2)$ and u satisfies the following Sturm–Liouville eigenvalue problem:

$$-u''(r) - \tfrac{(n-1) \sin_M'(r)}{\sin_M(r)} u'(r) + \tfrac{k(k+n-2)}{\sin_M^2(r)} u(r) = \mu u(r) \tag{2.1}$$

with the boundary conditions:

$$u'(R_1) = 0, \quad u'(R_2) = 0. \tag{2.2}$$

Notice that, for each $k \in \mathbb{N} \cup \{0\}$, the eigenvalues of the above Sturm–Liouville problem form an increasing sequence $0 \leq \mu_{k,1} < \mu_{k,2} < \mu_{k,3} < \cdots \nearrow \infty$ and each eigenvalue $\mu_{k,j}$ has multiplicity one and the corresponding eigenfunction vanishes exactly $(j-1)$ times in the interval (R_1, R_2). For $k \in \mathbb{N} \cup \{0\}$, and $u \in H^1((R_1, R_2); \sin_M^{n-1}(r)) \setminus \{0\}$, consider the Rayleigh quotient

$$\mathcal{R}_k(u) := \frac{\int_{R_1}^{R_2} \left((u'(r))^2 + \frac{k(k+n-2)}{\sin_M^2(r)} u^2(r) \right) \sin_M^{n-1}(r)\ dr}{\int_{R_1}^{R_2} u^2(r) \sin_M^{n-1}(r)\ dr}. \tag{2.3}$$

Now, a variational characterization of $\mu_{k,j}$ can be given as below:

$$\mu_{k,j} = \min_{E \in \mathcal{H}_j} \max_{u \in E \setminus \{0\}} \mathcal{R}_k(u),$$

where $\mathcal{H}_j$ is the set of all j-dimensional subspaces of $H^1\left((R_1, R_2); \sin_M^{n-1}(r)\right)$. Since $\mathcal{R}_{k+1}(u) > \mathcal{R}_k(u)$, for each $j \in \mathbb{N}$, we also observe that

$$\mu_{0,j} < \mu_{1,j} < \mu_{2,j} < \cdots \nearrow \infty. \tag{2.4}$$

Remark 2.2. For each $k \in \mathbb{N} \cup \{0\}$, Sturm–Liouville problem (2.1) with the boundary conditions $u(R_1) = u(R_2) = 0$ the set of eigenvalues form a sequence of the form $0 < \lambda_{k,1} < \lambda_{k,2} < \lambda_{k,3} < \cdots \nearrow \infty$. These eigenvalues have the following variational characterization:

$$\lambda_{k,j} = \min_{E \in \mathcal{H}_j} \max_{u \in E \setminus \{0\}} \mathcal{R}_k(u),$$

where $\mathcal{X}_j$ is the set of all j-dimensional subspaces of the Sobolev space $H_0^1((R_1, R_2); \sin_M^{n-1}(r))$. For each $k \in \mathbb{N} \cup \{0\}$ and $j \in \mathbb{N}$, from the variational characterizations, we clearly have $\mu_{k,j} \leq \lambda_{k,j}$. Moreover, using the simplicity of eigenvalues, one can show that

$$\mu_{k,j} < \lambda_{k,j}, \forall\, k \in \mathbb{N} \cup \{0\}, \forall\, j \in \mathbb{N}. \tag{2.5}$$

Remark 2.3. If $\lambda_i(S(r))$ denote the i^{th} eigenvalue of $\Delta_{S(r)}$, then $\lambda_0(S(r)) = 0$ and $\lambda_1(S(r)) = \frac{n-1}{\sin_M^2(r)}$. Also observe that $(\mathrm{tr}(A(r)))' = -\lambda_1(S(r))$.

The observations in the above remarks lead to the following lemma:

Lemma 2.4. *For $j \in \mathbb{N}$, $\mu_{0,j+1} = \lambda_{1,j}$ and hence $\mu_{1,j} < \mu_{0,j+1}$.*

Proof. Let f be an eigenfunction corresponding to the eigenvalue $\mu_{0,j+1}$ for some $j \in \mathbb{N}$. Thus f satisfy the following Sturm–Liouville problem:

$$-\frac{d^2 f}{dr^2} - \ \mathrm{tr}(A(r)) \frac{df}{dr} = \mu_{0,j+1} f,$$
$$f'(R_1) = 0\, f'(R_2) = 0.$$

By differentiating the above equation and using fact that $(\mathrm{tr}(A(r)))' = -\lambda_1(S(r))$, we obtain,

$$-\frac{d^2 f'(r)}{dr^2} - \ \mathrm{tr}(A(r)) \frac{df'(r)}{dr} + \lambda_1(S(r)) f'(r) = \mu_{0,j+1} f'(r).$$

Therefore, $u = f'$ is an eigenfunction corresponding to the eigenvalue $\mu_{0,j+1}$ of (2.1) with $k = 1$ and satisfies the boundary conditions $u(R_1) = u(R_2) = 0$. Thus $\mu_{0,j+1} \in \{\lambda_{1,i} : i \in \mathbb{N}\}$. From (2.4) and (2.5), also have

$$\mu_{0,j+1} < \mu_{1,j+1} < \lambda_{1,j+1}.$$

Thus, we must have $\mu_{0,2} = \lambda_{1,1}$ and subsequently $\mu_{0,i+1} = \lambda_{1,i}$ for $i = 2, 3, \dots, j-1$. □

The following lemma plays an integral part in proving our main theorem. For an analogue of this lemma for the Euclidean domains, see [2, Lemma 2.7].

Lemma 2.5. *For $k \in \mathbb{N}$, let $u(r)$ be the positive eigenfunction corresponding to the eigenvalue $\mu_{k,1}$ of the Sturm–Liouville problem (2.1) with the boundary conditions (2.2). Then*

(i) $\mu_{k,1} = \frac{k(k+n-2)}{\sin_M^2(b)}$ *for some* $b \in (R_1, R_2)$,
(ii) u *is strictly increasing on* (R_1, R_2),
(iii) *for each* $r \in (R_1, R_2)$,

$$\left(\frac{k(k+n-2)}{\sin_M^2(r)} - \mu_{k,1}\right) u^2(r) \geq \left(\frac{k(k+n-2)}{\sin_M^2(R_2)} - \mu_{k,1}\right) u^2(R_2). \tag{2.6}$$

Proof. (i) Notice that u satisfies

$$\begin{aligned} \frac{d}{dr}\left(\sin_M^{n-1}(r)\frac{d}{dr}u(r)\right) &= \left(\frac{k(k+n-2)}{\sin_M^2(r)} - \mu_{k,1}\right) u(r)\sin_M^{n-1}(r), r \in (R_1, R_2), \\ u'(R_1) = 0,\, u'(R_2) = 0. \end{aligned} \tag{2.7}$$

Let $\Phi(r) = \sin_M^{n-1}(r)u'(r)$. Then $\Phi(R_1) = 0 = \Phi(R_2)$. Thus there exists $b \in (R_1, R_2)$ such that $\Phi'(b) = 0$ and hence (2.7) yields $\mu_{k,1} = \frac{k(k+n-2)}{\sin_M^2(b)}$.

(ii) Since $\sin_M^{n-1}(r)u(r) > 0$, from (2.7) we obtain

$$\Phi'(r) > 0 \text{ for } r \in (R_1, b), \text{ and } \Phi'(r) < 0 \text{ for } r \in (b, R_2),$$

where b as given in (i). Now, as $\Phi(R_1) = 0 = \Phi(R_2)$, we easily conclude that $\Phi(r) > 0$ for every $r \in (R_1, R_2)$. Thus $u'(r) > 0$ for $r \in (R_1, R_2)$ as required.

(iii) For $r \in (R_1, b]$, (2.6) holds trivially. For $r \in (b, R_2)$, using (i), we get

$$0 > \frac{k(k+n-2)}{\sin_M^2(r)} - \mu_{k,1} > \frac{k(k+n-2)}{\sin_M^2(R_2)} - \mu_{k,1}.$$

Since u is strictly increasing and positive, the above inequality yields (2.6) for $r \in (b, R_2)$. □

Remark 2.6. The set of Neumann eigenvalues of $-\Delta$ on $(B_{R_2} \setminus \overline{B}_{R_1})$ is given by

$$\{\mu_i(B_{R_2} \setminus \overline{B}_{R_1})\}_{i\in\mathbb{N}} = \{\mu_{k,j}\}_{k\in\mathbb{N}\cup\{0\}, j\in\mathbb{N}}.$$

Therefore, $\mu_1(B_{R_2} \setminus \overline{B}_{R_1}) = \mu_{0,1} = 0$ and by Lemma 2.4 and Proposition 2.1, we conclude that

$$\mu_2(B_{R_2} \setminus \overline{B}_{R_1}) = \mu_3(B_{R_2} \setminus \overline{B}_{R_1}) = \cdots = \mu_{n+1}(B_{R_2} \setminus \overline{B}_{R_1}) = \min\{\mu_{0,2}, \mu_{1,1}\} = \mu_{1,1}.$$

The corresponding eigenfunctions are $u(r)\frac{X_i}{r}$, $1 \leq i \leq n$, where $(X_1, X_2, \ldots, X_n)$ is a geodesic normal coordinates centered at the point p and $u(r)$ is an eigenfunction corresponding to the eigenvalue $\mu_{1,1}$ of (2.1) with boundary conditions (2.2).

A proof for the next proposition can be given using the above lemma and similar arguments as given in the proof of Proposition 2.8 of [2].

Proposition 2.7. *Let Ω, R_1 and R_2 be as given in $(\mathbf{A_1})$ and $(\mathbf{A_2})$. For $k \in \mathbb{N}$, let u_k be a positive eigenfunction of (2.1) and (2.2) corresponding to the eigenvalue $\mu_{k,1}$. Then*

$$\frac{\int_\Omega \left((G_k'(r))^2 + \frac{k(k+n-2)}{\sin_M^2(r)} G_k^2(r)\right) dV}{\int_\Omega G_k^2(r)\ dV} \leq \mu_{k,1}, \tag{2.8}$$

where

$$G_k(r) = \begin{cases} u_k(r), & \text{if } r \in (R_1, R_2), \\ u_k(R_2), & \text{if } r \geq R_2. \end{cases}$$

Furthermore, equality holds in (2.8) if and only if Ω coincides with $B_{R_2} \setminus B_{R_1}$.

3. Geodesic normal coordinates on sphere and hyperbolic space

In this section, we will construct the geodesic normal coordinates on M that satisfy some properties similar to that of standard Euclidean coordinates.

3.1. Geodesic normal coordinates on the unit n-sphere $\mathbb{S}^n$

We consider the following parametrization for $q = (q_1, q_2, \ldots, q_{n+1}) \in \mathbb{S}^n$:

$$q_1 = \cos(\phi_1),\ q_2 = \sin(\phi_1)\cos(\phi_2), \ldots, q_n = \sin(\phi_1)\sin(\phi_2)\sin(\phi_3)\cdots\cos(\phi_n)$$
$$q_{n+1} = \sin(\phi_1)\sin(\phi_2)\sin(\phi_3)\cdots\sin(\phi_n),$$

with $\phi_1, \phi_2, \ldots, \phi_{n-1} \in [0, \pi]$ and $\phi_n \in [0, 2\pi]$. Then the tangent space $T_q\mathbb{S}^n$ at any point $q \in \mathbb{S}^n$ is spanned by $\frac{\partial}{\partial\phi_1}\big|_q, \frac{\partial}{\partial\phi_2}\big|_q, \ldots, \frac{\partial}{\partial\phi_n}\big|_q$, where

$$\left.\frac{\partial}{\partial\phi_1}\right|_q = \left(-\sin(\phi_1), \cos(\phi_1)\cos(\phi_2), \ldots, \cos(\phi_1)\sin(\phi_2)\sin(\phi_3)\cdots\sin(\phi_n)\right),$$
$$\left.\frac{\partial}{\partial\phi_2}\right|_q = \left(0, -\sin(\phi_1)\sin(\phi_2), \ldots, \sin(\phi_1)\cos(\phi_2)\sin(\phi_3)\cdots\sin(\phi_n)\right),$$
$$\vdots$$
$$\left.\frac{\partial}{\partial\phi_n}\right|_q = \left(0, 0, \ldots, 0, -\sin(\phi_1)\sin(\phi_2)\sin(\phi_3)\cdots\sin(\phi_n), \sin(\phi_1)\sin(\phi_2)\sin(\phi_3)\cdots\cos(\phi_n)\right).$$

This gives the Riemannian metric on $\mathbb{S}^n$,

$$g_{\mathbb{S}^n} = d\phi_1{}^2 + \sin^2(\phi_1)d\phi_2{}^2 + \cdots + \sin^2(\phi_1)\sin^2(\phi_2)\cdots\sin^2(\phi_{n-1})d\phi_n{}^2.$$

For $\mathbf{v}, \mathbf{w} \in T_p\mathbb{S}^n$, we denote $g_{\mathbb{S}^n}(p)(\mathbf{v}, \mathbf{w})$ and $g_{\mathbb{S}^n}(p)(\mathbf{v}, \mathbf{v})$ by $\langle \mathbf{v}, \mathbf{w}\rangle_p$ and $\|\mathbf{v}\|_p$, respectively.

Next we find a geodesic normal coordinates of a point $q \in \mathbb{S}^n$ centered at p. Let $\{e_i\}_{i=1}^{n+1}$ be the standard orthonormal basis of $\mathbb{R}^{n+1}$. For any $q \in \mathbb{S}^n$, there exist $\mathbf{v} \in T_p\mathbb{S}^n$, $\|\mathbf{v}\|_p = 1$ and $t \in \mathbb{R}^+ \cup \{0\}$ such that

$$q = \exp_p t\mathbf{v} = p\cos t + \mathbf{v}\sin t. \tag{3.1}$$

Now from the parametric representation of q, we easily get

$$t = \phi_1 \quad \text{and } \mathbf{v} = (0, \cos(\phi_2), \sin(\phi_2)\cos(\phi_3), \ldots, \sin(\phi_2)\sin(\phi_3)\cdots\sin(\phi_n)).$$

Therefore, with respect to the orthonormal basis $(e_2, e_3, \ldots, e_{n+1})$ of $T_p\mathbb{S}^n$ at p the geodesic normal coordinates of q are given by

$$X_i(q) = t\langle \mathbf{v}, e_{i+1}\rangle_p = \begin{cases} \phi_1\cos(\phi_2) & i = 1, \\ \phi_1\sin(\phi_2)\cdots\sin(\phi_i)\cos(\phi_{i+1}) & 2 \le i \le n-1, \\ \phi_1\sin(\phi_2)\cdots\sin(\phi_3)\sin(\phi_n) & i = n. \end{cases} \tag{3.2}$$

3.2. Geodesic normal coordinates on hyperbolic space

For n-dimensional hyperbolic space $\mathbb{H}^n$, consider the hyperboloid model

$$\{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} | -x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = -1, x_1 > 0\}$$

with the Riemannian metric

$$ds^2 = -dx_1^2 + dx_2^2 + \cdots + dx_{n+1}^2.$$

For $q = (q_1, q_2, \ldots, q_{n+1}) \in \mathbb{H}_n$, consider the following parametrization:

$$\begin{aligned} &q_1 = \cosh(\phi_1),\ q_2 = \sinh(\phi_1)\cos(\phi_2), \ldots, \\ &q_n = \sinh(\phi_1)\sin(\phi_2)\sin(\phi_3)\cdots\cos(\phi_n),\ q_{n+1} = \sinh(\phi_1)\sin(\phi_2)\sin(\phi_3)\cdots\sin(\phi_n), \end{aligned}$$

where $\phi_1 \in [0, \infty)$, $\phi_2, \ldots, \phi_{n-1} \in [0, \pi]$ and $\phi_n \in [0, 2\pi]$. Then its Riemannian metric will take form

$$\begin{aligned} ds^2 = {} & d\phi_1{}^2 + \sinh^2(\phi_1)d\phi_2{}^2 + \sinh^2(\phi_1)\sin^2(\phi_2)d\phi_3{}^2 + \cdots + \sinh^2(\phi_1)\sin^2(\phi_2)\cdots \\ & \cdots\sin^2(\phi_{n-1})d\phi_n{}^2. \end{aligned}$$

Now, any $q \in \mathbb{H}^n$ can be represented as

$$q = \exp_p t\mathbf{v} = p\cosh t + \mathbf{v}\sinh t,$$

for some $\mathbf{v} \in T_p\mathbb{H}^n$, $ds^2(\mathbf{v}, \mathbf{v})_p = 1$ and $t \in \mathbb{R}^+ \cup \{0\}$. Thus, by the parametric representation of q, as before we easily get

$$t = \phi_1 \quad \text{and } \mathbf{v} = (0, \cos(\phi_2), \sin(\phi_2)\cos(\phi_3), \ldots, \sin(\phi_2)\sin(\phi_3)\cdots\sin(\phi_n)).$$

Therefore, with respect to the orthonormal basis $(e_2, e_3, \ldots, e_{n+1})$ of $T_p\mathbb{H}^n$, the geodesic normal coordinates of $q \in \mathbb{H}^n$ centered at p also have the same expression as given in (3.2).

Remark 3.1. For $q \in M$, let $X_i(q), 1 \le i \le n$ be the geodesic normal coordinates of q. If $q = \exp_p(\mathbf{v})$ for some $\mathbf{v} \in T_pM$, then the geodesic distance $r_p(q)$ of q from p is $\|v\|_p$. Moreover,

$$\sum_{i=1}^{n} X_i^2(q) = \langle \mathbf{v}, \mathbf{v} \rangle_p = \phi_1(q).$$

Thus $r_p(q) = \sum_{i=1}^{n} X_i^2(q) = \phi_1(q)$. Therefore, we use r interchangeably with ϕ_1.

Remark 3.2. We can also define the notion of rotation on M via exponential map as follows: for $q \in M$ and a rotation R on T_pM,

$$R(q) := \exp_p\left(R\left(\exp_p^{-1}(q)\right)\right).$$

In particular, in terms of the geodesic normal coordinates, using (3.2) we get

$$\begin{aligned} R_{i,j}^{\frac{2\pi}{2}}(X_1, \dots, X_i, \dots, X_j, \dots, X_n) &= (X_1, \dots, -X_i, \dots, -X_j, \dots, X_n) \\ R_{i,j}^{\frac{2\pi}{4}}(X_1, \dots, X_i, \dots, X_j, \dots, X_n) &= (X_1, \dots, -X_j, \dots, X_i, \dots, X_n). \end{aligned} \tag{3.3}$$

Next, we compute the gradient of the geodesic normal coordinates (3.2). This result is used in the proof of the main result.

Lemma 3.3. *For $M = \mathbb{S}^n$ or $M = \mathbb{H}^n$, let $\{X_i : i = 1, 2, \dots, n\}$ be as given in (3.2). Let $L_M \le \frac{\pi}{2}$, if $M = \mathbb{S}^n$. Then for any smooth function $g : [0, L_M) \to \mathbb{R}$ and for $1 \le i < j \le n$, the followings hold:*

$$\left\langle \nabla\left(g(r)X_i\right), \nabla\left(g(r)X_j\right)\right\rangle = \frac{\left(rg'(r) + g(r)\right)^2}{r^2} X_iX_j - \frac{g^2(r)}{\sin_M^2(r)} X_iX_j, \tag{3.4}$$

$$\left|\nabla\left(g(r)\frac{X_i}{r}\right)\right|^2 = \left(g'(r)\right)^2 \left(\frac{X_i}{r}\right)^2 + \frac{g^2(r)}{\sin_M^2(r)}\left(1 - \left(\frac{X_i}{r}\right)^2\right). \tag{3.5}$$

Proof. First we consider $M = \mathbb{S}^n$. In view of Remark 3.1, we replace r with ϕ_1 and do the calculations. By a straightforward calculation, we can see that for $i \le n-1$,

$$\begin{aligned} \nabla\left(g(\phi_1)X_i\right) = {} & \frac{d}{d\phi_1}\left(\phi_1 g(\phi_1)\right)\frac{X_i}{\phi_1}\frac{\partial}{\partial \phi_1} + \sum_{k=2}^{i} g(\phi_1)\frac{\cos(\phi_k)}{\prod_{l=1}^{k-1}\sin^2(\phi_l)}\frac{X_i}{\sin(\phi_k)}\frac{\partial}{\partial\phi_k} \\ & - g(\phi_1)\frac{\sin(\phi_{i+1})}{\prod_{l=1}^{i}\sin^2(\phi_l)}\frac{X_i}{\cos(\phi_{i+1})}\frac{\partial}{\partial\phi_{i+1}}. \end{aligned} \tag{3.6}$$

For $i = n$,

$$\nabla\left(g(\phi_1)X_n\right) = \frac{d}{d\phi_1}\left(\phi_1 g(\phi_1)\right)\frac{X_n}{\phi_1}\frac{\partial}{\partial\phi_1} + \sum_{k=2}^{n} g(\phi_1)\frac{\cos(\phi_k)}{\prod_{l=1}^{k-1}\sin^2(\phi_l)}\frac{X_n}{\sin(\phi_k)}\frac{\partial}{\partial\phi_k}. \tag{3.7}$$

From (3.6) and (3.7), we conclude that for $1 \le i < j \le n$,

$$\begin{aligned} \left\langle \nabla\left(g(\phi_1)X_i\right), \nabla\left(g(\phi_1)X_j\right)\right\rangle = {} & \left(\frac{d}{d\phi_1}\left(\phi_1 g(\phi_1)\right)\right)^2 \frac{X_iX_j}{\phi_1^2} + \sum_{k=2}^{i} g^2(\phi_1)\frac{\cos^2(\phi_k)}{\prod_{l=1}^{k-1}\sin^2(\phi_l)}\frac{X_iX_j}{\sin^2(\phi_k)} \\ & - g^2(\phi_1)\frac{X_iX_j}{\prod_{l=1}^{i}\sin^2(\phi_l)}, \end{aligned}$$

$$= \left(\frac{d}{d\phi_1}\left(\phi_1 g(\phi_1)\right)\right)^2 \frac{X_iX_j}{\phi_1^2} + \sum_{k=2}^{i-1} g^2(\phi_1)\frac{\cos^2(\phi_k)}{\prod_{l=1}^{k-1}\sin^2(\phi_l)}\frac{X_iX_j}{\sin^2(\phi_k)}$$
$$- g^2(\phi_1)\frac{X_iX_j}{\prod_{l=1}^{i-1}\sin^2(\phi_l)},$$
$$= \left(\frac{d}{d\phi_1}\left(\phi_1 g(\phi_1)\right)\right)^2 \frac{X_iX_j}{\phi_1^2} - g^2(\phi_1)\frac{X_iX_j}{\sin^2(\phi_1)}.$$

Using similar calculations, we can establish (3.4) for $M = \mathbb{H}^n$, and (3.5) for both $M = \mathbb{S}^n$ and $M = \mathbb{H}^n$. □

Next, we prove some orthogonality results of test functions which are crucial for proving the main result.

3.3. Orthogonality of test functions

The orthogonality of test functions in $L^2(\Omega)$ and $H^1(\Omega)$ for $\Omega \subset \mathbb{R}^n$ has been proved in [2]. In the following proposition, we generalize this result for $\Omega \subset \mathbb{S}^n$ and $\mathbb{H}^n$.

Proposition 3.4. *Let Ω be a bounded domain in M and $\{X_i : i = 1, 2, \dots, n\}$ be as given in (3.2). Let $g : [0, \infty) \to \mathbb{R}$ be any smooth function. Then for any $i, j \in \{1, 2, \dots, n\}$ with $i \neq j$ and $m \in \mathbb{N} \cup \{0\}$, the following assertions hold:*

(i) If Ω is centrally symmetric, then

$$\int_\Omega g(r)X_iX_j^{2m}\, dV_X = 0 \text{ and } \int_\Omega g(r)X_i^{2m+1}\, dV_X = 0.$$

(ii) If $n \geq 3$ and Ω is symmetric of order 2, then

$$\int_\Omega g(r)X_iX_j^{m}\, dV_X = 0 \text{ and } \int_\Omega g(r)X_i^{2m+1}\, dV_X = 0.$$

(iii) If Ω is symmetric of order 4, then

$$\int_\Omega g(r)X_iX_j\, dV_X = 0.$$

(iv) If Ω is symmetric of order 4, then there exist constants A_1, A_2 such that

$$\int_\Omega g(r)X_i^2\, dV_X = A_1 \text{ and } \int_\Omega g(r)X_i^4\, dV_X = A_2 \text{ for all } i \in \{1, 2, \dots, n\}.$$

(v) If Ω is symmetric of order 4, then

$$\int_\Omega \left\langle \nabla\left(g(r)X_i\right), \nabla\left(g(r)X_j\right)\right\rangle dV_X = 0.$$

Proof. (i) For $X \in T_p(M)$. Using the transformation $Y = -X$, we obtain

$$\int_\Omega g(r)Y_iY_j^{2m}\, dV_Y = -\int_\Omega g(r)X_iX_j^{2m}\, dV_X \text{ and } \int_\Omega g(r)Y_i^{2m+1}\, dV_Y = \int_\Omega g(r)X_i^{2m+1}\, dV_X.$$

This implies

$$\int_\Omega g(r) X_i X_j^{2m} \, dV_X = 0 \text{ and } \int_\Omega g(r) X_i^{2m+1} \, dV_X = 0,$$

which proves our claim.

(ii) Since $n \geq 3$, choose k such that $k \neq i, j$. Then using the transformation

$$Y = R_{i,k}^{\frac{2\pi}{2}} X \; ((Y_1, \ldots, Y_i, \ldots, Y_k, \ldots, Y_n) = (X_1, \ldots, -X_i, \ldots, -X_k, \ldots, X_n)),$$

we get

$$\int_\Omega g(r) Y_i Y_j^m \, dV_Y = -\int_\Omega g(r) X_i X_j^m \, dV_X \text{ and } \int_\Omega g(r) Y_i^{2m+1} \, dV_Y = -\int_\Omega g(r) X_i^{2m+1} \, dV_X.$$

Hence the desired result follows.

(iii) The transformation

$$Y = R_{i,j}^{\frac{2\pi}{4}} X \; ((Y_1, \ldots, Y_i, \ldots, Y_j, \ldots, Y_n) = (X_1, \ldots, -X_j, \ldots, X_i, \ldots, X_n))$$

yields

$$\int_\Omega g(r) Y_i Y_j \, dV_Y = -\int_\Omega g(r) X_i X_j \, dV_X.$$

This implies the desired expression.

(iv) For any $i \neq 1$, applying the transformation

$$Y = R_{1,i}^{\frac{2\pi}{4}} X \; ((Y_1, \ldots, Y_i, \ldots, Y_n) = (-X_i, \ldots, X_1, \ldots, X_n)),$$

for all $i \in \{2, \ldots, n\}$, we can write

$$\int_\Omega g(r) X_i^2 \, dV_X = \int_\Omega g(r) Y_1^2 \, dV_Y,$$

$$\int_\Omega g(r) X_i^4 \, dV_X = \int_\Omega g(r) Y_1^4 \, dV_Y.$$

This provide the required constants $A_1 = \int_\Omega g(r) Y_1^2 \, dV_Y$ and $A_2 = \int_\Omega g(r) Y_1^4 \, dV_Y$.

(v) This follows easily from Lemma 3.3. □

4. Proof of Theorem 1.4 and a few remarks

We consider the Rayleigh quotient,

$$\mathcal{R}_\Omega(u) = \frac{\int_\Omega |\nabla u|^2 dV}{\int_\Omega u^2 dV}, \; u \in H^1(\Omega) \setminus \{0\}.$$

By the Courant-Fischer minimax formula, the Neumann eigenvalues of (1.1) are characterized by the following variational formula

$$\mu_{j+1}(\Omega) = \min_{E \in \mathcal{H}_j} \max_{0 \neq u \in E} \mathcal{R}_\Omega(u), \tag{4.1}$$

where $\mathcal{H}_j$ is the set of all j-dimensional subspaces of the Sobolev space $H^1(\Omega)$ that are orthogonal to the subspace of constant functions. In particular, the first positive Neumann eigenvalue $\mu_2(\Omega)$ is given by

$$\mu_2(\Omega) = \min_{u \in H^1(\Omega) \setminus \{0\}} \left\{ \mathcal{R}_\Omega(u) \middle| \int_\Omega u \, dV = 0 \right\}. \tag{4.2}$$

4.1. Proof of (1.8)

For $r > R_1$, let $G(r) = G_1(r)$, where G_1 is defined as in Proposition 2.7. Now by Proposition 3.4, we have

$$\int_\Omega \frac{G(r)}{r} X_i \, dV = 0 \text{ for all } i = 1, 2, \dots, n,$$

where $X_1, X_2, \dots, X_n$ are the geodesic normal coordinates mentioned in (3.2). Thus using $\frac{G(r)}{r} X_i$ as a test function in (4.2) and summing over $i = 1, 2, \dots, n$, we get

$$\mu_2(\Omega) \sum_{i=1}^n \int_\Omega \left(\frac{G(r)}{r} X_i \right)^2 dV \leq \sum_{i=1}^n \int_\Omega \left| \nabla \left(\frac{G(r)}{r} X_i \right) \right|^2 dV. \tag{4.3}$$

Using Lemma 3.3 and the fact $\sum_{i=1}^n X_i^2 = r^2$, we obtain

$$\sum_{i=1}^n \left| \nabla \left(\frac{G(r)}{r} X_i \right) \right|^2 = (G'(r))^2 + \frac{n-1}{\sin_M^2(r)} G^2(r).$$

Now from (4.3) and Proposition 2.7 we get

$$\mu_2(\Omega) \leq \frac{\int_\Omega \left((G'(r))^2 + \frac{n-1}{\sin_M^2(r)} G^2(r) \right) dV}{\int_\Omega G^2(r) dV} \leq \mu_{1,1}.$$

Since $\mu_{1,1} = \mu_2(B_{R_2} \setminus \overline{B}_{R_1})$, we get the desired result.

4.2. Proof of (1.9)

In view of Remark 2.6, it is enough to show that $\mu_{n+1}(\Omega) \leq \mu_{1,1}$. For this, consider the n-dimensional subspace of $H^1(\Omega)$

$$E = \text{ span } \left\{ \frac{G(r)}{r} X_i \;\middle|\; i = 1, 2, \dots, n \right\}.$$

By Proposition 3.4, for $i, j = 1, 2, \dots, n$ with $i \neq j$, we have

$$\begin{aligned} \int_\Omega \frac{G(r)}{r} X_i \, dV = 0, \\ \int_\Omega \left(\frac{G(r)}{r} X_i\right)\left(\frac{G(r)}{r} X_j\right) dV = 0, \\ \int_\Omega \left\langle \nabla\left(\frac{G(r)}{r} X_i\right), \nabla\left(\frac{G(r)}{r} X_j\right)\right\rangle dV = 0. \end{aligned} \tag{4.4}$$

Moreover, there exist constants C and D such that, for all $i = 1, 2, \ldots, n$,

$$\begin{aligned} &\int_\Omega \left(\frac{G(r)}{r} X_i\right)^2 dV = C, \\ &\int_\Omega \left|\nabla\left(\frac{G(r)}{r} X_i\right)\right|^2 dV = \int_\Omega \left[(G'(r))^2 \left(\frac{X_i}{r}\right)^2 + \frac{G^2(r)}{\sin_M^2(r)}\left(1 - \left(\frac{X_i}{r}\right)^2\right)\right] dV = D, \end{aligned} \tag{4.5}$$

where the second equality follows from Lemma 3.3. Now, by summing up the above equations over i, we get

$$C = \frac{1}{n}\int_\Omega G^2(r)\, dV, \qquad D = \frac{1}{n}\int_\Omega \left((G'(r))^2 + \frac{n-1}{\sin_M^2(r)} G^2(r)\right) dV. \tag{4.6}$$

Let $u \in E \setminus \{0\}$. Then there exists $c = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n \setminus \{0\}$ such that $u = \sum_{i=1}^n c_i \frac{G(r)}{r} X_i$. Now, using (4.4) and (4.5) we obtain:

$$\mathcal{R}_\Omega(u) = \frac{\int_\Omega |\nabla u|^2\, dV}{\int_\Omega u^2\, dV} = \frac{\int_\Omega \left|\sum_{i=1}^n c_i \nabla\left(\frac{G(r)}{r} X_i\right)\right|^2 dV}{\int_\Omega \left(\sum_{i=1}^n c_i \frac{G(r)}{r} X_i\right)^2 dV} = \frac{\sum_{i=1}^n c_i^2 \int_\Omega \left|\nabla\left(\frac{G(r)}{r} X_i\right)\right|^2 dV}{\sum_{i=1}^n c_i^2 \int_\Omega \left(\frac{G(r)}{r} X_i\right)^2 dV} = \frac{D}{C}.$$

Therefore, from (4.6) and Proposition 2.7, we conclude that

$$\mathcal{R}_\Omega(u) = \frac{\int_\Omega \left((G'(r))^2 + \frac{n-1}{\sin_M^2(r)} G^2(r)\right) dV}{\int_\Omega G^2(r)\, dV} \le \mu_{1,1} = \mu_2(B_{R_2} \setminus \overline{B}_{R_1}), u \in E \setminus \{0\}.$$

Thus (4.1) yields:

$$\mu_{n+1}(\Omega) \le \max_{u \in E \setminus \{0\}} \mathcal{R}_\Omega(u) \le \mu_2(B_{R_2} \setminus \overline{B}_{R_1}).$$

Remark 4.1. Our proof also work for $M = \mathbb{R}^n$ by considering standard cartesian coordinates as normal coordinates centered at the origin with Riemannian metric $g = dr^2 + r^2 g_{\mathbb{S}^{n-1}}$.

Remark 4.2. It is natural to anticipate the main Theorem 1.4 for symmetric domains in smooth Riemannian manifold $M = [0, R) \times \mathbb{S}^{n-1}$ equipped with the warped product metric $g = dr^2 + h^2(r) g_{\mathbb{S}^{n-1}}$. Here function $h \in C^\infty([0, R))$, $h(r) > 0$ for $r \in (0, R)$, $h'(0) = 1$ and $h^{(2k)}(0) = 0$ for all integers $k \ge 0$. For such manifolds

$$\mathrm{tr}(\mathrm{A(r)}) = \frac{1}{h^{n-1}(r)} \frac{d}{dr}\left(h^{n-1}(r)\right) \text{ and } \lambda_1(S(r)) = \frac{n-1}{h^2(r)}$$

Thus, for proving Lemma 2.4, we need $(\mathrm{tr}(A(r)))' = -\lambda_1(S(r))$. This is true if and only if

$$h(r)h''(r) - h'^2(r) = -1,$$

and the only solution of this ODE are $h(r) = r, \sin r$ and $\sinh r$. Thus our proof works only for $h(r) = r, \sin r$ and $\sinh r$.

Remark 4.3. Next we list some related open problems.

(1) For $M = \mathbb{S}^n$, we proved the main result only for then domains contained in the hemisphere. Is it possible to extend the main results for domains that are not contained in the hemisphere?
(2) Rank-1 symmetric spaces naturally generalize the space forms as the isometry group acts transitively on the unit tangent bundle. Thus establishing Szegö-Weinberger inequality for the higher Neumann eigenvalues for the bounded domains in rank-1 symmetric spaces and also in manifolds with bounded curvature seems to be some interesting problems. The challenging part, in these cases, is to find the appropriate geodesic normal coordinates.

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