Chapter 6

Stability analysis of delayed neural network using new delay-product based functionals

6.1 Introduction

Neural networks have been successfully applied in many practical applications such as associative memory, signal processing and optimization problems [121, 126]. Generally, the neural networks are implemented by integrated electronic circuits that contains operational amplifiers. Due to the communication speed between neurons and limited switching speed of the amplifiers, time-delay often occurs in its implementation that deteriorate performance and even causes instability. Therefore, stability analysis of Delayed Neural Network (DNN) is an important issue in the study of performance of neural networks.

The Lyapunov-Krasovskii theory is often used for delay-dependent stability analysis of neural networks with time varying delay. A number of stability criteria have been proposed in the terms of Linear-Matrix Inequalities (LMIs) [122, 125]. The primary focus in these approaches is to develop stability criteria by ensuring derivative of the LKF to be negative definite. An important index in the measurement of conservativeness of these criterion is the Largest Admissible Upper Bound (LAUB) of time-varying delay. To get LAUB of delay for DNN, two main issues are encountered to find a precise bound of the integral function in derivative of LKF and to construct a suitable LKF. However, these two issues are complementary because the structure of LKF depends on states present in the integral function.

Regarding the second issue bounding of integral function, Jensen's inequality [1] has been extremely used. Later on many other inequalities, such as, Wirtinger inequality [77], Auxiliary Function Inequality [78], Bessel-Legendre Inequality (BLI) [94] and Free-Matrix Inequality [81] have been introduced.

To obtain less conservative results for stability analysis of DNN choice of suitable LKFs is crucial. In the last few years, many LKFs have been proposed, such as augmented LKF [124], delay-partition LKF [140], multiple integral LKF [139], piecewise LKF [133] and other functionals like [85, 128, 132, 141]. For the stability analysis of DNN, some of these are extensively used. Still there are scope to constitute new types of LKFs to improve the existing results and to obtain largest upper bound of delay.

To this end, in order to exploit the information of delays and its derivative, in defining LKFs known as delay-product-type functional (DPF) [138] has been introduced, which contains delay as a co-efficient in the non-integral quadratic terms. It has been extended in [90,91] to construct DPFs by using double integral of the states introduced in AFI based analysis. In [120], a DPF has been formulated using matrix based functions. In [92], new DPFs have been proposed by modifying WI and FMI to exploit the advantages of single integral state vectors. Further, the concept of DPFs has been extended to systems with two additive time varying delay in [144] and [135]. In [113, 114], a zero equality has been used to get less conservative H_∞ performance. Extending this idea, in [127] and [131], a flexible LKF containing more states has been proposed and the effectiveness of the zero equality to obtain less conservative results has been demonstrated.

This chapter considers development of new DPFs. Main contributions of this chapter are on the following observations and corresponding developments.

1. The second order BLI introduces two new states, which are the combination of interval normalized single and double integrals.These states have been included in the augmented vectors of the DPF and Lyapunov matrix based quadratic terms.

2. Further, an extended DPF is formulated by including single integral states and its interval normalized forms. This facilitates construction and incorporation of a new zero equality so that it helps to reduce conservatism by increasing the feasibility region. In addition, the time-derivative of this DPF introduces product of delay and its derivative dependent terms and exploit the same to obtain better results.

On the basis of these two proposed DPFs, improved stability criteria for DNN are derived. The improvement in terms of LAUB of delay is demonstrated by considering numerical examples that shows the proposed criterion less conservative even though less number of variables are to solved in the LMIs.

6.2 System Description and Preliminaries

A Neural network having equilibrium point shifted into origin can be expressed as

$$\dot{x}(t) = -Ax(t) + A_1 f(Wx(t)) + A_2 f(Wx(t - \tau(t))) \tag{6.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector representing n number of neurons; $A = diag\{a_1, ..., a_{n-1}, a_n\} > 0$, A_1, A_2 and $W \in \mathbb{R}^{n\times n}$ are the known interconnecting weight matrices for neurons; $\tau(t)$ denotes delay function with the following properties:

$$h \geq \tau(t) \geq 0, \qquad \mu_2 \geq \dot{\tau}(t) \geq -\mu_1 \tag{6.2}$$

where μ_1, μ_2 and h are real scalars greater than zero.
The activation function of neuron is defined by $f(x(t)) = \{f(x_1(t)), ..., f(x_{n-1}(t)), f(x_n(t))\}$ and satisfies

$$\sigma_i^- \leq \frac{f_i(p_1) - f_i(p_2)}{p_1 - p_2} \leq \sigma_i^+, p_1 \neq p_2, (i = 1, 2, ..., n) \tag{6.3}$$

where σ_i^+ and σ_i^- are real scalars with known values. In this chapter, we represent $\Sigma_1 = diag\{\sigma_1^-, ..., \sigma_n^-\}$ and $\Sigma_2 = diag\{\sigma_1^+, ..., \sigma_n^+\}$.

The aim of this chapter is to construct new LKFs to obtain improved stability criteria for DNN of (6.1). To obtain the stability results, some lemmas are required for the derivation that are recalled as follows.

Lemma 12 *[82] For matrices $X_i > 0$ and $S_i, i = 1, 2,$ positive scalars α, β related by $\alpha + \beta = 1$, the following holds:*

$$\begin{bmatrix} \frac{1}{\alpha}X_1 & 0 \\ 0 & \frac{1}{\beta}X_2 \end{bmatrix} \geq \begin{bmatrix} (1+\beta)X_1 - T_1 & \beta S_1 + \alpha S_2 \\ * & (1+\alpha)X_2 - T_2 \end{bmatrix} \tag{6.4}$$

where $T_1 = \beta S_2 R_2^{-1} S_2^T$ and $T_2 = \alpha S_1^T R_1^{-1} S_1$

Lemma 13 *[94] For any constant matrix 0≤R and continuous function $w \in [a,b] \to \mathbb{R}^n$, such that following holds:*

$$\int_a^b \dot{w}^T(s)R\dot{w}(s)ds \geq \frac{1}{(b-a)}\sum_{q=0}^{N}(2q+1)\theta_q^T R\theta_q] \tag{6.5}$$

In inequality (6.1), when $q = 0, 1, 2$ it represents JI, WI and second order BLI respectively. Note that, to reduce conservatism an additional term has been introduced and each term and each one also introduces a new higher order state. For q=0,1,2 one can write θ_q as $\theta_0 = w(b) - w(a), \theta_1 = w(a) + w(b) - \frac{2}{(b-a)}\int_a^b w(s)ds,$
$\theta_2 = \theta_1 - \frac{6}{(b-a)}\int_a^b \delta_{b,a}(s)w(s)ds,$ *and* $\delta_{b,a}(s) = 2\left(\frac{s-a}{b-a}\right) - 1.$

The state vector $\frac{1}{(b-a)}\int_a^b \delta_{b,a}(s)\omega(s)ds$ in θ_2 has been incorporated as an additional state to overcome the conservatism provided by Wirtinger based inequality. In this work, this state is utilized as augmented states in Lyapunov matrix related quadratic term and also in the formulation of delay-product type functionals for the new LKFs defined later.

Following the definition of the $\delta_{b,a}(s)$ in Lemma 13, two limiting functions $\delta_1(s)$ and $\delta_2(s)$ for the intervals $[t, t-\tau(t)]$ and $[t-\tau(t), t-h]$, respectively, are defined as:

$$\delta_1(s) = 2\left(\frac{s+\tau(t)}{\tau(t)}\right) - 1,\ \delta_2(s) = 2\left(\frac{s+h}{h-\tau(t)}\right) - 1$$

Further, for simplified representation, the following notations are used in this paper.

$$\begin{cases} h_\tau(t) = h - \tau(t) & x_\tau(t) = x(t-\tau(t)), \\ x_h(t) = x(t-h), & \tilde{\tau}(t) = 1 - \dot{\tau}(t) \\ w_1(t) = \int_{-\tau(t)}^{0} x_t(s)ds, & w_2(t) = \int_{-h}^{-\tau(t)} x_t(s)ds, \\ w_3(t) = \int_{-\tau(t)}^{0} \delta_1(s)x_t(s)ds, & w_4(t) = \int_{-h}^{-\tau(t)} \delta_2(s)x_t(s)ds \end{cases}$$

6.3 Main Results

In this section two new DPFs are presented. Then using these DPFs new stability criteria are derived that are the main results of this work.

6.3.1 New delay-product type functionals

In WI two new states in the form of interval normalized single integral have been introduced. These states are utilized to formulate DPF with co-efficient $\tau(t)$ and $h_\tau(t)$

in [88, 138] to exploit the relationship between time-delay and its range. By extending this idea, the DPFs of [90] includes double integral states introduced in second order BLI. In [91], two types of DPFs are introduced that contain the following terms:

$$\bar{V}_a(t) = x^T(t)[\tau(t)Y_1 + h_\tau(t)Y_2]x(t) + \theta^T(t)Y_3\theta(t)$$

$$and$$

$$\bar{V}_b(t) = \begin{bmatrix} x(t) & \theta(t) \end{bmatrix} [\tau(t)Y_1 + h_\tau(t)Y_2] \begin{bmatrix} x(t) & \theta(t) \end{bmatrix}$$

where $\theta(t) = Col\{\int_{t-\tau(t)}^{t} x(s)ds, \int_{t-\tau(t)}^{t} \frac{(t-s)}{\tau(t)} x(s)ds, \int_{t-h}^{t-\tau(t)} x(s)ds, \int_{t-h}^{t-\tau(t)} \frac{(t-\tau(t)-s)}{h_\tau(t)} x(s)ds\}$, with $Y_j, Z_k > 0, (j = 1, 2, 3$ and $k = 1, 2)$.

It is observed that the states $x(t)$ and $\theta(t)$ in $\bar{V}_a(t)$ are independent, whereas these states are augmented in $\bar{V}_b(t)$ leading to less conservative results.

On basis of these DPFs, by using the states of (6.5) we introduce two delay-product type functional candidates as:

$$V_a(x_t) = \eta_0^T(t)P\eta_0(t) + \tau(t)\eta_1^T(t)P_1\eta_1(t) + h_\tau(t)\eta_2^T(t)P_2\eta_2(t) \tag{6.6}$$

$$V_b(x_t) = \eta_0^T(t)P\eta_0(t) + \tau(t)\tilde{\eta}_1^T(t)\tilde{P}_1\tilde{\eta}_1(t) + h_\tau(t)\tilde{\eta}_2^T(t)\tilde{P}_2\tilde{\eta}_2(t) \tag{6.7}$$

where

$$\begin{cases} \eta_0(t) = col\{x(t), x_\tau(t), w_1(t), w_2(t), w_3(t), w_4(t)\} \\ \eta_1(t) = col\{x(t), x_\tau(t), \frac{1}{\tau(t)}w_1(t), \frac{1}{\tau(t)}w_3(t)\} \\ \eta_2(t) = col\{x(t), x_\tau(t), \frac{1}{h_\tau(t)}w_2(t), \frac{1}{h_\tau(t)}w_4(t)\} \\ \tilde{\eta}_1(t) = col\{\eta_1(t), w_1(t)\}, \ \tilde{\eta}_2(t) = col\{\eta_2(t), w_2(t), \} \end{cases}$$

The DPFs formulated in (6.6) and (6.7) incorporate two new states $\frac{1}{\tau(t)}w_3(t)$ and $\frac{1}{h_\tau(t)}w_4(t)$. In addition, the DPF (6.7) contains two additional single integral states. This type of augmentation was considered first in [131] and its ability to yield better results is shown therein. The advantage of these proposed LKFs are two folds: (i) Reconstruction of the DPF, Lyapunov matrix based term and integral inequality using new combined states. (ii) Including both single integral states and its normalized form, new zero-equalities can be constructed.

6.3.2 Stability analysis

This section presents new stability criteria for (6.1) by incorporating DPFs (6.6) and (6.7) in formulation of the LKF in following Theorems. First, the DPF (6.6) is included in the formation of LKF in the following Theorem.

Theorem 8 *DNN (6.1) is asymptotically stable if there exist* $0 < P \in \mathbb{R}^{6n\times 6n}$, $0 < P_i \in \mathbb{R}^{4n\times 4n}$, $0 < Q_1 \in \mathbb{R}^{3n\times 3n}$, $0 < Q_2 \in \mathbb{R}^{2n\times 2n}$, $0 < R \in \mathbb{R}^{n\times n}$, *diagonal matrices* $0 < L_k, M_j, N_j \in \mathbb{R}^{n\times n}$ *and any matrices* $U_i \in \mathbb{R}^{3n\times 3n}$ *for i=1,2; j=1,2,3 and k=1,2,...,6 such that:*

$$\begin{bmatrix} \Upsilon(0,\mu_i) & E_1^T U_2 \\ * & -\tilde{R} \end{bmatrix} < 0 \tag{6.8}$$

$$\begin{bmatrix} \Upsilon(h,\mu_i) & E_2^T U_1 \\ * & -\tilde{R} \end{bmatrix} < 0 \tag{6.9}$$

where

$$\Upsilon(\dot{\tau}(t),\tau(t)) = \Phi_0(\dot{\tau}(t),\tau(t)) + \Phi_1(\dot{\tau}(t)) + \Phi_2(\dot{\tau}(t)) + \Phi_3(\tau(t)) + \Phi_4 \tag{6.10}$$

$$\Phi_0(\tau(t),\dot{\tau}(t)) = Sym\{\mathit{\Pi}_0^T P \mathit{\Pi}_1\} + \dot{\tau}(t)\mathit{\Pi}_2^T P_1 \mathit{\Pi}_2 + Sym\{\mathit{\Pi}_2^T P_1 \mathit{\Pi}_3\} - \dot{\tau}(t)\mathit{\Pi}_4^T P_2 \mathit{\Pi}_4$$
$$+ Sym\{\mathit{\Pi}_4^T P_2 \mathit{\Pi}_5\} \tag{6.11}$$

$$\Phi_1(\dot{\tau}(t)) = \mathit{\Pi}_6^T Q_1 \mathit{\Pi}_6 - \tilde{\tau}(t)\mathit{\Pi}_7^T Q_1 \mathit{\Pi}_7 + \tilde{\tau}(t)\mathit{\Pi}_8^T Q_1 \mathit{\Pi}_8 - \mathit{\Pi}_9^T Q_2 \mathit{\Pi}_9 \tag{6.12}$$

$$\Phi_2(\dot{\tau}(t)) = Sym\{[(e_{10} - \mathit{\Sigma}_1 W e_1)^T L_1 + (\mathit{\Sigma}_2 W e_1 - e_{10})^T L_2] W e_s$$
$$+ [\tilde{\tau}(t)(e_{11} - \mathit{\Sigma}_1 W e_2)^T L_3 + \tilde{\tau}(t)(\mathit{\Sigma}_2 W e_2 - e_{11})^T L_4] W e_5$$
$$+ [(e_{12} - \mathit{\Sigma}_1 W e_3)^T L_5 + (\mathit{\Sigma}_2 W e_3 - e_{12})^T L_6] W e_6\} \tag{6.13}$$

$$\Phi_3(\tau(t)) = h^2 e_s^T R e_s - E_1^T[\beta U_1 + \alpha U_2]E_2 - E_1^T(1+\beta)\tilde{R}E_1 - E_2^T(1+\alpha)\tilde{R}E_2 \tag{6.14}$$

$$\Phi_4 = \sum_{i=1}^{3} Sym\{(e_{8+i} - \mathit{\Sigma}_1 W e_i)^T \times M_i(\mathit{\Sigma}_2 W e_i - e_{8+i})\}$$
$$+ \sum_{i=1}^{2} Sym\{[(e_{8+i} - e_{9+i}) - \mathit{\Sigma}_1 W(e_i - e_{1+i})]^T N_i \times [\mathit{\Sigma}_2 W(e_i - e_{1+i}) - (e_{8+i} - e_{9+i})]\}$$
$$+ Sym\{[(e_9 - e_{11}) - \mathit{\Sigma}_1 W(e_1 - e_3)]^T N_3 \times [\mathit{\Sigma}_2 W(e_1 - e_3) - (e_9 - e_{11})]\} \tag{6.15}$$

The column vectors $\mathit{\Pi}_i, i = 1, 2, ..., 9$ utilized in the above expressions can be defined as

$$\mathit{\Pi}_0 = [e_1^T,\ e_2^T,\ \tau(t)e_5^T,\ h_\tau(t)e_6^T,\ \tau(t)e_7^T,\ h_\tau(t)e_8^T]^T$$

$$\mathit{\Pi}_1 = [e_s^T,\ \tilde{\tau}(t)e_4^T,\ e_1^T - \tilde{\tau}(t)e_2^T,\ \tilde{\tau}(t)e_2^T - e_3^T, e_1^T + \tilde{\tau}(t)e_2^T - (1+\tilde{\tau}(t))e_5^T - \dot{\tau}(t)e_7^T,$$
$$\tilde{\tau}(t)e_2^T + e_3^T - (1+\tilde{\tau}(t))e_6^T + \dot{\tau}(t)e_8^T]^T$$

$$\mathit{\Pi}_2 = [e_1^T, e_2^T,\ e_5^T,\ e_7^T]^T, \mathit{\Pi}_4 = [e_1^T, e_2^T, e_6^T, e_8^T]^T$$

$$\mathit{\Pi}_3 = [\tau(t)e_s^T,\ \tilde{\tau}(t)\tau(t)e_4^T, e_1^T - \tilde{\tau}(t)e_2^T - \dot{\tau}(t)e_5^T, e_1^T + \tilde{\tau}(t)e_2^T - (1+\tilde{\tau}(t))e_5^T - 2\dot{\tau}(t)e_7^T]^T$$

$$\Pi_5 = [h_\tau(t)e_s^T, h_\tau(t)\tilde{\tau}(t)e_4^T, \tilde{\tau}(t)e_2^T - e_3^T + \dot{\tau}(t)e_6^T, \tilde{\tau}(t)e_2^T + e_3^T - (1+\tilde{\tau}(t))e_6^T + 2\dot{\tau}(t)e_8^T]^T$$

$$\Pi_6 = [e_1^T,\ e_s^T,\ e_9^T]^T, \qquad \Pi_7 = [e_2^T,\ e_4^T,\ e_{10}^T]^T,$$

$$\Pi_8 = [e_2^T,\ e_{10}^T]^T, \qquad \Pi_9 = [e_3^T,\ e_{11}^T]^T$$

$$E_1 = Col\{e_1 - e_2,\ e_1 + e_2 - 2e_5,\ e_1 - e_2 - 6e_7\}$$

$$E_2 = Col\{e_2 - e_3,\ e_2 + e_3 - 2e_6,\ e_2 - e_3 - 6e_8\} \tag{6.16}$$

$$\zeta_1(t) = col\{x(t), x_\tau(t), x_h(t), \dot{x}_\tau(t), \frac{1}{\tau(t)}w_1(t), \frac{1}{h_\tau(t)}w_2(t), \frac{1}{\tau(t)}w_3(t), \frac{1}{h_\tau(t)}w_4(t), f(Wx(t)),$$
$$f(Wx_\tau(t)), f(Wx_h(t)), \dot{x}_h(t)\}$$

$$e_s = Ae_1 + A_1e_9 + A_2e_{10}$$

with $e_1, e_2, ..., e_{11} \in \mathbb{R}^{n\times 11n}$ are basic block matrices. **Proof :** Consider the LKF as:

$$V(t) = V_a(t) + \sum_{i=0}^{2} V_i(t) \tag{6.17}$$

where $V_a(t)$ is defined in (6.6), and

$$V_0(t) = \int_{t-\tau(t)}^{t} \eta_3^T(s)Q_1\eta_3(s)ds + \int_{t-h}^{t-\tau(t)} \eta_4^T(s)Q_2\eta_4(s)ds$$

$$V_1(t) = 2\sum_{i=1}^{n}\int_0^{W_ix(t)} [l_{1i}f_i^-(s) + l_{2i}f_i^+(s)]ds$$
$$+ 2\sum_{i=1}^{n}\int_0^{W_ix_d(t)} [l_{3i}f_i^-(s) + l_{4i}f_i^+(s)]ds$$
$$+ 2\sum_{i=1}^{n}\int_0^{W_ix_h(t)} [l_{5i}f_i^-(s) + l_{6i}f_i^+(s)]ds$$

$$V_2(t) = h\int_{-h}^{0}\int_{t+u}^{t} \dot{x}^T(s)R\dot{x}(s)dsdu$$

$$\eta_3(s) = col\{x(s), \dot{x}(s), f(Wx(s))\},\ \eta_4(s) = col\{x(s), f(Wx(s)), \dot{x}(s)\}$$

Next, time-derivative of the individual terms in (6.17) along the solution of DNN (6.1) are derived. One can write

$$\dot{V}_a(t) = \zeta_1^T(t)\Phi_0(\tau(t), \dot{\tau}(t))\zeta_1(t) \tag{6.18}$$

where $\Phi_0(\tau(t), \dot{\tau}(t))$ defined in (11). Similarly,

$$\dot{V}_0(t) = \zeta_1^T(t)\Phi_1(\dot{\tau}(t))\zeta_1(t) \tag{6.19}$$

$$\dot{V}_1(t) = \zeta_1^T(t)\Phi_2(\dot{\tau}(t))\zeta_1(t) \tag{6.20}$$

$$\dot{V}_2(t) = \zeta_1^T(t)h^2e_s^TRe_s\zeta_1(t) - h\int_{t-h}^{t} \dot{x}^T(s)R\dot{x}(s)ds \tag{6.21}$$

where $\Phi_1(\dot{\tau}(t))$ and $\Phi_2(\dot{\tau}(t))$ are defined in (6.12) and (6.13). An upper bound of the integral function (6.21) can be obtained by using the inequality (6.5) of Lemma 13 as:

$$-h\int_{t-h}^{t} \dot{x}^T(s)R\dot{x}(s)ds \leq -\zeta_1(t)^T\left(\frac{h}{\tau(t)}E_1^T\tilde{R}E_1 + \frac{h}{h_\tau(t)}E_2^T\tilde{R}E_2\right)\zeta_1(t) \tag{6.22}$$

where E_1, E_2 are defined in (16) and $\tilde{R} = diag\{R, 3R, 5R\}$. Now, one can use Lemma 1 with $\frac{\tau(t)}{h} = \alpha$, $\frac{h_\tau(t)}{h} = \beta$ on the right hand side (RHS) of (6.22) and substituting it into (6.21), one can write

$$\dot{V}_2(t) \leq \zeta_1^T(t)[\Phi_3(\tau(t)) + \Gamma(\tau(t))]\zeta_1(t) \tag{6.23}$$

where $\Phi_3(\tau(t))$ is defined in (14)

$$\Gamma(\tau(t)) = \beta E_1^T U_2 \tilde{R}^{-1} U_2^T E_1 + \alpha E_2^T U_1^T \tilde{R}^{-1} U_1 E_2 \tag{6.24}$$

From (6.3), with $M = diag\{m_1, m_2, ..., m_n\} \geq 0$ and $N = diag\{n_1, n_2, ..., n_n\} \geq 0$, the following inequalities hold for $s, s_1, s_2 \in \mathbb{R}$:

$$\Theta(s, M) \geq 0, \qquad \Psi(s_1, s_2, N) \geq 0 \tag{6.25}$$

where

$$\begin{aligned}
\Theta(s, M) &= 2[f(Wx(s)) - \Sigma_2 Wx(s)]^T \times M[\Sigma_1 Wx(s) - f(Wx(s))] \\
\Psi(s_1, s_2, N) &= 2[f(Wx(s_1)) - f(Wx(s_2)) \\
&\quad - \Sigma_2 W(x(s_1) - x(s_2))]^T \times N[\Sigma_1 W(x(s_1) - x(s_2)) - f(Wx(s_1)) + f(Wx(s_2))]
\end{aligned}$$

Now, by substituting $\Theta(s, M) = \Theta(t, M_1), \Theta(t-\tau(t), M_2), \Theta(t-h, M_3)$ and $\Psi(s_1, s_2, N) = \Psi(t, t-\tau(t), N_1), \Psi(t-\tau(t), t-h, N_2), \Psi(t, t-h, N_3)$ we have

$$\begin{aligned}
&\Theta(t, M_1) \geq 0, \Theta(t-\tau(t), M_2) \geq 0, \Theta(t-h, M_3) \geq 0 \\
&\Psi(t, t-\tau(t), N_1) \geq 0, \Psi(t-\tau(t), t-h, N_2) \geq 0 \\
&\Psi(t, t-h, N_3) \geq 0
\end{aligned}$$

Further, by addition of these inequalities one obtains

$$\zeta_1^T(t)\Phi_4\zeta_1(t) \geq 0 \tag{6.26}$$

where Φ_4 is defined in (6.15). Then, using (18), (6.19), (6.20), (6.23) and (6.26), one obtain

$$\dot{V}(t) \leq \zeta_1^T(t)[\Upsilon(\dot{\tau}(t), \tau(t)) + \Gamma]\zeta_1(t) \tag{6.27}$$

where $\Upsilon(\dot{\tau}(t), \tau(t))$ is defined in (10).

If the expression $\Upsilon(\dot{\tau}(t), \tau(t)) + \Gamma$, which is linear in both $\tau(t)$ and its variation with time is negative definite for all $\tau(t) \in [0, h]$ and $\dot{\tau}(t) \in [\mu_1, \mu_2]$, then $\dot{V}(t) < 0$. Finally, using Schur complement one can transform $\Upsilon(\dot{\tau}(t), \tau(t)) + \Gamma$ into LMIs (6.8) and (6.9). □

Remark 10 *As discussed in section 6.3.1, $V_a(t)$ in LKF (6.17) is newly formulated by introducing two new states. In addition, $V_0(t)$ is a cross-term functional based on the states $x(s), \dot{x}(s)$ and $f(Wx(s))$ so defined that, in the process of estimating bound of $\dot{V}_0(t)$, more information regarding activation function can be used. Also, additional information interms of the activation function in $\dot{V}(t)$ has been provided by the inequalities (6.25). In order to exploit the characteristics of activation function further $V_1(t)$ is constructed.*

Next, the DPF (6.7) is considered instead of (6.6). It may be noted that (6.7) contains more states as compared to (6.6) and hence introduces more LMI variables in the matrices of the functional. A new zero equality is used to take care of the new states.

Theorem 9 *DNN (6.1) is asymptotically stable if there exist $0 < P \in \mathbb{R}^{6n\times 6n}$, $0 < \tilde{P}_i \in \mathbb{R}^{5n\times 5n}$, $0 < Q_1 \in \mathbb{R}^{3n\times 3n}$, $0 < Q_2 \in \mathbb{R}^{2n\times 2n}$, $0 < R \in \mathbb{R}^{n\times n}$, diagonal matrices $L_k, M_j, N_j \in \mathbb{R}^{n\times n}$ and any matrices $U_i \in \mathbb{R}^{3n\times 3n}$, G_i of appropriate dimension with (i=1,2), (j=1,2), (k=1,2,...,6) satisfying the following LMIs:*

$$\begin{bmatrix} \tilde{\Upsilon}(0, \mu_i) & \tilde{E}_1^T U_2 \\ * & -\tilde{R} \end{bmatrix} < 0 \tag{6.28}$$

$$\begin{bmatrix} \tilde{\Upsilon}(h, \mu_i) & \tilde{E}_2^T U_1 \\ * & -h\tilde{R} \end{bmatrix} < 0 \tag{6.29}$$

where

$$\tilde{\Upsilon}(\dot{\tau}(t),\tau(t)) = \tilde{\Phi}_0(\dot{\tau}(t), \tau(t)) + \tilde{\Phi}_1(\dot{\tau}(t)) + \tilde{\Phi}_2(\dot{\tau}(t)) + \tilde{\Phi}_3(\tau(t)) + \tilde{\Phi}_4 \tag{6.30}$$

$$\begin{aligned}\tilde{\Phi}_0(\tau(t), \dot{\tau}(t)) = {} & Sym\{\tilde{\Pi}_0^T P \tilde{\Pi}_1\} + \dot{\tau}(t)\tilde{\Pi}_2^T \tilde{P}_1 \tilde{\Pi}_2 + Sym\{\tilde{\Pi}^T \tilde{P}_1 \tilde{\Pi}_3\} - \dot{\tau}(t)\tilde{\Pi}_4^T \tilde{P}_2 \tilde{\Pi}_4 \\ & + Sym\{\tilde{\Pi}_4^T \tilde{P}_2 \tilde{\Pi}_5\} + Sym\{\lambda_1 G_1 \wp_1 + \lambda_2 G_2 \wp_2\}\end{aligned} \tag{6.31}$$

$$\tilde{\Phi}_1(\dot{\tau}(t)) = \tilde{\Pi}_6^T Q_1 \tilde{\Pi}_6 - \tilde{\tau}(t)\tilde{\Pi}_7^T Q_1 \tilde{\Pi}_7 + \tilde{\tau}(t)\tilde{\Pi}_8^T Q_2 \tilde{\Pi}_8 - \tilde{\Pi}_9^T Q_2 \tilde{\Pi}_9 \tag{6.32}$$

$$\begin{aligned}\tilde{\Phi}_2(\dot{\tau}(t)) = {} & Sym\{[(\tilde{e}_9 - \Sigma_1 W \tilde{e}_1)^T L_1 + (\Sigma_2 W \tilde{e}_1 - \tilde{e}_9)^T L_2] W \tilde{e}_s \\ & + [\tilde{\tau}(t)(\tilde{e}_{10} - \Sigma_1 W \tilde{e}_2)^T L_3 + \tilde{\tau}(t)(\Sigma_2 W \tilde{e}_2 - \tilde{e}_{10})^T L_4] W \tilde{e}_5 \\ & + [(\tilde{e}_{11} - \Sigma_1 W \tilde{e}_3)^T L_5 + (\Sigma_2 W \tilde{e}_3 - \tilde{e}_{11})^T L_6] W \tilde{e}_6\}\end{aligned} \tag{6.33}$$

$$\tilde{\Phi}_3(\tau(t)) = h^2 \tilde{e}_s^T R \tilde{e}_s - \tilde{E}_1^T (1+\beta)\tilde{R}\tilde{E}_1 - \tilde{E}_2^T (1+\alpha)\tilde{R}\tilde{E}_2 - 2\tilde{E}_1^T[\alpha U_1 + \beta U_2]\tilde{E}_2 \tag{6.34}$$

$$\tilde{\Phi}_4 = \sum_{i=1}^{3} Sym\{(\tilde{e}_{8+i} - \Sigma_1 W \tilde{e}_i)^T \times M_i(\Sigma_2 W \tilde{e}_i - \tilde{e}_{8+i})\}$$
$$+ \sum_{i=1}^{2} Sym\{[(\tilde{e}_{8+i} - \tilde{e}_{9+i}) - \Sigma_1 W(\tilde{e}_i - \tilde{e}_{1+i})]^T N_i \times [\Sigma_2 W(\tilde{e}_i - \tilde{e}_{1+i}) - (\tilde{e}_{8+i} - \tilde{e}_{9+i})]\}$$
$$+ Sym\{[(\tilde{e}_9 - \tilde{e}_{11}) - \Sigma_1 W(\tilde{e}_1 - \tilde{e}_3)]^T N_3 \times [\Sigma_2 W(\tilde{e}_1 - \tilde{e}_3) - (\tilde{e}_9 - \tilde{e}_{11})]\} \quad (6.35)$$

The column vectors $\tilde{\Pi}_i, (i = 1, 2, ..., 9)$ used in the $\tilde{\Phi}_i, (i = 1, 2, ..., 4)$ expressions can be defined as

$$\begin{aligned}
\tilde{\Pi}_1 =& [\tilde{e}_s^T,\ \tilde{\tau}(t)\tilde{e}_4^T,\ \tilde{e}_1^T - \tilde{\tau}(t)\tilde{e}_2^T,\ \tilde{\tau}(t)\tilde{e}_2^T - \tilde{e}_3^T, \tilde{e}_1^T + \tilde{\tau}(t)\tilde{e}_2^T - (1 + \tilde{\tau}(t))\tilde{e}_5^T - \dot{\tau}(t)\tilde{e}_7^T,\\
&\tilde{\tau}(t)\tilde{e}_2^T + \tilde{e}_3^T - (1 + \tilde{\tau}(t))\tilde{e}_6^T + \dot{\tau}(t)\tilde{e}_8^T]^T\\
\tilde{\Pi}_0 =& [\tilde{e}_1^T,\ \tilde{e}_2^T,\ \tau(t)\tilde{e}_5^T,\ h_\tau(t)\tilde{e}_6^T,\ \tau(t)\tilde{e}_7^T,\ h_\tau(t)\tilde{e}_8^T]^T\\
\tilde{\Pi}_2 =& [\tilde{e}_1^T,\ \tilde{e}_2^T,\ \tilde{e}_5^T,\ \tilde{e}_7^T,\ \tilde{e}_{12}^T]^T,\ \tilde{\Pi}_4 = [\tilde{e}_1^T,\ \tilde{e}_2^T,\ \tilde{e}_6^T,\ \tilde{e}_8^T,\ \tilde{e}_{13}^T]^T\\
\tilde{\Pi}_3 =& [\tau(t)\tilde{e}_s^T,\ \tilde{\tau}(t)\tau(t)\tilde{e}_4^T, \tilde{e}_1^T - \tilde{\tau}(t)\tilde{e}_2^T - \dot{\tau}(t)\tilde{e}_5^T, \tilde{e}_1^T + \tilde{\tau}(t)\tilde{e}_2^T - (1 + \tilde{\tau}(t))\tilde{e}_5^T - 2\dot{\tau}(t)\tilde{e}_7^T,\\
&\tau(t)(\tilde{e}_1^T - \tilde{\tau}(t)\tilde{e}_2^T)]^T\\
\tilde{\Pi}_5 =& [h_\tau(t)\tilde{e}_s^T, h_\tau(t)\tilde{\tau}(t)\tilde{e}_4^T, \tilde{\tau}(t)\tilde{e}_2^T - \tilde{e}_3^T + \dot{\tau}(t)\tilde{e}_6^T, \tilde{\tau}(t)\tilde{e}_2^T + \tilde{e}_3^T - (1 + \tilde{\tau}(t))\tilde{e}_6^T + 2\dot{\tau}(t)\tilde{e}_8^T,\\
&h_\tau(t)(\tilde{\tau}(t)\tilde{e}_2^T - \tilde{e}_3^T)]^T\\
\tilde{\Pi}_6 =& [\tilde{e}_1^T,\ \tilde{e}_s^T,\ \tilde{e}_9^T]^T, \qquad \tilde{\Pi}_7 = [\tilde{e}_2^T,\ \tilde{e}_4^T,\ \tilde{e}_{10}^T]^T,\\
\tilde{\Pi}_8 =& [\tilde{e}_2^T,\ \tilde{e}_{10}^T]^T, \qquad \tilde{\Pi}_9 = [\tilde{e}_3^T,\ \tilde{e}_{11}^T]^T\\
\tilde{E}_1 =& Col\{\tilde{e}_1 - \tilde{e}_2,\ \tilde{e}_1 + \tilde{e}_2 - 2\tilde{e}_5,\ \tilde{e}_1 - \tilde{e}_2 - 6\tilde{e}_7\}\\
\tilde{E}_2 =& Col\{\tilde{e}_2 - \tilde{e}_3,\ \tilde{e}_2 + \tilde{e}_3 - 2\tilde{e}_6,\ \tilde{e}_2 - \tilde{e}_3 - 6\tilde{e}_8\}\\
\zeta_2(t) =& col\{x(t), x_\tau(t), x_h(t), \dot{x}_\tau(t), \frac{1}{\tau(t)}w_1(t), \frac{1}{h_\tau(t)}w_2(t), \frac{1}{\tau(t)}w_3(t), \frac{1}{h_\tau(t)}w_4(t), f(Wx(t)),\\
&f(Wx_\tau(t)), f(Wx_h(t)), w_1(t), w_2(t), \dot{x}_h(t)\}\\
\tilde{e}_s =& A\tilde{e}_1 + A_1\tilde{e}_9 + A_2\tilde{e}_{10},
\end{aligned}$$

with $\tilde{e}_1, \tilde{e}_2, ..., \tilde{e}_{13} \in \mathbb{R}^{n\times 13n}$ are basic block-entry matrices. **Proof :** The new LKF formulated by incorporating DPF (6.7) as:

$$\tilde{V}(t) = V_b(t) + \sum_{i=0}^{2} V_i(t) \quad (6.36)$$

where $V_0(t), V_1(t)$ and $V_2(t)$ as defined in (17).

Inspired by the idea of zero equality in [131], the following zero equality has been formed by using integral vectors $w_1(t)$, $w_2(t)$ and its interval normalized form with any matrices G_1 and G_2 as:

$$2\zeta_2^T(t)[\lambda_1 G_1 \wp_1(t) + \lambda_2 G_2 \wp_2(t)]\zeta_2(t) = 0 \quad (6.37)$$

where, $\wp_1 = \tau(t)\tilde{e}_5 - \tilde{e}_{12}$ $\wp_2 = h_\tau(t)\tilde{e}_6 - \tilde{e}_{13}$ and $\lambda_i (i = 1, 2)$ are delay weighted elementary matrices which provides flexibility to select $\zeta_2(t)$ or part of it.
By, including zero-equality (6.37), the derivative of $V_b(t)$ can be expressed as

$$\dot{V}_b(t) = \zeta_2^T(t)\tilde{\Phi}_0(\tau(t), \dot{\tau}(t))\zeta_2(t) \tag{6.38}$$

where $\tilde{\Phi}_0(\tau(t), \dot{\tau}(t))$ is defined in (6.31). Finally, using similar steps as in Theorem 8, one can obtain

$$\dot{\tilde{V}}(t) \le \zeta_2^T(t)[\tilde{\Upsilon}(\dot{\tau}(t), \tau(t)) + \tilde{\Gamma}(\tau(t))]\zeta_2(t) \tag{6.39}$$

where $\tilde{\Upsilon}(\dot{\tau}(t), \tau(t))$ is defined in (30) and

$$\tilde{\Gamma}(\tau(t)) = \beta\tilde{E}_1^T U_2 \tilde{R}^{-1} U_2^T \tilde{E}_1 + \alpha\tilde{E}_2^T U_1^T \tilde{R}^{-1} U_1 \tilde{E}_2 \tag{6.40}$$

The expression $\tilde{\Upsilon}(\dot{\tau}(t), \tau(t)) + \tilde{\Gamma}$ is linear in both $\tau(t)$ and its derivative. If it is negative definite for all $\tau(t) \in [0, h]$ and $\dot{\tau}(t) \in [\mu_1, \mu_2]$ then $\dot{\tilde{V}}(t) < 0$. Thereby, employing Schur complement, one can rewrite $\tilde{\Upsilon}(\dot{\tau}(t), \tau(t)) + \tilde{\Gamma}$ into LMIs (6.28) and (6.29). □
Choice of $\lambda_i(i = 1, 2)$: Generally, one can expect more relaxed stability criterion, if more states are involved in λ_i. However, by doing so leads to involving more free matrix variables and the conservativeness of the stability criterion may not improved. Therefore, it is important to choose suitable states to achieve trade-off between the computational complexity and conservatism. So, we have considered the following cases in expressions of λ_i.
Case 1:
$\lambda_1 = col\{\tilde{e}_1, \tilde{e}_4, \tilde{e}_{12}, \tilde{e}_{10}, \tilde{e}_5\}$, $\lambda_2 = col\{\tilde{e}_1, \tilde{e}_4, \tilde{e}_{13}, \tilde{e}_{10}, \tilde{e}_6\}$
Case 2:
$\lambda_1 = \lambda_2 = col\{\tilde{e}_1, \tilde{e}_2, ..., \tilde{e}_{13}\}$
In case 1, the dimension of matrix variables $G_1, G_2 \in \mathbb{R}^{5n\times n}$, whereas in Case 2 $G_1, G_2 \in \mathbb{R}^{13n\times n}$.

In order to compare the result with corresponding ones not using the DPFs, the following result can be drawn from Theorem 8. The proof is straight forward and in the similar lines as in Theorem 8, but considering $\bar{V}(t) = \sum_{i=0}^{2} V_i(t)$.

Corollary 1 *For positive constants h and μ_i, DNN (6.1) is asymptotically stable if there exists $0 < P \in \mathbb{R}^{6n\times 6n}$, $0 < Q_1 \in \mathbb{R}^{3n\times 3n}$, $0 < Q_2 \in \mathbb{R}^{2n\times 2n}$, $0 < R \in \mathbb{R}^{n\times n}$, diagonal* *matrices* $0 < L_k, M_j, N_j \in \mathbb{R}^{n\times n}$ *and any matrices* $U_i \in \mathbb{R}^{3n\times 3n}$ *for* $i = 1, 2; j = 1, 2, 3$ *and* $k = 1, 2, ..., 6$ *such that:*

$$\begin{bmatrix} \Upsilon_1(0, \mu_i) & E_1^T U_2 \\ * & -\tilde{R} \end{bmatrix} < 0 \tag{6.41}$$

$$\begin{bmatrix} \Upsilon 1(h, \mu_i) & E_2^T U_1 \\ * & -\tilde{R} \end{bmatrix} < 0 \tag{6.42}$$

where

$$\Upsilon_1(\dot{\tau}(t), \tau(t)) = \Phi_{01}(\dot{\tau}(t), \tau(t)) + \Phi_1(\dot{\tau}(t)) + \Phi_2(\dot{\tau}(t)) + \Phi_3(\tau(t)) + \Phi_4 \tag{6.43}$$

$$\Phi_{01}(\dot{\tau}(t), \tau(t)) = Sym\{\Pi_0^T P \Pi_1\} \tag{6.44}$$

6.4 Numerical Analysis

Two numerical examples are considered in this section to demonstrate the efficacy of the proposed results in comparison to the earlier works. MATLAB LMI Toolbox [123] is used for obtaining the result.

6.4.1 Examples

Example 3 *Consider DNN (6.1) with*

$$A = diag\{1.2769, 0.6231, 0.9230, 0.4480\}, W = I$$

$$A_1 = \begin{bmatrix} -0.0373 & 0.4852 & -0.3351 & 0.2336 \\ -1.6033 & 0.5988 & -0.3224 & 1.2352 \\ 0.3394 & -0.0860 & -0.3824 & -0.5785 \\ -0.1311 & 0.3253 & -0.9534 & -0.5015 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0.8674 & -1.2405 & -0.5325 & 0.0220 \\ 0.0474 & -0.9164 & 0.0360 & 0.9816 \\ 1.8495 & 2.6117 & -0.3788 & 0.8428 \\ -2.0413 & 0.5179 & 1.1734 & -0.2775 \end{bmatrix}$$

$$\Sigma_1 = 0, \ \Sigma_2 = diag\{0.1137, 0.1279, 0.7994, 0.2368\}$$

Table 6.1: LAUB of delay h for various values of μ

Methods	$\mu = -\mu_1 = \mu_2$			NLVs	MOL
	0.1	**0.5**	**0.9**		
Theorem 2 [129]	4.5072	3.8984	3.4437	$75n^2 + 21n$	44
Theorem 1 [136]	4.5086	3.8091	3.2895	$153n^2 + 22n$	52
Proposition 1 [143]	4.5382	3.9313	3.4763	$60n^2 + 22n$	56
Theorem 1(N=2) [119]	4.5470	3.9749	3.5052	$112.5n^2 + 28.5n$	56
Proposition 3 [142]	4.5432	3.9754	3.5791	$131n^2 + 24n$	48
Corollary 1	4.5198	3.8169	3.3185	$43n^2 + 10n$	44
Theorem 8	4.5327	3.9050	3.4749	$56n^2 + 22n$	44
Theorem 9 (case 1)	4.5510	4.0105	3.5464	$72n^2 + 22n$	52
Theorem 9 (case 2)	4.5648	4.0467	3.5885	$81n^2 + 22n$	52

The LAUB of the time-varying delay h obtained for different $\mu(= 0.1, 0.5, 0.9)$ using the proposed criterion are listed in Table 6.1 along with the existing results.

(a) It is observed that Corollary 1 and Theorem 8 are less conservative than of [129] and [136] but more conservative than other methods. However, Theorem 9 in both the cases gives better result in comparison to all the existing methods listed in Table 1.

(b) Further, one can note that, by addition of DPFs (6.6) and (6.7) in the LKF of proposed Theorems the conservativeness can be improved.

Example 4 *Consider another DNN in the form of (6.1), where*

$$A = diag\{7.3458, 6.9987, 5.5949\}, A_1 = 0, A_2 = I$$

$$W = \begin{bmatrix} 13.6014 & -2.9616 & -0.6936 \\ 7.4736 & 21.6810 & 3.2100 \\ 0.7920 & -2.6334 & -20.1300 \end{bmatrix}$$

with $\Sigma_1 = 0, \Sigma_2 = diag\{0.368, 0.1795, 0.2876\}$. Also, the time-varying delay and the activation function satisfy (6.2) and (6.3), respectively.

LAUB h for various $\mu = -\mu_1 = \mu_2$ using the proposed criterion and existing ones are stated in Table 6.2. The results obtained in Theorem 8 is better than Theorem 3 of [138],

Table 6.2: LAUB of delay h for various values of μ

Methods	$\mu = -\mu_1 = \mu_2$			NLVs	MOL
	0.1	**0.5**	**0.9**		
Theorem 3 [138]	1.1135	0.4922	0.4701	$79n^2 + 15n$	30
Theorem 1 [137]	1.1240	0.5689	0.4737	$151n^2 + 23n$	39
Proposition 1 [143]	1.1488	0.5864	0.4899	$60n^2 + 22n$	42
Proposition 3(N=2) [142]	1.1511	0.5835	-	$131n^2 + 24n$	36
Theorem 1(N=2) [119]	1.1545	0.6010	-	$83.5n^2 + 26.5n$	42
Corollary 1	1.1420	0.5684	0.4745	$25n^2 + 13n$	33
Theorem 8	1.1480	0.5787	0.4877	$56n^2 + 21n$	33
Theorem 9 (Case1)	1.1561	0.6001	0.5027	$72n^2 + 23n$	39
Theorem 9 (Case2)	1.1576	0.6049	0.5064	$81n^2 + 23n$	39

Theorem 1 of [137], but it is conservative than other existing criterion. However, Theorem 9 provides better results as compared to all works listed in Table 6.2.

Remark 11 *The implementation complexity and convergence speed of LMI mainly depends on the two factors such as maximum order of LMIs (MOL) and number of LMI variables (NLV) [130]. Hence, these factors are listed in Table 6.1 and 6.2 for various methods. On the basis of these factors, Corollary 1 and Theorem 8 have high convergence speed, since these methods reqiures less NLV and MOL. Similarly, Theorem 9 requires less NLVs in most methods listed in Table 6.1 except Theorem 9 [129] and Proposition 1 [143] and more MOL except Proposition 1 [143] and Theorem 1(N=2) [119]. Therefore, Proposed methods involves lesser computational time and more suited for systems with higher dimension.*

6.4.2 Performance comparison

For stability analysis of DNN using LKF interms of solving LMIs, the conservativeness of a criterion is conventionally assessed using the LAUB of delay. However, not only the conservatism but also the numerical complexity must be assessed for comparing performances of different stability criteria. Performance measures of stability criterion interms

Table 6.3: PCR and PNI of Theorem 8 and 9 with respect to, Corollary 1

Method (a)	**Proposed Theorems (PT)** / **Method (b)**	**PCR (%)** $\mu(Ex1)$ **0.1**	**0.5**	**0.9**	$\mu(Ex2)$ **0.1**	**0.5**	**0.9**	**PNI (%)** **Ex 1**	**Ex 2**
	Tm 1	0.29	2.3	4.71	0.52	1.8	2.78	38.28	37.68
Corollary 1	Theorem 9(Case1)	0.69	5.07	6.86	1.23	5.57	5.94	74.71	71.01
	Theorem 9(Case2)	0.99	6.02	8.13	1.36	6.42	6.72	94.94	92.75

of these two aspects can be recalled from [139] as follows.

(1) Percentage of Numerical complexity increment (PNI):

$$\text{PNI of (a) from (b)} = \frac{(NLV)_a - (NLV)_b}{(NLV)_b} \times 100\% \tag{6.45}$$

The PNI in (6.45) presents the increased rate of numerical complexity by criterion (a) from (b).

(2) Percentage of conservatism reduction (PCR):

$$\text{PCR of (a) from (b)} = \frac{(LAUB)_a - (LAUB)_b}{(LAUB)_b} \times 100\% \tag{6.46}$$

The PCR in (6.46) presents reduction in conservativeness by criterion (a) from (b).

The PNR and PCR values are computed next for different methods for the numerical examples to make a detailed comparison.

(1). Comparison between the proposed methods: (a) Note that Theorem 8 and 9 incorporate additional matrices as compared to Corollary 1 by employing DPFs (6.6) and (6.7). The PNI and PCR of these Theorems are compared with respect to Corollary 1 as shown in Table 6.3. Positive values of PCR for all the cases show that Theorem 8 and 9 are less conservative due to the use of DPFs though there are increase in PNI index as well for all the cases.

(b) From Table 6.3, it can be seen that PCI of Theorem 9 is considerably more as compared to Theorem 8 due to the presence of additional states in the DPF (6.7). So, the effect of these states enhances the result.

(2) Comparison to the existing works: The PNI and PCR of the existing criterion with respect to Theorem 9 (cases 1 and 2) are listed in Table 6.4.

Table 6.4: PCR and PNI of Theorem 9 from the existing works

Method (a)	Existing Methods (EM)	PCR of PT from EM						PNI of PT from EM	
		$\mu(Ex1)$			$\mu(Ex2)$			Ex 1	Ex 2
	Method (b)	0.1	0.5	0.9	0.1	0.5	0.9		
Theorem 9 (Case 1)	Theorem 2 [129]	0.97	2.87	2.98	-	-	-	-1.32	-
	Theorem 1 [136]	0.94	5.28	7.80	-	-	-	-50.90	-
	Theorem 3 [138]	-	-	-	3.82	21.90	6.93	-	-5.90
	Theorem 1 [137]	-	-	-	2.85	5.30	6.12	-	-49.78
	Proposition 1 [143]	0.28	2.01	2.05	0.63	0.01	2.60	18.7	18.3
	Proposition 3(N=2) [142]	1.72	0.89	1.18	0.43	2.84	-	-43.2	-42.7
	Theorem 1(N=2) [119]	0.88	0.89	1.18	0.14	-0.15	-	-35.0	-34.7
Theorem 9 (Case2)	Theorem 2 [129]	1.27	3.80	4.20	-	-	-	8.09	-
	Theorem 1 [136]	1.24	6.23	9.08	-	-	-	-45.26	-
	Theorem 3 [138]	-	-	-	3.96	22.89	7.72	-	4.72
	Theorem 1 [137]	-	-	-	2.98	6.16	6.89	-	-44.11
	Proposition 1 [143]	0.59	2.90	3.27	0.77	3.15	3.36	32.4	31.6
	Proposition 3(N=2) [142]	0.48	1.80	0.26	0.43	3.67	-	-36.7	-36.2
	Theorem 1(N=2) [119]	0.39	1.8	2.37	0.27	0.65	-	-27.5	-27.3

PT $\rightarrow$ Proposed Methods, EM $\rightarrow$ Existing Methods

(a) From Table 6.4, one can observe that proposed results yield positive PCR. Further, it can be noted that Theorem 9 (Case2) has larger PCR than Theorem 9 (case 1) for both the examples.

(b) One can note from Table 6.4 that there are cases with negative PNI and positive PCR. It implies that proposed Theorems are better in reducing conservatism yet with reduction in computational cost in comparison these existing methods.

From the above numerical examples and comparison vis-a-vis to the existing methods, it can be concluded that reduction in conservatism of the proposed stability criterion is due to the use of new DPFs involving states from the second order BLI.Extending the idea to construct DPFs using the states from higher order BLI may lead to further reduction in conservatism with increased computational cost.

Neural networks have applications in practical control systems. As in [134], a neural network has been successfully utilized to design reliable intelligent path following control method for robotic airship subject to sensor faults. In [133], the problem of stabilization of sampled data neural network based systems with state quantization has been addressed. In near future, we will apply the proposed methods may be used for stabilization of sampled data systems and robotic applications, where neural network is used.

6.5 Summary

In this paper, stability analysis of generalized DNN is studied. By constructing LKFs using newly developed DPFs, two stability criterion are derived in which the information on delay and its time-derivative are considerably utilized to get improved results. The effectiveness of the developed criterion are shown by considering the numerical examples. A detailed numerical comparison has been made to demonstrate trade-off of conservatism reduction with increased computational burden.

The next chapter summarizes the work of the thesis, points out contributions and limitations and also forecasts the possible endeavours in the context of Lyapunov stability methods.