Chapter 3

Multivariate α-Fractal Functions: Dimension and Fractional Integral

In this chapter, we examine the concept of an α-fractal function, which corresponds to a multivariate continuous function defined on a hypercube $[0,1] \times \cdots \times [0,1]$ (repeated N times). The parameters are chosen so that the resulting fractal version retains certain properties of the original function.

3.1 Introduction

Numerous theories related to the fractal dimensions and fractional integrals of univariate and bivariate fractal functions have been explored. The concept of fractal dimension and fractional integral of a univariate fractal function is well-established in the literature (see, for instance, [1, 9, 14, 19, 28, 37, 54]) and references therein.

However, there is a lack of research on higher dimensional fractal functions. In this chapter, we examine the box dimension and the Hausdorff dimensions of the graph of the multivariate α-fractal function and its restriction. We also investigate the behavior of the α-fractal function when it is restricted to the coordinate axis. Furthermore, we prove that the mixed-Riemann-Liouville fractional integral of a fractal function follows a self-referential equation.

3.1.1 Delineation

The current chapter is structured as follows. Section 3.2 focused on examining the box and the Hausdorff dimensions of the constructed multivariate α-fractal function. We observed that if the given function is Hölder continuous, the associated α-fractal function is also Hölder continuous for appropriate parameters. Moreover, in Subsection 3.2.1, we studied the restrictions of α-fractal function on the axis. In Section 3.3, we proved that the mixed Riemann-Liouville fractional integral of the fractal interpolation function satisfies a self-referential equation.

3.2 Box Dimension of the Graph of Multivariate Fractal Function

By estimating the oscillation of functions, we shall seek to compute the box dimension of the graph of the multivariate fractal function. The main ideas of [15, 48] are modified and applied to our method.

Lemma 3.1. *Let $f : I^N \to \mathbb{R}$ be a continuous function on I^N. Then,* $\dim_H \big(\mathcal{G}(f)\big) \geq N$.

Proof. Define a function $T : \mathcal{G}(f) \to I^N$ such that

$$T\big((x_1, \ldots, x_N, f(x_1, \ldots, x_N))\big) = (x_1, \ldots, x_N).$$

It is easy to observe that T is a Lipschitz function. Now recall that if $g : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is any Lipschitz function, then $\dim_H(g(A)) \leq \dim_H(A)$. Hence, we get $\dim_H \big(T(\mathcal{G}(f))\big) \leq \dim_H \big(\mathcal{G}(f)\big)$. It is straightforward to verify that the function T is surjective, and thus the conclusion. $\square$

Definition 3.2. Let $A \subset \mathbb{R}^N$ be a closed bounded hypercube and $f : A \to \mathbb{R}$. The oscillation of f over the rectangle A is defined as

$$R_f[A] = \sup_{(x_1,\ldots,x_N),(y_1,\ldots,y_N)\in A} |f(x_1,\ldots,x_N) - f(y_1,\ldots,y_N)|.$$

Next, we shall provide a multivariate analog of Proposition 11.1 in [36].

Lemma 3.3. *Let $f : I^N \to \mathbb{R}$ be a continuous function. For $M_1 < \cdots < M_N < M$, let us take cuboid of size $\frac{1}{M_1^m} \times \cdots \times \frac{1}{M_N^m} \times \frac{1}{M^m}$ for some $M_k, M \geq 2$, where $1 \leq k \leq N$ and for some $m \in \mathbb{N}$. Let $\delta = \frac{1}{M^m}$ be the height of the cuboid. If $N_\delta(\mathcal{G}(f))$ is the number of δ-cuboid that intersect the graph of f, then*

$$\begin{aligned}\sum_{i_N=1}^{M_N^m} \cdots \sum_{i_1=1}^{M_1^m} \max\left\{1, \lceil M^m R_f[A_{i_1\ldots i_N}]\rceil\right\} \leq & N_\delta(\mathcal{G}(f)) \\ \leq & [M_1 \ldots M_N]^m + \sum_{i_1=1}^{M_1^m} \cdots \sum_{i_N=1}^{M_N^m} \lceil M^m R_f[A_{i_1\ldots i_N}]\rceil,\end{aligned}$$

where $\lceil\cdot\rceil$ denotes the ceiling function and $A_{i_1\ldots i_N} = [i_1\delta, (i_1+1)\delta] \times \cdots \times [i_N\delta, (i_N+1)\delta]$.

Proof. Using the continuity of f, it follows that the number of δ-cuboids in the part above $A_{i_1\ldots i_N}$ that intersect the graph of f is at least $\max\left\{1, \lceil M^m R_f[A_{i_1\ldots i_N}]\rceil\right\}$ and at most $1 + \lceil M^m R_f[A_{i_1\ldots i_N}]\rceil$. The appropriate bounds are obtained by summing all of these components. □

Definition 3.4. A function $f : I^N \to \mathbb{R}$ is said to be Hölder continuous with exponent σ and Hölder constant K_f if

$$|f(x_1,\ldots,x_N) - f(y_1,\ldots,y_N)| \leq K_f \|(x_1,\ldots,x_N) - (y_1,\ldots,y_N)\|_2^\sigma$$

for all $(x_1, \ldots, x_N), (y_1, \ldots, y_N) \in I^N$, and for some $K_f > 0$.

We define the Hölder space

$$\mathcal{H}^\sigma(I^N) := \{g : I^N \to \mathbb{R} : \text{ g is Hölder continuous with exponent } \sigma\}.$$

If we equip the space $\mathcal{H}^\sigma(I^N)$ with norm $\|g\| = \|g\|_\infty + [g]_\sigma$, where

$$[g]_\sigma = \sup_{(x_1,\ldots,x_N)\neq(y_1,\ldots,y_N)} \frac{|g(x_1, \ldots, x_N) - g(y_1, \ldots, y_N)|}{\|(x_1, \ldots, x_N) - (y_1, \ldots, y_N)\|_2^\sigma},$$

then it forms a Banach space.

Now let $\left\{(x_{1,i_1}, \ldots, x_{N,i_N}) \in \mathbb{R}^N : i_k \in \Sigma_{M_k,0},\ k \in \Sigma_N\right\}$ such that $0 = x_{k,0} < \cdots < x_{k,M_k} = 1$ with $x_{k,i_k} - x_{k,i_k-1} = \frac{1}{M_k}$, where $k \in \Sigma_N$. Further, we denote

$$\begin{aligned} D_{i_1\ldots i_N} &= u_{1,i_1}(I_1) \times \cdots \times u_{N,i_N}(I_N), \\ D_{(i_1)_1\ldots(i_1)_m\ldots(i_N)_1\ldots(i_N)_m} &= u_{(i_1)_1\ldots(i_1)_m}(I_1) \times \cdots \times u_{(i_N)_1\ldots(i_N)_m}(I_N), \\ u_{(i_k)_1\ldots(i_k)_m} &= u_{k,(i_k)_1} \circ \cdots \circ u_{k,(i_k)_m}, \end{aligned} \tag{3.1}$$

where $(i_k)_j \in \Sigma_{M_k}$ with $j \in \Sigma_m$ and $k \in \Sigma_N$.

Next, define

$$\begin{aligned} G_{i_1\ldots i_N} :=& \{(x_1, \ldots, x_N, f^\alpha(x_1, \ldots, x_N)) : (x_1, \ldots, x_N) \in D_{i_1\ldots i_N}\} \\ \overline{\alpha}_{i_1\ldots i_N} =& \max\{|\alpha(x_1, \ldots, x_N)| : (x_1, \ldots, x_N) \in D_{i_1\ldots i_N}\}, \\ \overline{\alpha} =& \max\{\overline{\alpha}_{i_1\ldots i_N} : (i_1, \ldots, i_N) \in \prod_{k=1}^{N} \Sigma_{M_k}\} \\ \alpha_{\min} =& \min\{|\alpha(x_1, \ldots, x_N)| : (x_1, \ldots, x_N) \in I^N\}, \\ \alpha_{\max} =& \max\{|\alpha(x_1, \ldots, x_N)| : (x_1, \ldots, x_N) \in I^N\}. \end{aligned} \tag{3.2}$$

Remark 3.5. Let $\overline{\gamma} = \sum_{(i_1,\ldots,i_N)\in\Sigma_{M_1}\times\cdots\times\Sigma_{M_N}} \overline{\alpha}_{i_1,\ldots,i_N}$. For $m \in \mathbb{N}$, we have

$$\overline{\gamma}^m = \sum_{((i_1)_m\ldots,(i_N)_m)\in\Sigma_{M_1}\times\cdots\times\Sigma_{M_N}} \overline{\alpha}_{(i_1)_1,\ldots,(i_N)_1}\cdots\overline{\alpha}_{(i_1)_m,\ldots,(i_N)_m}.$$

Lemma 3.6. *Let f, α, and s be Hölder continuous functions with Hölder exponents σ_1, σ_2, and σ_3 and Hölder constants K_f, K_α, and K_s, respectively.*

Furthermore, we suppose that $M_1 < \cdots < M_N$ and $x_{k,i_k} - x_{k,i_k-1} = \frac{1}{M_k}$ for each $k \in \Sigma_N$. Then, we have

$$\begin{aligned} &R_{F_{(i_1)_m\cdots(i_N)_m}}[G_{(i_1)_1\ldots(i_1)_{m-1}\ldots(i_N)_1\ldots(i_N)_{m-1}}] \\ \leq &N\overline{\alpha}R_{f^\alpha}[D_{(i_1)_1\ldots(i_1)_{m-1}\ldots(i_N)_1\ldots(i_N)_{m-1}}] + \left(\frac{K^*}{M_1^{m\sigma}} + \frac{\overline{\alpha}K_s}{M_1^{(m-1)\sigma}}\right)N^{\frac{\sigma}{2}+1}, \end{aligned} \tag{3.3}$$

where $\sigma = \min\{\sigma_1, \sigma_2, \sigma_3\}$ and $K^* := K_\alpha\big(\|f\|_\infty + \|s\|_\infty\big) + K_f$.

Proof. We know that for $(x_1, \ldots, x_N) \in I^N$, there exists $\prod_{k=1}^N I_{k,i_k}$, where $i_k \in \Sigma_{M_k}$, such that $(x_1, \ldots, x_N) \in \prod_{k=1}^N I_{k,i_k}$, then using (1.13), we have

$$F_{i_1\ldots i_N}(x_1,\ldots,x_N,z) = f\left(u_{1,i_1}(x_1),\ldots,u_{N,i_N}(x_N)\right) + \alpha\left(x_1,\ldots,x_N\right)\left(z - s(x_1,\ldots,x_N)\right). \tag{3.4}$$

Now consider $(x_1, \ldots, x_N), (y_1, \ldots, y_N) \in I^N$ such that $x_k, y_k \in I_{k,(i_k)_1}$ for $(i_k)_1 \in \Sigma_{M_k}$, then using above equation, we have

$$\begin{aligned} &\left|F_{(i_1)_1\ldots(i_N)_1}\left(x_1,\ldots,x_N,f^\alpha(x_1,\ldots,x_N)\right) - F_{(i_1)_1\ldots(i_N)_1}\left(y_1,\ldots,y_N,f^\alpha(y_1,\ldots,y_N)\right)\right| \\ =&\left|f\left(u_{1,(i_1)_1}(x_1),\ldots,u_{N,(i_N)_1}(x_N)\right) + \alpha\left(x_1,\ldots,x_N\right)\left(f^\alpha(x_1,\ldots,x_N) - s(x_1,\ldots,x_N)\right)\right. \\ &\left. - f\left(u_{1,(i_1)_1}(y_1),\ldots,u_{N,(i_N)_1}(y_N)\right) - \alpha\left(y_1,\ldots,y_N\right)\left(f^\alpha(y_1,\ldots,y_N) - s(y_1,\ldots,y_N)\right)\right| \\ \leq&|\alpha(x_1,\ldots,x_N)(f^\alpha(x_1,\ldots,x_N) - f^\alpha(y_1,\ldots,y_N))| + |\alpha(x_1,\ldots,x_N)(s(x_1,\ldots,x_N) - s(y_1,\ldots,y_N))| \\ &+ \left(|f^\alpha(y_1,\ldots,y_N)| + |s(y_1,\ldots,y_N)|\right)|\alpha\left(x_1,\ldots,x_N\right) - \alpha\left(y_1,\ldots,y_N\right)| \\ &+ \left|f\left(u_{1,(i_1)_1}(x_1),\ldots,u_{N,(i_N)_1}(x_N)\right) - f\left(u_{1,(i_1)_1}(y_1),\ldots,u_{N,(i_N)_1}(y_N)\right)\right| \\ \leq&\overline{\alpha}_{(i_1)_1\ldots(i_N)_1}|f^\alpha(x_1,\ldots,x_N) - f^\alpha(y_1,\ldots,y_N)| + \overline{\alpha}_{(i_1)_1\ldots(i_N)_1}K_s\|(x_1,\ldots,x_N) - (y_1,\ldots,y_N)\|_2^{\sigma_3} \\ &+ K_\alpha\left(\|f^\alpha\|_\infty + \|s\|_\infty\right)\|(x_1,\ldots,x_N) - (y_1,\ldots,y_N)\|_2^{\sigma_2} \end{aligned}$$

$$
\begin{aligned}
&+K_f\left\|\left(u_{1,(i_1)_1}(x_1),\ldots,u_{N,(i_N)_1}(x_N)\right)-\left(u_{1,(i_1)_1}(y_1),\ldots,u_{N,(i_N)_1}(y_N)\right)\right\|_2^{\sigma_1}\\
\leq&\overline{\alpha}_{(i_1)_1\ldots(i_N)_1}R_{f^\alpha}[I^N]+\overline{\alpha}_{(i_1)_1\ldots(i_N)_1}K_s\|(x_1,\ldots,x_N)-(y_1,\ldots,y_N)\|_2^\sigma\\
&+\left(K_\alpha\left(\|f^\alpha\|_\infty+\|s\|_\infty\right)+K_f\right)\max\left\{\frac{1}{M_1^\sigma},\ldots,\frac{1}{M_N^\sigma}\right\}\|(x_1,\ldots,x_N)-(y_1,\ldots,y_N)\|_2^\sigma\\
\leq&\overline{\alpha}_{(i_1)_1\ldots(i_N)_1}R_{f^\alpha}[I^N]+\left(\overline{\alpha}_{(i_1)_1\ldots(i_N)_1}K_s+\frac{K^*}{M_1^\sigma}\right)N^{\sigma/2}\\
\leq&N\overline{\alpha}R_{f^\alpha}[I^N]+\left(\overline{\alpha}K_s+\frac{K^*}{M_1^\sigma}\right)N^{\frac{\sigma}{2}+1}.
\end{aligned}
$$

Further iterations lead to the following result:

$$
\begin{aligned}
&R_{F_{(i_1)_m\ldots(i_N)_m}}[G_{(i_1)_1\ldots(i_1)_{m-1}\ldots(i_N)_1\ldots(i_N)_{m-1}}]\\
\leq&N\overline{\alpha}R_{f^\alpha}[D_{(i_1)_1\ldots(i_1)_{m-1}\ldots(i_N)_1\ldots(i_N)_{m-1}}]+\left(\frac{K^*}{M_1^{m\sigma}}+\frac{\overline{\alpha}K_s}{M_1^{(m-1)\sigma}}\right)N^{\frac{\sigma}{2}+1},
\end{aligned}
\tag{3.5}
$$

completing the claim. □

Note 3.7. In the proof of Lemma 3.6, it was assumed that x_k and y_k belong to the interval $I_{k,(i_k)_1}$ for $k\in\Sigma_N$ and $(i_k)_1\in\Sigma_{M_k}$. However, if there exists a $k\in\Sigma_N$ such that x_k is in $I_{k,(i_k)_1}$ and y_k is in $I_{k,(j_k)_1}$, where $(i_k)_1$ and $(j_k)_1$ are elements of Σ_{M_k} and $(i_k)_1\neq(j_k)_1$, then Equation (1.13) can not be applied directly. In this case, Equation (1.13) can be applied by first utilizing the triangle inequality with the joints (i.e., $\{x_{k,1},\ldots,x_{k,M_k}\}$) between the intervals $I_{k,(i_k)_1}$ and $I_{k,(j_k)_1}$, and subsequently employing Equation (1.13). For example, let us consider $x_1\in I_{1,2}$ and $y_1\in I_{1,3}$, then we have

$$
\begin{aligned}
&\left|F_{2\ldots(i_N)_1}\left(x_1,\ldots,x_N,f^\alpha(x_1,\ldots,x_N)\right)-F_{3\ldots(i_N)_1}\left(y_1,\ldots,y_N,f^\alpha(y_1,\ldots,y_N)\right)\right|\\
\leq&\left|F_{2\ldots(i_N)_1}\left(x_1,\ldots,x_N,f^\alpha(x_1,\ldots,x_N)\right)-F_{2\ldots(i_N)_1}\left(x_{1,2},\ldots,y_N,f^\alpha(y_1,\ldots,y_N)\right)\right|\\
&+\left|F_{3\ldots(i_N)_1}\left(x_{1,2},\ldots,x_N,f^\alpha(x_1,\ldots,x_N)\right)-F_{3\ldots(i_N)_1}\left(y_1,\ldots,y_N,f^\alpha(y_1,\ldots,y_N)\right)\right|.
\end{aligned}
$$

Now, using (1.13) into both terms of the right-hand side and following the same steps as in the previous lemma, the desired outcome can be obtained.

Remark 3.8. In the course of this thesis, we have employed the same technique to prove the results that were utilized in Lemma 3.6, whenever we made use of

Equation 1.13 or the self-referential equation, i.e., we have considered $x_k, y_k \in I_{k,i_k}$ for $k \in \Sigma_N$. Any other scenarios can be addressed by referring to Note 3.7.

The following theorem establishes bounds for the graph dimension of the multivariate α-fractal function f^α.

Theorem 3.9. *Let f^α be the multivariate α-fractal function. Further, suppose that $M_1 < \cdots < M_N < M$ for some $M \geq 2$ and $x_{k,i_k} - x_{k,i_k-1} = \frac{1}{M_k}$ for each $k \in \Sigma_N$. Then, we have the following:*

(i) If $\alpha_{\min} > \frac{1}{M_1^\sigma}$, then $N \leq \dim_H\big(\mathcal{G}(f^\alpha)\big) \leq \underline{\dim}_B\big(\mathcal{G}(f^\alpha)\big) \leq \overline{\dim}_B\big(\mathcal{G}(f^\alpha)\big) \leq 1 + \frac{\log(\overline{\gamma})}{\log(M)}$.

(ii) If $\frac{1}{M_1^\sigma} < \alpha_{\max} < 1$, then $N \leq \dim_H\big(\mathcal{G}(f^\alpha)\big) \leq \underline{\dim}_B\big(\mathcal{G}(f^\alpha)\big) \leq \overline{\dim}_B\big(\mathcal{G}(f^\alpha)\big) \leq N + 1 + \frac{\log(\alpha_{\max})}{\log(M)}$.

(iii) If $\alpha_{\max} \leq \frac{1}{M_1^\sigma}$, then $N \leq \dim_H\big(\mathcal{G}(f^\alpha)\big) \leq \underline{\dim}_B\big(\mathcal{G}(f^\alpha)\big) \leq \overline{\dim}_B\big(\mathcal{G}(f^\alpha)\big) \leq N + 1 - \sigma$.

Proof. Note that for all $(x_1, \ldots, x_N) \in \prod_{k=1}^N I_{k,i_k}$ the functional equation is satisfied by the function f^α,

$$f^\alpha(x_1, \ldots, x_N) = F_{i_1 \ldots i_N}\Big(u_{1,i_1}^{-1}(x_1), \ldots, u_{N,i_N}^{-1}(x_N), f^\alpha\big(u_{1,i_1}^{-1}(x_1), \ldots, u_{N,i_N}^{-1}(x_N)\big)\Big).$$

For $(x_1, \ldots, x_N), (y_1, \ldots, y_N) \in D_{(i_1)_1 \ldots (i_N)_1}$, we obtain

$$\begin{aligned}
&\big|f^\alpha(x_1, \ldots, x_N) - f^\alpha(y_1, \ldots, y_N)\big| \\
=&\Big|F_{(i_1)_1 \ldots (i_N)_1}\Big(u_{1,(i_1)_1}^{-1}(x_1), \ldots, u_{N,(i_N)_1}^{-1}(x_N), f^\alpha\big(u_{1,(i_1)_1}^{-1}(x_1), \ldots, u_{N,(i_N)_1}^{-1}(x_N)\big)\Big) \\
&- F_{(i_1)_1 \ldots (i_N)_1}\Big(u_{1,(i_1)_1}^{-1}(y_1), \ldots, u_{N,(i_N)_1}^{-1}(y_N), f^\alpha\big(u_{1,(i_1)_1}^{-1}(y_1), \ldots, u_{N,(i_N)_1}^{-1}(y_N)\big)\Big)\Big|
\end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{(x_1,\ldots,x_N),(y_1,\ldots,y_N)\in D_{(i_1)_1\ldots(i_N)_1}} \Big| F_{(i_1)_1\ldots(i_N)_1}\Big(u^{-1}_{1,(i_1)_1}(x_1),\ldots,u^{-1}_{N,(i_N)_1}(x_N),\\
&\quad f^{\alpha}\big(u^{-1}_{1,(i_1)_1}(x_1),\ldots,u^{-1}_{N,(i_N)_1}(x_N)\big)\Big)\\
&\quad - F_{(i_1)_1\ldots(i_N)_1}\Big(u^{-1}_{1,(i_1)_1}(y_1),\ldots,u^{-1}_{N,(i_N)_1}(y_N), f^{\alpha}\big(u^{-1}_{1,(i_1)_1}(y_1),\ldots,u^{-1}_{N,(i_N)_1}(y_N)\big)\Big)\Big|\\
&= \sup_{(\tilde{x_1},\ldots,\tilde{x_N}),(\tilde{y_1},\ldots,\tilde{y_N})\in I^N} \Big| F_{(i_1)_1\ldots(i_N)_1}\big(\tilde{x_1},\ldots,\tilde{x_N}, f^{\alpha}(\tilde{x_1},\ldots,\tilde{x_N})\big)\\
&\quad - F_{(i_1)_1\ldots(i_N)_1}\big(\tilde{y_1},\ldots,\tilde{y_N}, f^{\alpha}(\tilde{y_1},\ldots,\tilde{y_N})\big)\Big|\\
&= R_{F_{(i_1)_1\ldots(i_N)_1}}[\mathcal{G}(f^{\alpha})].
\end{aligned}
$$

Since $(x_1,\ldots,x_N)$ and $(y_1,\ldots,y_N)$ are arbitrary, we get

$$R_{f^{\alpha}}[D_{(i_1)_1\ldots(i_N)_1}] \leq R_{F_{(i_1)_1\ldots(i_N)_1}}[\mathcal{G}(f^{\alpha})].$$

Using iteration, it follows that

$$R_{f^{\alpha}}[D_{(i_1)_1\ldots(i_1)_m\ldots(i_N)_1\ldots(i_N)_m}] \leq \; R_{F_{(i_1)_m\ldots(i_N)_m}}[G_{(i_1)_1\ldots(i_1)_{m-1}\ldots(i_N)_1\ldots(i_N)_{m-1}}].$$

Using Lemma 3.6, we obtain

$$
\begin{aligned}
&R_{f^{\alpha}}[D_{(i_1)_1\ldots(i_1)_m\ldots(i_N)_1\ldots(i_N)_m}]\\
\leq&\overline{\alpha}_{(i_1)_m\ldots(i_N)_m}\cdots\overline{\alpha}_{(i_1)_1\ldots(i_N)_1}R_{f^{\alpha}}[I^N]\\
&+2^{\sigma/2}K^*\Big(\frac{1}{M_1^{m\sigma}}+\frac{\overline{\alpha}_{(i_1)_m\ldots(i_N)_m}}{M_1^{(m-1)\sigma}}+\cdots+\frac{\overline{\alpha}_{(i_1)_m\ldots(i_N)_m}\cdots\overline{\alpha}_{(i_1)_2\ldots(i_N)_2}}{M_1^{\sigma}}\Big)\\
&+2^{\sigma/2}K_s\Big(\frac{\overline{\alpha}_{(i_1)_m\ldots(i_N)_m}}{M_1^{(m-1)\sigma}}+\frac{\overline{\alpha}_{(i_1)_m\ldots(i_N)_m}\overline{\alpha}_{i_{m-1}j_{m-1}}}{M_1^{(m-2)\sigma}}+\cdots+\overline{\alpha}_{(i_1)_m\ldots(i_N)_m}\cdots\overline{\alpha}_{(i_1)_1\ldots(i_N)_1}\Big).
\end{aligned}
$$

(i) If $\alpha_{\min} > \frac{1}{M_1^{\sigma}}$, then we get

$$
\begin{aligned}
&R_{f^{\alpha}}[D_{(i_1)_1\ldots(i_1)_m\ldots(i_N)_1\ldots(i_N)_m}]\\
\leq\;&\overline{\alpha}_{(i_1)_m\ldots(i_N)_m}\cdots\overline{\alpha}_{(i_1)_1\ldots(i_N)_1}R_{f^{\alpha}}[I^N]\\
&+\frac{2^{\sigma/2}K^*(\overline{\alpha}_{(i_1)_m\ldots(i_N)_m}\cdots\overline{\alpha}_{(i_1)_2\ldots(i_N)_2})}{M_1^{\sigma}}\Big(1+\frac{1}{\alpha_{\min}M_1^{\sigma}}+\cdots+\frac{1}{\alpha_{\min}^{m-1}M_1^{(m-1)\sigma}}\Big)
\end{aligned}
$$

$$+ 2^{\sigma/2} K_s \overline{\alpha}_{(i_1)_m \dots (i_N)_m} \dots \overline{\alpha}_{(i_1)_1 \dots (i_N)_1} \left(1 + \frac{1}{\alpha_{\min} M_1^{\sigma}} + \dots + \frac{1}{\alpha_{\min}^{m-1} M_1^{(m-1)\sigma}}\right)$$
$$\leq \overline{\alpha}_{(i_1)_m \dots (i_N)_m} \dots \overline{\alpha}_{(i_1)_1 \dots (i_N)_1} \left(R_{f^\alpha}[I^N] + \left(\frac{2^{\sigma/2} K^*}{M_1^{\sigma} \alpha_{\min}} + 2^{\sigma/2} K_s\right) \left(\frac{1}{1 - \frac{1}{\alpha_{\min} M_1^{\sigma}}}\right)\right).$$

We write

$$R_{f^\alpha}[D_{(i_1)_1 \dots (i_1)_m \dots (i_N)_1 \dots (i_N)_m}] \leq K_1 \overline{\alpha}_{(i_1)_m \dots (i_N)_m} \dots \overline{\alpha}_{(i_1)_1 \dots (i_N)_1},$$

where $K_1 = R_{f^\alpha}[I^N] + \left(\frac{2^{\sigma/2} K^*}{M_1^{\sigma} \alpha_{\min}} + 2^{\sigma/2} K_s\right) \left(\frac{1}{1 - \frac{1}{\alpha_{\min} M_1^{\sigma}}}\right)$.
From Lemma 3.3, we have

$$\begin{aligned}
N_\delta(\mathcal{G}(f^\alpha)) &\leq [M \dots M]^m \\
&+ \sum_{((i_1)_k, \dots, (i_N)_k) \in \prod_{k=1}^{N} \Sigma_{M_k}} \left\lceil M^m R_{f^\alpha}[D_{(i_1)_1 \dots (i_1)_m \dots (i_N)_1 \dots (i_N)_m}] \right\rceil \\
&\leq M^{mN} + \sum_{((i_1)_k, \dots, (i_N)_k) \in \prod_{k=1}^{N} \Sigma_{M_k}} \left\lceil M^m K_1 \overline{\alpha}_{(i_1)_m \dots (i_N)_m} \dots \overline{\alpha}_{(i_1)_1 \dots (i_N)_1} \right\rceil \\
&\leq 2M^{mN} + M^m K_1 \sum_{((i_1)_k, \dots, (i_N)_k) \in \prod_{k=1}^{N} \Sigma_{M_k}} \overline{\alpha}_{(i_1)_m \dots (i_N)_m} \dots \overline{\alpha}_{(i_1)_1 \dots (i_N)_1} \\
&\leq 2M^{mN} + M^m K_1 \overline{\gamma}^m,
\end{aligned}$$

where $\overline{\gamma} = \sum_{i_1, \dots, i_N = 1}^{M_1, \dots, M_N} \overline{\alpha}_{i_1, \dots, i_N}$. Moreover,

$$\begin{aligned}
\overline{\lim}_{\delta \to 0} \frac{\log\left(N_\delta(\mathcal{G}(f^\alpha))\right)}{-\log(\delta)} &\leq \overline{\lim}_{m \to \infty} \frac{\log\left(2M^{mN} + M^m K_1 \overline{\gamma}^m\right)}{m \log(M)} \\
&= 1 + \frac{\log(\overline{\gamma})}{\log(M)}.
\end{aligned}$$

(ii) Assuming that $\frac{1}{M_1^{\sigma}} < \alpha_{\max} < 1$, then one get

$$R_{f^\alpha}[D_{(i_1)_1 \dots (i_1)_m \dots (i_N)_1 \dots (i_N)_m}]$$

$$
\begin{aligned}
&\leq \alpha_{\max}^m R_{f^\alpha}[I^N] + \frac{2^{\sigma/2}K^*\alpha_{\max}}{M_1^\sigma}\left(1+\frac{1}{\alpha_{\max}M_1^\sigma}+\cdots+\frac{1}{\alpha_{\max}^{m-1}M_1^{(m-1)\sigma}}\right)\\
&\quad + 2^{\sigma/2}K_s\alpha_{\max}^m\left(1+\frac{1}{\alpha_{\max}M_1^\sigma}+\cdots+\frac{1}{\alpha_{\max}^{m-1}M_1^{(m-1)\sigma}}\right)\\
&\leq \alpha_{\max}^m\left(R_{f^\alpha}[I^N] + \left(\frac{2^{\sigma/2}K^*}{M_1^\sigma\alpha_{\max}}+2^{\sigma/2}K_s\right)\left(\frac{1}{1-\frac{1}{\alpha_{\max}M_1^\sigma}}\right)\right).
\end{aligned}
$$

Hence,

$$
R_{f^\alpha}[D_{(i_1)_1\ldots(i_1)_m\ldots(i_N)_1\ldots(i_N)_m}] \leq K_2\alpha_{\max}^m,
$$

where $K_2 = R_{f^\alpha}[I^N] + \left(\frac{2^{\sigma/2}K^*}{M_1^\sigma\alpha_{\max}}+2^{\sigma/2}K_s\right)\left(\frac{1}{1-\frac{1}{\alpha_{\max}M_1^\sigma}}\right)$. On similar lines to that in (i), we deduce

$$
\begin{aligned}
\overline{\lim}_{\delta\to 0}\frac{\log\left(N_\delta(\mathcal{G}(f^\alpha))\right)}{-\log(\delta)} &\leq \overline{\lim}_{m\to\infty}\frac{\log\left(2M^{mN}+M^{m(N+1)}M_2\alpha_{\max}^m\right)}{m\log(M)}\\
&= N+1+\frac{\log(\alpha_{\max})}{\log(M)}.
\end{aligned}
$$

(iii) If $\alpha_{\max} \leq \frac{1}{M_1^\sigma}$, then we have

$$
R_{f^\alpha}[D_{(i_1)_1\ldots(i_1)_m\ldots(i_N)_1\ldots(i_N)_m}] \leq \frac{R_{f^\alpha}[I^N]+m2^{\sigma/2}(K^*+K_s)}{M_1^{m\sigma}}.
$$

We get the following in the same way as in the above two cases:

$$
\begin{aligned}
&\overline{\lim}_{\delta\to 0}\frac{\log\left(N_\delta(\mathcal{G}(f^\alpha))\right)}{-\log(\delta)}\\
\leq&\overline{\lim}_{m\to\infty}\frac{\log\left(2M^{mN}+M^{mN}M_1^{(1-\sigma)m}\left(R_{f^\alpha}[I^N]+m2^{\sigma/2}(K^*+K_s)\right)\right)}{m\log(M)}\\
=&N+1-\sigma.
\end{aligned}
$$

□

Remark 3.10. The above theorem can be compared with [54]. It should be noted that Malysz gives an exact value of box dimension for a special class of multivariate FIFs. However, our result gives bounds of box dimension for a more general class of FIFs.

Theorem 3.11. *Let f, α and s be Hölder continuous with exponent σ. If $a := \min\{|a_{k,i_k}| : k \in \Sigma_N, i_k \in \Sigma_{M_k}\}$ and $\max\left\{\frac{\|\alpha\|_\infty}{a^\sigma}, \|\alpha\|\right\} < 1$, then f^α is Hölder continuous with exponent σ. Moreover, $N \leq \dim_H\big(\mathcal{G}(f^\alpha)\big) \leq \underline{\dim}_B\big(\mathcal{G}(f^\alpha)\big) \leq \overline{\dim}_B\big(\mathcal{G}(f^\alpha)\big) \leq N + 1 - \sigma$.*

Proof. Let $\mathcal{H}_f^\sigma(I^N) = \big\{g \in \mathcal{H}^\sigma(I^N) : g(x_{1,i_1}, \ldots, x_{N,i_N}) = f(x_{1,i_1}, \ldots, x_{N,i_N})$ for all $i_k \in \partial\Sigma_{M_k,0},\ k \in \Sigma_N\big\}$. We note that the space $\mathcal{H}_f^\sigma(I^N)$ is a closed subset of $\mathcal{H}^\sigma(I^N)$ and consequently $\mathcal{H}_f^\sigma(I^N)$ is a complete metric space concerning the metric induced by norm $\|\cdot\|$. Define a function $T : \mathcal{H}_f^\sigma(I^N) \to \mathcal{H}_f^\sigma(I^N)$ by

$$(T_f g)(x_1, \ldots, x_N) = f(x_1, \ldots, x_N) + \alpha(x_1, \ldots, x_N)\ (g - s)(u_{1,i_1}^{-1}(x_1), \ldots, u_{N,i_N}^{-1}(x_N))$$

for all $(x_1, \ldots, x_N) \in \prod_{k=1}^N I_{k,i_k}$, where $(i_1, \ldots, i_N) \in \prod_{k=1}^N \Sigma_{M_k}$. Note that

$$\begin{aligned}
&[T_f g]_\sigma \\
&= \max_{(i_1,\ldots,i_N)\in\prod_{k=1}^N \Sigma_{M_k}} \sup_{\substack{(x_1,\ldots,x_N)\neq(y_1,\ldots,y_N)\\ (x_1,\ldots,x_N),(y_1,\ldots,y_N)\in\prod_{k=1}^N I_{k,i_k}}} \frac{|T_f g(x_1, \ldots, x_N) - T_f g(y_1, \ldots, y_N)|}{\|(x_1, \ldots, x_N) - (y_1, \ldots, y_N)\|_2^\sigma} \\
&\leq \max_{(i_1,\ldots,i_N)\in\prod_{k=1}^N \Sigma_{M_k}} \Bigg[\sup_{\substack{(x_1,\ldots,x_N)\neq(y_1,\ldots,y_N)\\ (x_1,\ldots,x_N),(y_1,\ldots,y_N)\in\prod_{k=1}^N I_{k,i_k}}} \frac{|f(x_1, \ldots, x_N) - f(y_1, \ldots, y_N)|}{\|(x_1, \ldots, x_N) - (y_1, \ldots, y_N)\|_2^\sigma} \\
&\quad + \sup_{\substack{(x_1,\ldots,x_N)\neq(y_1,\ldots,y_N)\\ (x_1,\ldots,x_N),(y_1,\ldots,y_N)\in\prod_{k=1}^N I_{k,i_k}}} \\
&\qquad\qquad \frac{\|\alpha\|_\infty \Big|(g - s)(u_{1,i_1}^{-1}(x_1), \ldots, u_{N,i_N}^{-1}(x_N)) - (g - s)(u_{1,i_1}^{-1}(y_1), \ldots, u_{N,i_N}^{-1}(y_N))\Big|}{\|(x_1, \ldots, x_N) - (y_1, \ldots, y_N)\|_2^\sigma} \\
&\quad + \sup_{\substack{(x_1,\ldots,x_N)\neq(y_1,\ldots,y_N)\\ (x_1,\ldots,x_N),(y_1,\ldots,y_N)\in\prod_{k=1}^N I_{k,i_k}}} \frac{\|g - s\|_\infty \Big|\alpha(x_1, \ldots, x_N) - \alpha(y_1, \ldots, y_N)\Big|}{\|(x_1, \ldots, x_N) - (y_1, \ldots, y_N)\|_2^\sigma} \Bigg]
\end{aligned}$$

$$\leq [f]_\sigma + \frac{\|\alpha\|_\infty}{a^\sigma}\left([g]_\sigma + [s]_\sigma\right) + \|g-s\|_\infty [\alpha]_\sigma.$$

For $g, h \in \mathcal{H}_f^\sigma(I^N)$

$$\begin{aligned}
\|T_f g - T_f h\| &= \|T_f g - T_f h\|_\infty + [T_f g - T_f h]_\sigma \\
&\leq \|\alpha\|_\infty \|g-h\|_\infty + \frac{|\alpha|}{a^\sigma}[g-h]_\sigma + \|g-h\|_\infty [\alpha]_\sigma \\
&= \frac{\|\alpha\|_\infty}{a^\sigma}[g-h]_\sigma + \|g-h\|_\infty \|\alpha\| \\
&\leq \frac{\|\alpha\|_\infty}{a^\sigma}\|g-h\| + \|g-h\|_\infty \|\alpha\| \\
&\leq \max\left\{\frac{\|\alpha\|_\infty}{a^\sigma}, \|\alpha\|\right\}\|g-h\|.
\end{aligned}$$

It follows from the hypothesis that T is a contraction function on $\mathcal{H}_f^\sigma(I^N)$. Hence, a unique fixed point of T is obtained by the Banach contraction principle. In particular, $f^\alpha \in \mathcal{H}_f^\sigma(I^N)$.

Let K_{f^α} be the Hölder constant for a Hölder continuous function $f^\alpha : I^N \to \mathbb{R}$ with exponent σ. For $\delta = \frac{1}{M^m}$ and using the Lemma 3.3 we get

$$\overline{\dim}_B\left(\mathcal{G}(f^\alpha)\right) = \overline{\lim_{\delta\to 0}} \frac{\log\left(N_\delta\left(\mathcal{G}\left(f^\alpha\right)\right)\right)}{-\log(\delta)} \leq \overline{\lim_{m\to\infty}} \frac{\log\left(2M^{mN} + M^{m(1-\sigma)}M^{mN}K_{f^\alpha}2^{\frac{\sigma}{2}}\right)}{m\log(M)} = 1 + N - \sigma,$$

and this completes the proof. □

Example 3.12. Consider the following function in the domain $[-1,1]\times[-1,1]$ as the germ function

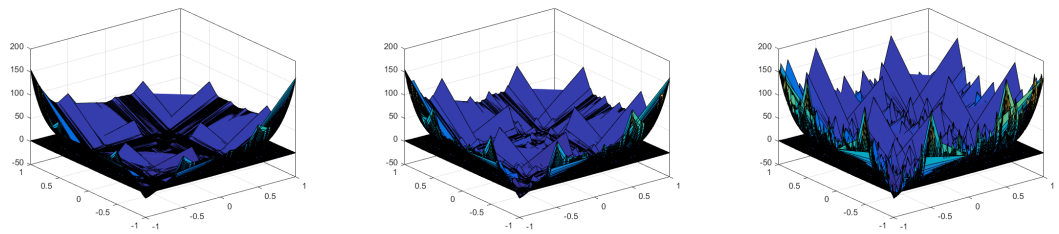
$$f(x,y) = 41(y^3 - x^5)^2 + (y - x^2)^3.$$

Let us consider the following parameters:

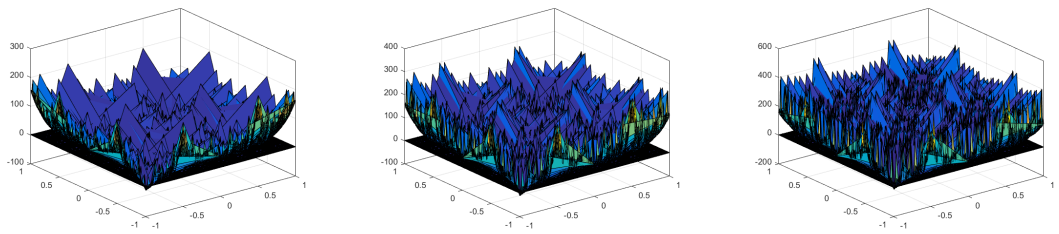
(i) A net Δ determined by the partition $\{-1, -0.5, 0, 0.5, 1\}$ of $[-1,1]$.

(ii) Scaling function $\alpha(x,y) = 0.1, 0.2, 0.4, 0.5, 0.7, 0.9$, respectively for all $(x,y) \in [-1,1]\times[-1,1]$.

(iii) Base function $s(x, y) = x^3y^5f(x, y)$.



$\alpha(x, y) = 0.1$ $\alpha(x, y) = 0.2$ $\alpha(x, y) = 0.4$



$\alpha(x, y) = 0.5$ $\alpha(x, y) = 0.7$ $\alpha(x, y) = 0.9$

FIGURE 3.1: $\mathcal{G}(f^\alpha)$ on different Scaling function

3.2.1 Restrictions of α-Fractal Function on Axis

Let f^α be a fractal interpolation function as in Subsection 1.3.4. Now define a function $g^{\alpha^g} : I_{1,i_1} \to \mathbb{R}$ such that

$$g^{\alpha^g}(x_1) = f^\alpha\left(x_1, u_{2,1}(0), \ldots, u_{N,1}(0)\right) \text{ for all } x_1 \in I_{1,i_1}, \tag{3.6}$$

where $\alpha^g(x_1) = \alpha\left(x_1, u_{2,1}(0), \ldots, u_{N,1}(0)\right)$ and $s^g(x_1, \ldots, x_N) = s(x_1, u_{2,1}(0), \ldots, u_{N,1}(0))$.

Theorem 3.13. *Let $f \in \mathcal{C}(I^N)$ be a continuous function and f^α the α-fractal function corresponding to f, then g^{α^g} as defined in (3.6) is the induced α-fractal function corresponding to a continuous function $g : I \to \mathbb{R}$, where $g(x^1) = f(x^1, u_{2,1}(0), \ldots, u_{N,1}(0))$ for all $x^1 \in I$. Moreover,* $\dim\left(\mathcal{G}\left(g^{\alpha^g}\right)\right) \leq \dim\left(\mathcal{G}\left(f^\alpha\right)\right)$.

Proof. To prove that g^{α^g} is an α-fractal function corresponding to g, it is sufficient to prove that g^{α^g} satisfies a self-referential equation. With the definition of u_{k,i_k}, it is clear that for all $k \in \Sigma_N$, $u_{k,1}(0) = 0$, then we have

$$\begin{aligned}
&g^{\alpha^g}(x_1)\\
=&f^{\alpha}\left(x_1, u_{2,1}(0), \ldots, u_{N,1}(0)\right)\\
=&f(x_1, u_{2,1}(0), \ldots, u_{N,1}(0)) + \alpha\left(u_{1,i_1}^{-1}(x_1), 0, \ldots, 0\right)\left[f^{\alpha}\left(u_{1,i_1}^{-1}(x_1), 0, \ldots, 0\right) - s\left(u_{1,i_1}^{-1}(x_1), 0, \ldots, 0\right)\right]\\
=&g(x_1) + \alpha^g(u_{1,i_1}^{-1}(x))\left[g^{\alpha^g}\left(u_{1,i_1}^{-1}(x_1)\right) - s^{\alpha^g}\left(u_{1,i_1}^{-1}(x_1)\right)\right].
\end{aligned}$$

It satisfies a self-referential equation, hence g^{α^g} is an α-fractal function corresponding to g.

Moreover, $\mathcal{G}\left(g^{\alpha^g}\right) \subseteq \mathcal{G}\left(f^{\alpha}\right)$, implies that

$$\dim\left(\mathcal{G}\left(g^{\alpha^g}\right)\right) \leq \dim\left(\mathcal{G}\left(f^{\alpha}\right)\right).$$

This completes the proof. □

Example 3.14. Let $f : I^2 \to \mathbb{R}$ be a real-valued function. Using f, we define $g : I \to \mathbb{R}$ as follows:

$$g(x) = f(x, 0). \tag{3.7}$$

It can be seen that

$$\dim \mathcal{G}(g) \leq \dim \mathcal{G}(f).$$

Consider $f(x, y) = x^2 + y^2$, then the function $g(x) = x^2$ is obtained by using (3.7). It can be easily seen that $\dim \mathcal{G}(f) = 2$ and $\dim \mathcal{G}(g) = 1$, that is, $\dim \mathcal{G}(g) \leq \dim \mathcal{G}(f)$. Now, take $f(x, y) = W(x) + y$, where $W : I \to \mathbb{R}$ is the Weierstrass function. Similarly, we get $g(x) = W(x)$. Finally, $\dim \mathcal{G}(f) = 1 + \dim(W)$ and

$\dim \mathcal{G}(g) = \dim \mathcal{G}(W)$. Hence, $\dim \mathcal{G}(g) \leq \dim \mathcal{G}(f)$. Now, select $f(x,y) = W(x)y$, then $g(x) = 0$. Thus, we get $\dim \mathcal{G}(fg) = 1$.

Theorem 3.15. *Let f, α and s be Hölder continuous with exponent σ. If $a := \min\{|a_{k,i_k}| : k \in \Sigma_N, i_k \in \Sigma_{M_k}\}$ and $\max\left\{\frac{\|\alpha\|_\infty}{a^\sigma}, \|\alpha\|\right\} < 1$, then g^{α^g} is Hölder continuous with exponent σ. Moreover, $1 \leq \dim_H\left(\mathcal{G}(g^{\alpha^g})\right) \leq \underline{\dim}_B\left(\mathcal{G}(g^{\alpha^g})\right) \leq \overline{\dim}_B\left(\mathcal{G}(g^{\alpha^g})\right) \leq 2 - \sigma$.*

Proof. In view of Theorem 3.11, the fractal function f^α is Hölder continuous with exponent σ. Since

$$g^{\alpha^g}(x_1) = f^\alpha\left(x_1, u_{2,1}(0), \ldots, u_{N,1}(0)\right) \text{ for all } x_1 \in I_{1,i_1},$$

where $\alpha^g(x_1) = \alpha\left(x_1, u_{2,1}(0), \ldots, u_{N,1}(0)\right)$, $s^g(x_1, \ldots, x_N) = s(x_1, u_{2,1}(0), \ldots, u_{N,1}(0))$, we have

$$|g^{\alpha^g}(x) - g^{\alpha^g}(y)| = |f^\alpha\left(x, u_{2,1}(0), \ldots, u_{N,1}(0)\right) - f^\alpha\left(y, u_{2,1}(0), \ldots, u_{N,1}(0)\right)| \leq |x-y|^\sigma.$$

That is, g^{α^g} is Hölder continuous with exponent σ.
Now, by [36, Corollary 11.2] it is obvious that $\overline{\dim}_B\left(\mathcal{G}(g^{\alpha^g})\right) \leq 2-\sigma$. This completes the proof. □

3.3 Mixed Riemann-Liouville Fractional Integral

Definition 3.16. The Mixed Riemann-Liouville fractional integral of a multivariate continuous function $f : I_N \to \mathbb{R}$ is defined as

$$\left(\mathfrak{J}^{(\mu_1,\ldots,\mu_N)} f\right)(x_1, \ldots, x_N)$$

$$=\frac{1}{\Gamma(\mu_1)\dots\Gamma(\mu_N)}\int_0^{x_1}\cdots\int_0^{x_N}(x_1-t_1)^{\mu_1-1}\dots(x_N-t_N)^{\mu_N-1}f(t_1,\dots,t_N)\,\mathrm{d}t_1\dots\mathrm{d}t_N,$$

where $\mu_1,\dots\mu_N>0$ and $(x_1,\dots,x_N)\neq(-1,\dots,-1)$ for all $(x_1,\dots,x_N)\in I^N$.

Theorem 3.17. *Let g^* be the FIF defined as in (1.12) which is determined by the IFS defined in (1.10), then $\mathfrak{J}^{(\mu_1,\dots,\mu_N)}g^*$ will satisfy the following self-referential equation*

$$\mathfrak{J}^{(\mu_1,\dots,\mu_N)}g^*(x_1,\dots,x_N)=\widehat{F}_{i_1\dots i_N}\left(u_{1,i_1}^{-1}(x_1),\dots,u_{N,i_N}^{-1}(x_N),\ \mathfrak{J}^{(\mu_1,\dots,\mu_N)}g^*\left(u_{1,i_1}^{-1}(x_1),\dots,u_{N,i_N}^{-1}(x_N)\right)\right),$$

where $\widehat{F}_{i_1\dots i_N}:X\to\mathbb{R}$ is defined as,

$$\widehat{F}_{i_1\dots i_N}(x_1,\dots,x_N,z)=\delta a_{N,i_N}^{\mu_N}\dots a_{1,i_1}^{\mu_1}z+\widehat{\mathcal{B}}_{i_1\dots i_N}(x_1,\dots,x_N),$$

here

$$\begin{aligned}
&\widehat{\mathcal{B}}_{i_1\dots i_N}(x_1,\dots,x_N)\\
&=\mathcal{M}\Bigg(\int_0^{u_{1,i_1}(x_1)}\cdots\int_0^{u_{N,i_N}(0)}(u_{1,i_1}(x_1)-t_1)^{\mu_1-1}\dots(u_{N,i_N}(x_N)-t_N)^{\mu_N-1}g^*(t_1,\dots,t_N)\,\mathrm{d}t_1\dots\mathrm{d}t_N\\
&\quad+a_{N,i_N}^{\mu_N}\int_0^{u_{1,i_1}(x_1)}\cdots\int_0^{u_{N-1,i_{N-1}}(0)}\int_0^{x_N}(u_{1,i_1}(x_1)-t_1)^{\mu_1-1}\dots(x_N-w_N)^{\mu_N-1}\\
&\qquad\qquad g^*(t_1,\dots,u_{N,i_N}(w_N))\,\mathrm{d}t_1\dots\mathrm{d}t_{N-1}\mathrm{d}w_N\\
&\quad+a_{N,i_N}^{\mu_N}a_{N-1,i_{N-1}}^{\mu_{N-1}}\int_0^{u_{1,i_1}(x_1)}\cdots\int_0^{x_{N-1}}\int_0^{x_N}(u_{1,i_1}(x_1)-t_1)^{\mu_1-1}\dots(x_{N-1}-w_{N-1})^{\mu_{N-1}-1}(x_N-w_N)^{\mu_N-1}\\
&\qquad\qquad g^*(t_1,\dots,t_{N-2},u_{N-1,i_{N-1}}(w_{N-1}),u_{N,i_N}(w_N))\,\mathrm{d}t_1\dots\mathrm{d}t_{N-2}\mathrm{d}w_{N-1}\mathrm{d}w_N\\
&\quad+\dots+a_{N,i_N}^{\mu_N}\dots a_{2,i_2}^{\mu_2}\int_0^{u_{1,i_1}(0)}\int_0^{x_2}\cdots\int_0^{x_N}(u_{1,i_1}(x_1)-t_1)^{\mu_1-1}(x_2-w_2)^{\mu_2-1}\dots(x_N-w_N)^{\mu_N-1}\\
&\qquad\qquad g^*(t_1,u_{2,i_2}(w_2),\dots,u_{N,i_N}(w_N))\,\mathrm{d}t_1\mathrm{d}w_2\dots\mathrm{d}w_N\\
&\quad+a_{N,i_N}^{\mu_N}\dots a_{1,i_1}^{\mu_1}\int_0^{x_1}\cdots\int_0^{x_N}(x_1-w_1)^{\mu_1-1}\dots(x_N-w_N)^{\mu_N-1}\mathcal{B}_{i_1\dots i_N}(w_1,\dots,w_N)\,\mathrm{d}w_1\dots\mathrm{d}w_N\Bigg),
\end{aligned}$$

where $\mathcal{M}=\dfrac{1}{\Gamma(\mu_1)\dots\Gamma(\mu_N)}$.

Proof. For $(x_1,\dots,x_N)\in I^N$, we have

$$
\begin{aligned}
&\mathfrak{J}^{(\mu_1,\dots,\mu_N)}g^*\left(u_{1,i_1}(x_1),\dots,u_{N,i_N}(x_N)\right)\\
=&\mathcal{M}\int_0^{u_{1,i_1}(x_1)}\cdots\int_0^{u_{N,i_N}(x_N)}(u_{1,i_1}(x_1)-t_1)^{\mu_1-1}\dots(u_{N,i_N}(x_N)-t_N)^{\mu_N-1}g^*(t_1,\dots,t_N)\,\mathrm{d}t_1\dots\mathrm{d}t_N\\
=&\mathcal{M}\left(\int_0^{u_{1,i_1}(x_1)}\cdots\int_0^{u_{N,i_N}(0)}(u_{1,i_1}(x_1)-t_1)^{\mu_1-1}\dots(u_{N,i_N}(x_N)-t_N)^{\mu_N-1}g^*(t_1,\dots,t_N)\,\mathrm{d}t_1\dots\mathrm{d}t_N\right.\\
&\left.+\int_0^{u_{1,i_1}(x_1)}\cdots\int_{u_{N,i_N}(0)}^{u_{N,i_N}(x_N)}(u_{1,i_1}(x_1)-t_1)^{\mu_1-1}\dots(u_{N,i_N}(x_N)-t_N)^{\mu_N-1}g^*(t_1,\dots,t_N)\,\mathrm{d}t_1\dots\mathrm{d}t_N\right)
\end{aligned}
$$

put $t_N=u_{N,i_N}(w_N)$ in the second integral, then we get

$$
\begin{aligned}
=&\mathcal{M}\left(\int_0^{u_{1,i_1}(x_1)}\cdots\int_0^{u_{N,i_N}(0)}(u_{1,i_1}(x_1)-t_1)^{\mu_1-1}\dots(u_{N,i_N}(x_N)-t_N)^{\mu_N-1}g^*(t_1,\dots,t_N)\,\mathrm{d}t_1\dots\mathrm{d}t_N\right.\\
&+a_{N,i_N}\int_0^{u_{1,i_1}(x_1)}\cdots\int_0^{x_N}(u_{1,i_1}(x_1)-t_1)^{\mu_1-1}\dots(u_{N,i_N}(x_N)-u_{N,i_N}(w_N))^{\mu_N-1}\\
&\left.g^*(t_1,\dots,u_{N,i_N}(w_N))\,\mathrm{d}t_1\dots\mathrm{d}t_{N-1}\mathrm{d}w_N\right)\\
=&\mathcal{M}\left(\int_0^{u_{1,i_1}(x_1)}\cdots\int_0^{u_{N,i_N}(0)}(u_{1,i_1}(x_1)-t_1)^{\mu_1-1}\dots(u_{N,i_N}(x_N)-t_N)^{\mu_N-1}g^*(t_1,\dots,t_N)\,\mathrm{d}t_1\dots\mathrm{d}t_N\right.\\
&+a_{N,i_N}^{\mu_N}\int_0^{u_{1,i_1}(x_1)}\cdots\int_0^{u_{N-1,i_{N-1}}(0)}\int_0^{x_N}(u_{1,i_1}(x_1)-t_1)^{\mu_1-1}\dots(x_N-w_N)^{\mu_N-1}\\
&g^*(t_1,\dots,u_{N,i_N}(w_N))\,\mathrm{d}t_1\dots\mathrm{d}t_{N-1}\mathrm{d}w_N\\
&+a_{N,i_N}^{\mu_N}\int_0^{u_{1,i_1}(x_1)}\cdots\int_{u_{N-1,i_{N-1}}(0)}^{u_{N-1,i_{N-1}}(x_{N-1})}\int_0^{x_N}(u_{1,i_1}(x_1)-t_1)^{\mu_1-1}\dots(x_N-w_N)^{\mu_N-1}\\
&\left.g^*(t_1,\dots,u_{N,i_N}(w_N))\,\mathrm{d}t_1\dots\mathrm{d}t_{N-1}\mathrm{d}w_N\right)
\end{aligned}
$$

put $t_{N-1}=u_{N-1,i_{N-1}}(w_{N-1})$ in the third integral, then

$$
\begin{aligned}
=&\mathcal{M}\left(\int_0^{u_{1,i_1}(x_1)}\cdots\int_0^{u_{N,i_N}(0)}(u_{1,i_1}(x_1)-t_1)^{\mu_1-1}\dots(u_{N,i_N}(x_N)-t_N)^{\mu_N-1}g^*(t_1,\dots,t_N)\,\mathrm{d}t_1\dots\mathrm{d}t_N\right.\\
&+a_{N,i_N}^{\mu_N}\int_0^{u_{1,i_1}(x_1)}\cdots\int_0^{u_{N-1,i_{N-1}}(0)}\int_0^{x_N}(u_{1,i_1}(x_1)-t_1)^{\mu_1-1}\dots(x_N-w_N)^{\mu_N-1}\\
&g^*(t_1,\dots,u_{N,i_N}(w_N))\,\mathrm{d}t_1\dots\mathrm{d}t_{N-1}\mathrm{d}w_N\\
&+a_{N,i_N}^{\mu_N}a_{N-1,i_{N-1}}\int_0^{u_{1,i_1}(x_1)}\cdots\int_0^{x_{N-1}}\int_0^{x_N}(u_{1,i_1}(x_1)-t_1)^{\mu_1-1}\dots\big(u_{N-1,i_{N-1}}(x_{N-1})\\
&-u_{N-1,i_{N-1}}(w_{N-1})\big)^{\mu_{N-1}-1}(x_N-w_N)^{\mu_N-1}g^*(t_1,\dots,t_{N-2},u_{N-1,i_{N-1}}(w_{N-1}),u_{N,i_N}(w_N))\\
&\left.\mathrm{d}t_1\dots\mathrm{d}t_{N-2}\mathrm{d}w_{N-1}\mathrm{d}w_N\right)
\end{aligned}
$$

$$
\begin{aligned}
=&\mathcal{M}\Bigg(\int_0^{u_{1,i_1}(x_1)}\cdots\int_0^{u_{N,i_N}(0)}(u_{1,i_1}(x_1)-t_1)^{\mu_1-1}\dots(u_{N,i_N}(x_N)-t_N)^{\mu_N-1}g^*(t_1,\dots,t_N)\ \mathrm{d}t_1\dots\mathrm{d}t_N\\
&+a_{N,i_N}^{\mu_N}\int_0^{u_{1,i_1}(x_1)}\cdots\int_0^{u_{N-1,i_{N-1}}(0)}\int_0^{x_N}(u_{1,i_1}(x_1)-t_1)^{\mu_1-1}\dots(x_N-w_N)^{\mu_N-1}\\
&\qquad g^*(t_1,\dots,u_{N,i_N}(w_N))\mathrm{d}t_1\dots\mathrm{d}t_{N-1}\mathrm{d}w_N\\
&+a_{N,i_N}^{\mu_N}a_{N-1,i_{N-1}}^{\mu_{N-1}}\int_0^{u_{1,i_1}(x_1)}\cdots\int_0^{x_{N-1}}\int_0^{x_N}(u_{1,i_1}(x_1)-t_1)^{\mu_1-1}\dots(x_{N-1}-w_{N-1})^{\mu_{N-1}-1}\\
&(x_N-w_N)^{\mu_N-1}g^*(t_1,\dots,t_{N-2},u_{N-1,i_{N-1}}(w_{N-1}),u_{N,i_N}(w_N))\ \mathrm{d}t_1\dots\mathrm{d}t_{N-2}\mathrm{d}w_{N-1}\mathrm{d}w_N\Bigg)
\end{aligned}
$$

going on with similar steps, we get

$$
\begin{aligned}
=&\mathcal{M}\Bigg(\int_0^{u_{1,i_1}(x_1)}\cdots\int_0^{u_{N,i_N}(0)}(u_{1,i_1}(x_1)-t_1)^{\mu_1-1}\dots(u_{N,i_N}(x_N)-t_N)^{\mu_N-1}g^*(t_1,\dots,t_N)\ \mathrm{d}t_1\dots\mathrm{d}t_N\\
&+a_{N,i_N}^{\mu_N}\int_0^{u_{1,i_1}(x_1)}\cdots\int_0^{u_{N-1,i_{N-1}}(0)}\int_0^{x_N}(u_{1,i_1}(x_1)-t_1)^{\mu_1-1}\dots(x_N-w_N)^{\mu_N-1}\\
&\qquad g^*(t_1,\dots,u_{N,i_N}(w_N))\mathrm{d}t_1\dots\mathrm{d}t_{N-1}\mathrm{d}w_N\\
&+a_{N,i_N}^{\mu_N}a_{N-1,i_{N-1}}^{\mu_{N-1}}\int_0^{u_{1,i_1}(x_1)}\cdots\int_0^{x_{N-1}}\int_0^{x_N}(u_{1,i_1}(x_1)-t_1)^{\mu_1-1}\dots(x_{N-1}-w_{N-1})^{\mu_{N-1}-1}\\
&(x_N-w_N)^{\mu_N-1}g^*(t_1,\dots,t_{N-2},u_{N-1,i_{N-1}}(w_{N-1}),u_{N,i_N}(w_N))\ \mathrm{d}t_1\dots\mathrm{d}t_{N-2}\mathrm{d}w_{N-1}\mathrm{d}w_N\\
&+\dots+a_{N,i_N}^{\mu_N}\dots a_{2,i_2}^{\mu_2}\int_0^{u_{1,i_1}(0)}\int_0^{x_2}\cdots\int_0^{x_N}(u_{1,i_1}(x_1)-t_1)^{\mu_1-1}(x_2-w_2)^{\mu_2-1}\dots(x_N-w_N)^{\mu_N-1}\\
&\qquad g^*(t_1,u_{2,i_2}(w_2),\dots,u_{N,i_N}(w_N))\ \mathrm{d}t_1\mathrm{d}w_2\dots\mathrm{d}w_N+a_{N,i_N}^{\mu_N}\dots a_{1,i_1}^{\mu_1}\\
&\int_0^{x_1}\cdots\int_0^{x_N}(x_1-w_1)^{\mu_1-1}\dots(x_N-w_N)^{\mu_N-1}g^*(u_{1,i_1}(w_1),\dots,u_{N,i_N}(w_N))\ \mathrm{d}w_1\dots\mathrm{d}w_N\Bigg)\\
=&\mathcal{M}\Bigg(\int_0^{u_{1,i_1}(x_1)}\cdots\int_0^{u_{N,i_N}(0)}(u_{1,i_1}(x_1)-t_1)^{\mu_1-1}\dots(u_{N,i_N}(x_N)-t_N)^{\mu_N-1}g^*(t_1,\dots,t_N)\ \mathrm{d}t_1\dots\mathrm{d}t_N\\
&+a_{N,i_N}^{\mu_N}\int_0^{u_{1,i_1}(x_1)}\cdots\int_0^{u_{N-1,i_{N-1}}(0)}\int_0^{x_N}(u_{1,i_1}(x_1)-t_1)^{\mu_1-1}\dots(x_N-w_N)^{\mu_N-1}\\
&\qquad g^*(t_1,\dots,u_{N,i_N}(w_N))\mathrm{d}t_1\dots\mathrm{d}t_{N-1}\mathrm{d}w_N\\
&+a_{N,i_N}^{\mu_N}a_{N-1,i_{N-1}}^{\mu_{N-1}}\int_0^{u_{1,i_1}(x_1)}\cdots\int_0^{x_{N-1}}\int_0^{x_N}(u_{1,i_1}(x_1)-t_1)^{\mu_1-1}\dots(x_{N-1}-w_{N-1})^{\mu_{N-1}-1}\\
&\qquad(x_N-w_N)^{\mu_N-1}g^*(t_1,\dots,t_{N-2},u_{N-1,i_{N-1}}(w_{N-1}),u_{N,i_N}(w_N))\ \mathrm{d}t_1\dots\mathrm{d}t_{N-2}\mathrm{d}w_{N-1}\mathrm{d}w_N\\
&+\dots+a_{N,i_N}^{\mu_N}\dots a_{2,i_2}^{\mu_2}\int_0^{u_{1,i_1}(0)}\int_0^{x_2}\cdots\int_0^{x_N}(u_{1,i_1}(x_1)-t_1)^{\mu_1-1}(x_2-w_2)^{\mu_2-1}\dots(x_N-w_N)^{\mu_N-1}\\
&\qquad g^*(t_1,u_{2,i_2}(w_2),\dots,u_{N,i_N}(w_N))\ \mathrm{d}t_1\mathrm{d}w_2\dots\mathrm{d}w_N\\
&+a_{N,i_N}^{\mu_N}\dots a_{1,i_1}^{\mu_1}\int_0^{x_1}\cdots\int_0^{x_N}(x_1-w_1)^{\mu_1-1}\dots(x_N-w_N)^{\mu_N-1}\delta g^*(w_1,\dots,w_N)\ \mathrm{d}w_1\dots\mathrm{d}w_N\\
&+a_{N,i_N}^{\mu_N}\dots a_{1,i_1}^{\mu_1}\int_0^{x_1}\cdots\int_0^{x_N}(x_1-w_1)^{\mu_1-1}\dots(x_N-w_N)^{\mu_N-1}\mathcal{B}_{i_1\dots i_N}(w_1,\dots,w_N)\ \mathrm{d}w_1\dots\mathrm{d}w_N\Bigg)
\end{aligned}
$$

$$=\delta a_{N,i_N}^{\mu_N} \dots a_{1,i_1}^{\mu_1} \mathfrak{J}^{(\mu_1,\dots,\mu_N)} g^*(x_1,\dots,x_N) + \widehat{\mathcal{B}}_{i_1\dots i_N}(x_1,\dots,x_N),$$

establishing the claim. □

Remark 3.18. The above theorem can be compared with Theorem 5.3 of [28], wherein the calculus of fractal surfaces has been studied.

Remark 3.19. The fractal integral of a FIF has also been discussed previously. See, for instance, [42, 43]. In [43], the authors have studied the F^α-integral of a FIF. However, we have discussed the Riemann-Liouville fractional integral of a FIF.

3.4 Conclusion

In this chapter, we have discussed the construction of a multivariate fractal interpolation function based on a given data set (Subsection 1.3.3). We have observed that the graph of this constructed function acts as an attractor for an IFS (Theorem 1.16) and satisfies a self-referential equation (Equation (1.12)). Additionally, we have proven the existence of multivariate α-fractal functions corresponding to continuous functions defined on I^N (Subsection 1.3.4). Furthermore, we have examined the fractal dimension of these multivariate α-fractal functions (Theorems 3.9, 3.11), as well as their restrictions on the coordinate axis (Subsection 3.2.1). Finally, we have demonstrated that the mixed Riemann-Liouville fractional integral of a multivariate fractal interpolation function satisfies the self-referential equation (Theorem 3.17).

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