Optimality Concepts and Methods for Interval-Valued and Set-Valued Optimization Problems

Thesis submitted in partial fulfillment

for the Award of Degree

Doctor of Philosophy

by

Krishan Kumar

DEPARTMENT OF MATHEMATICAL SCIENCES

INDIAN INSTITUTE OF TECHNOLOGY

(BANARAS HINDU UNIVERSITY)

VARANASI - 221005

Roll No. 19121001 **October 2024**

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26/10/24

Dr. Debdas Ghosh
(Supervisor)
Associate Professor
Department of Mathematical Sciences
Indian Institute of Technology
(Banaras Hindu University)
Varanasi-221005

पर्यवेक्षक /Supervisor
Indian Institute of Technology
(Banaras Hindu University)
Varanasi-221005

DECLARATION BY THE CANDIDATE

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Place: Varanasi

Krishan Kr. 26/10/2024
(Krishan Kumar)

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26/10/24
(Dr. Debdas Ghosh)
Associate Professor
Department of Mathematical Sciences
Indian Institute of Technology
(Banaras Hindu University)
Varanasi-221005

पर्यवेक्षक/Supervisor
गणितीय विज्ञान विभाग
Department of Mathematical Sciences
भारतीय प्रौद्योगिकी संस्थान
Indian Institute of Technology
(काशी हिन्दू विश्वविद्यालय)
(Banaras Hindu University)
वाराणसी/Varanasi-221005

26.10.2024
(Prof. Sanjay Kumar Pandey)
Professor and Head
Department of Mathematical Sciences
Indian Institute of Technology
(Banaras Hindu University)
Varanasi-221005

विभागाध्यक्ष/HEAD
गणितीय विज्ञान विभाग
Department of Mathematical Sciences
भारतीय प्रौद्योगिकी संस्थान
Indian Institute of Technology
(काशी हिन्दू विश्वविद्यालय)
Banaras Hindu University)
वाराणसी/Varanasi-22100

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Place: Varanasi

(Krishan Kumar)

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I lovingly devote this research to my parents, the foundation of my life, whose unwavering support and love have been my greatest source of strength.

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Krishan Kr. 26/10/2024

Date: 26/10/2024

Place: Varanasi **(Krishan Kumar)**

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List of Symbols

Symbol	Description
$\mathbb{R}$	Set of real numbers
$\mathbb{R}^+$	Set of nonnegative real numbers
$\mathbb{N}$	Set of natural numbers
$\mathbb{R}^m$	$\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ (m-times)
$\mathbb{R}^m_+$	$\mathbb{R}_+ \times \mathbb{R}_+ \times \cdots \times \mathbb{R}_+$
$\lVert\cdot\rVert$	Euclidean norm on $\mathbb{R}^n$
$\langle \cdot, \cdot \rangle$	Standard inner product on $\mathbb{R}^n$
$I(\mathbb{R})$	Set of all closed and bounded intervals in $\mathbb{R}$
$\overline{\mathbb{R}}$	$\mathbb{R} \cup \{-\infty, +\infty\}$
$\overline{I(\mathbb{R})}$	$I(\mathbb{R}) \cup \{-\infty, +\infty\}$
$I(\mathbb{R})^n$	Set of interval vectors
$\mathbb{B}$	Closed unit ball in $\mathbb{R}^n$
$\psi^*_S(x)$	Support function of a subset S of $\mathbb{R}^n$ at $x \in \mathbb{R}^n$
$\mathrm{cl}(S)$	Closure of the set S
$\mathscr{P}(\mathbb{R}^m)$	Class of all nonempty subsets of $\mathbb{R}^m$
$\mathrm{int}(A)$	Interior of the set A
$\mathrm{conv}(A)$	Convex hull of the set A
$K \in \mathscr{P}(\mathbb{R}^m)$	Closed, convex, pointed, and solid cone
K^*	Set $\{v \in \mathbb{R}^m : v^\top u \geq 0 \text{ for all } u \in K\}$, i.e., dual cone of K
$[p]$	Set $\{1, 2, \ldots, p\}$, where $p \in \mathbb{N}$
$\mathcal{S}_x$	Set $\mathcal{S} - x = \{s - x : s \in \mathcal{S}\}$, where $\mathcal{S} \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$
$\mathrm{J}\, f(x)$	Jacobian of the function f at x and range or image space of $\mathrm{J}\, f(x)$, where $f : \mathbb{R}^n \to \mathbb{R}^m$ is a differentiable function at x

List of Abbreviations

Abbreviation	**Description**
IVF	Interval-Valued Function
IOP	Interval Optimization Problem
KKT	Karush-Kuhn-Tucker
WSM	Weak Sharp Minima
ILPP	Interval Linear Programming Problem
MRPIOP	Minimum Risk Portfolio Interval Optimization Problem
SOP	Set Optimization Problem
VOP	Vector Optimization Problem
CSOP	Constrained Set Optimization Problem
CVOP	Constrained Vector Optimization Problem
FR	Fletcher-Reeves
CD	Conjugate Descent
SD	Steepest Descent

PREFACE

Optimization is a fundamental tool that plays an essential role across a wide range of disciplines, including science, engineering, economics, and management. It serves as the backbone of decision-making processes that seek to identify the most effective solutions within specific constraints. Whether the objective is to enhance efficiency, reduce costs, or reconcile competing goals, optimization offers a systematic framework for finding the optimal solution. By formulating problems to allow the calculation of maximum or minimum values for a given function under various conditions, optimization provides a precise methodology for tackling complex challenges. Utilizing advanced mathematical models and algorithms, optimization empowers decision-makers to navigate the complexities of real-world problems, where resources are constrained and objectives are conflicting in nature.

Classical mathematics often struggles to capture accurately real-world situations due to the inherent uncertainty of events and environments. Commonly, the collected data is imprecise or inexact, influenced by factors such as measurement errors or random occurrences. In some cases, the data may only be rough estimates, adding to the challenge of representing complex systems with precision. To handle this uncertainty in problems, some specific areas of optimization have been developed. These areas include robust optimization, fuzzy optimization, stochastic optimization, interval optimization, multiobjective optimization, and set optimization. Among these potential fields, this thesis focuses on interval and set optimization.

Optimization involving interval and set-valued problems represents a significant advancement in the field, addressing the complexities that arise when uncertainty and imprecision are inherent in the decision-making process. Unlike traditional optimization, where decisions are based on precise values, interval optimization deals with uncertain data by representing it as intervals, thus allowing for solutions that are robust to fluctuations in the input. Similarly, set-valued optimization extends this concept further by considering solutions that are not single values but entire sets, accommodating a wider range of possible outcomes. These approaches are particularly useful in real-world applications, where data is imprecise or subject to variation, such as in engineering design, financial modeling, and robust decision-making under uncertainty. The ability to optimize over intervals and sets ensures that solutions are not only theoretically sound but also practical and adaptable in dynamic environments.

In this thesis, we define the concept of a support function for interval vectors and subdifferentiablity for interval-valued functions using generalized Hukuhara difference (known as gH-subdifferentiability). Next, the notion of weak sharp minima for interval-valued functions is given. Using the support function and subdifferentiability, primal and dual characterization of weak sharp minima for interval-valued functions are presented. Further, a more general form of gH-subdifferentiability, ϵ-subdifferentiability (gH_ϵ-subdifferentiability), is defined for interval-valued functions. Subsequently, the difference between gH-subdifferentiability and gH_ϵ-subdifferentiability is also mentioned. The nonemptiness, closedness, boundedness, and convexity of both gH and gH_ϵ-subdifferential set of interval-valued functions are proved. Next, the definition of an approximate solution to the interval optimization problem is given, and with the help of gH_ϵ-subdifferentiability, two necessary and sufficient optimality conditions are proved. It is also given how gH_ϵ-subdifferentiability can be used to find an approximate solution to interval minimax optimization problems.

After the theoretical analysis and optimality conditions for interval optimization, we move to the numerical development for set optimization, which is a more general framework, in this thesis. Recently, numerical algorithms for set optimization problems have gained significant interest. Researchers are providing different methods for different types of set optimization problems nowadays. Recently, a steepest descent method for a particular class of set optimization problems, in which the objective function has finite cardinality, has been proposed. We extend the numerical development for this class of set optimization problems in this thesis. After the steepest descent method, conjugate gradient and projected gradient methods are the most fundamental and prominent first-order methods, which are commonly used by researchers. Therefore, in this thesis, we propose the conjugate gradient method and projected gradient method for unconstrained and constrained set optimization problems whose objective function has the collection of finite vector-valued functions, respectively. Subsequently, the well-definedness, convergence and numerical demonstration of proposed methods are also shown. Moreover, a numerical comparison with the existing steepest descent method for the considered unconstrained set optimization problem is also given.

We hope that the contributions in this thesis will help move the fields of interval and set optimization forward. By providing new ideas and methods, this work aims to tackle the challenges of uncertainty and inexact data in real-world problems. We believe these contributions will encourage more research and lead to better and more practical solutions in both theory and applications of optimization.