A Priori and A Posteriori Error Estimates for Singularly Perturbed Differential and Integro-Differential Equations

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by

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Conclusions

The primary focus of this thesis lies in the development and analysis of parameter-uniform convergent schemes on layer-adapted meshes for various important classes of singularly perturbed problems. A significant emphasis is placed on establishing a general error analysis framework that allows to deduce uniform convergence on various layer adapted meshes conveniently within a single framework. Recognizing the ubiquitous need for error estimates that do not rely on prior knowledge of the exact solution and its derivatives, providing a posteriori error estimates is another vital contribution of this work.

This chapter provides a concise summary of the key finding, emphasizing the contributions of the thesis and the methodologies employed to achieve these results. The conclusion and notable observations are outlined as follows:

- Chapter 1 serves as the introduction to the thesis, laying out the analytical and mathematical foundations of singular perturbation theory. It includes a comprehensive literature review of uniformly convergent numerical methods for various classes of SPPs, offering insights into past and contemporary research in these domains. Furthermore, the chapter elucidates the motivation of the research work.

- Chapter 2 focuses on the construction of high-order uniformly convergent numerical methods for the class of nonlinear singularly perturbed problems with integral boundary condition. The discretization of the problem consists of a hybrid scheme defined on an arbitrary nonuniform mesh. A unified error analysis framework is introduced, and uniform convergence on various layer-adapted meshes is proved. Further, the adaptive generation of meshes is proposed based on a suitable monitor function and the mesh equidistribution principle. The method is shown to be uniformly convergent of $\mathcal{O}(N^{-2}\ln N^2)$ on Shishkin meshes and $\mathcal{O}(N^{-2})$ on Bakhvalov and equidistributed meshes.

- In Chapter 3, we examine a first-order nonlinear singularly perturbed parameterized problem with integral boundary condition. To discretize this problem, we employ the implicit Euler scheme for the nonlinear problem, whereas a composite right rectangle rule is applied to the integral boundary condition. Both a priori and a posteriori error analysis is developed for the proposed scheme, demonstrating optimal first-order uniform convergence on a priori and a posteriori meshes.

- In Chapter 4, we address the first-order linear singularly perturbed delay Volterra integro-differential equation that exhibits multiple-layer phenomena. The discretization consists of an implicit difference scheme for the derivative term and a composite numerical integration rule for the integral term. A priori and a posteriori error analysis for the proposed discrete scheme is carried out. Additionally, the comparison of uniformly accurate results on these meshes are shown.

- Chapter 5 focuses on the development of a high-order convergent adaptive numerical method for a coupled system of first-order singularly perturbed nonlinear differential equations with distinct perturbation parameters. The problem is discretized by a hybrid finite difference scheme, for which a posteriori error estimate in the maximum norm is derived. The layer-adapted meshes are generated using the equidistribution of the monitor function, chosen based on the derived a posteriori error estimate. The method is shown to produce optimal second-order accurate results. Further, it is observed that the proposed method can be easily extended for a system of arbitrary number of equations.

- Finally, in Chapter 6, we examine a system of singularly perturbed Volterra integro-differential equations with initial conditions. The derivative term in these equations is multiplied with distinct small positive parameters, giving rise to overlapping layers. We propose a numerical scheme that avoids the extra condition on the problem's data required by the scheme of Liang et al. [1]. Then, we derive a priori and a posteriori error bounds for the proposed scheme and further rectify the shortcomings of a posteriori error estimation in Liang et al. [1].