Chapter 5

Oscillations of Fourier coefficients of product of L-functions at integers in a sparse set

In this chapter, we study the asymptotic behaviour of higher moments of generalized divisor functions associated to the Fourier coefficients of Rankin-Selberg L-function associated to a normalized Hecke eigenform over $SL_2(\mathbb{Z})$, supported at the integers represented by primitive integral positive definite binary quadratic forms (reduced forms) of a fixed negative discriminant.

5.1 Introduction

Study of the asymptotic behaviour of summatory functions of arithmetical functions is one of the classical problems in analytic number theory. Such problems are referred as divisor problems ([36, p. 27], [50, p. 149], [69, p. 510]). For ℓ be a positive integer, the generalized divisor function $d_\ell(n)$ (defined in 1.2) is closely related with $\zeta^\ell(s)$, where $\zeta(s)$ is the famous Riemann zeta function. Several authors have considered divisor problems and their analogues in the case of various automorphic forms. In particular, the following divisor problem was considered by several authors:

$$S_w(X) := \sum_{n \leq X} \lambda_{w, f\times f}(n), \quad w \geq 1, \tag{5.1}$$

where $\lambda_{w,f\times f}(n) = \sum_{n=n_1\dots n_w} \lambda_{f\times f}(n_1)\cdots\lambda_{f\times f}(n_w)$ and $\lambda_{f\times f}(n) = \lambda_f^2(n)$ is the n-th normalized Fourier coefficient (defined in Chapter 1, page 17) of Rankin-Selberg L-functions $R(f\times f, s)$ associated to a Hecke eigenform $f \in S_k(SL_2(\mathbb{Z}))$ defined in [30, p. 118, Chapter 6]. The study of this type of sum was initiated independently by Rankin [62] and Selberg [64], and the following estimate (for $w = 1$) was obtained:

$$S_1(X) := \sum_{n\leq X} \lambda_f^2(n) = C_f X + O_{f,\epsilon}(X^{\frac{3}{5}+\epsilon}), \tag{5.2}$$

where C_f is a positive constant depending on f. Later, Kanemitsu, Sankaranarayanan and Tanigawa [36] extended the above result for $w \geq 2$ and obtained the estimate for the sum $S_w(X)$. More precisely, they proved the following:

$$S_w(X) := \sum_{n\leq X} \lambda_{w,f\times f}(n) = M_w(X) + O_{f,\varepsilon}(X^{1-\frac{1}{2w}+\varepsilon}). \tag{5.3}$$

Here $M_w(X) = \operatorname*{Res}_{s=1}\left(\frac{L(f\times f,s)^w}{s} X^s\right)$ which is of the form $XP_{w-1}(\log(X))$, where $P_w(t)$ is a polynomial of degree w. The estimate in (5.3) has been refined and generalized by several authors for $GL(2)$-forms (see [33, 50, 69] for more details). Such types of problems become more interesting if one considers the summatory function over a sparse sequence. More precisely, $\{x_n\}$ is a sequence of positive integers given by $n = \mathcal{Q}(x_1, x_2)$ (defined later on page 70), and if n is not represented by $\mathcal{Q}$, then $x_n = 0$ and an arithmetical function $A(n)$, it would be interesting to consider the summatory function $\sum_{n\leq X} A^\ell(x_n)$, where $\ell \geq 1$. In this direction, several interesting results for $A(n) = \lambda_f(n)$ (n-th normalized Fourier coefficient (as defined in Chapter 1 on page 17) of a normalized Hecke eigenform f) have been obtained. For details, we refer to the introduction in [51]. One interesting sequence is the sequence supported at the sum of two squares. Zhai [73] considered the sequence supported at the sum of two squares and obtained the estimates of the following sums (for $2 \leq r \leq 8$ and

$X \geq 1$):

$$\sum_{\substack{(x_1,x_2)\in\mathbb{Z}^2 \\ x_1^2+x_2^2\leq X}} (\lambda_f(x_1^2+x_2^2))^r.$$

More precisely, Zhai [73] proved that

$$\sum_{\substack{(x_1,x_2)\in\mathbb{Z}^2 \\ x_1^2+x_2^2\leq X}} (\lambda_f(x_1^2+x_2^2))^r = XP_r(log\ X) + O_{f,\epsilon}(X^{\theta_r+\epsilon}), \tag{5.4}$$

where $P_2(t), P_4(t), P_6(t), P_8(t)$ are polynomials of degree $0, 1, 4$ and 13, respectively, and $P_r(t) \equiv 0$ for $r = 3, 5, 7$ and $\theta_2 = \frac{8}{11}, \theta_3 = \frac{17}{20}, \theta_4 = \frac{43}{46}, \theta_5 = \frac{83}{86}, \theta_6 = \frac{184}{187}, \theta_7 = \frac{355}{358}, \theta_8 = \frac{752}{755}$. Later, by using the analytic properties of symmetric power L-functions, Xu [70] has extended the work of Zhai [73] for general $r \in \mathbb{N}$ and proved the estimate for the following sum associated to Hecke eigenform $f \in S_k(SL_2(\mathbb{Z}))$:

$$\sum_{x_1^2+x_2^2\leq X} \lambda_f^r(x_1^2+x_2^2).$$

In another direction, Vaishya [67, 68] studied the following sum $\sum_{\substack{\mathcal{Q}(x_1,x_2)\leq X \\ (x_1,x_2)\in\mathbb{Z}^2}} (\lambda_f(\mathcal{Q}(x_1,x_2)))^r$ for $r \leq 8$, where $\mathcal{Q}(x_1, x_2)$ is a reduced form of a fixed negative discriminant D (defined in Section 5.2) with certain conditions and obtained several interesting results. Further, Vaishya and Pandey [59] obtained the estimates for the more general power moments associated to the Fourier coefficients $\lambda_f(n)$, where n is represented by a reduced form of a fixed negative discriminant D. Also, the behaviour of sign change of the sequence $\lambda_f(\mathcal{Q}(x_1, x_2))_{(x_1,x_2)\in\mathbb{Z}^2}$ has been studied in [67, 68].

Recently, Hua [29] obtained the asymptotic behaviour of the following sum:

$$S^{i_1,\ldots,i_l}_{w_1,\ldots,w_i}(X) := \sum_{\substack{n=x_1^2+x_2^2\leq X \\ x_1,x_2\in\mathbb{Z}}} \lambda_{w_1,f\times f}(n)^{i_1}\lambda_{w_2,f\times f}(n)^{i_2}\ldots\lambda_{w_\ell,f\times f}(n)^{i_\ell}, \tag{5.5}$$

for $w_j, i_j \geq 1$ with $1 \leq j \leq \ell$, where $\ell \geq 2$, in the form of the following result [29, Theorem 1.1].

Let $\omega = \prod_{j=1}^{\ell} w_j^{i_j}$ and $\theta = \sum_{j=1}^{\ell} i_j$. For any $\epsilon > 0$, we have the following estimate for $S^{i_1,\ldots,i_\ell}_{w_1,\ldots,w_\ell}(x)$ given by

$$S^{i_1,\ldots,i_\ell}_{w_1,\ldots,w_\ell}(x) = xP_{\frac{\omega}{\theta}\binom{2\theta}{\theta-1}-1}(\log x) + O\left(X^{1-\frac{1}{2^{2\theta}\omega}}\right),$$

where $P_m(x)$ is a polynomial of degree m.

For $w \geq 1$, we define an L-function $R(f \times f, s)^w$ associated to f as follows

$$R(f \times f, s)^w = \sum_{n=1}^{\infty} \frac{\lambda_{w,f\times f}(n)}{n^s}, \tag{5.6}$$

where $\lambda_{w,f\times f}(n) := \sum_{n=n_1\ldots n_w} \lambda_{f\times f}(n_1)\cdots\lambda_{f\times f}(n_w)$. In this chapter, result of Hua [29] has been extended for any reduced form of discriminant D. We also improve certain previous estimates of Hua [29] when $\mathcal{Q}(x_1, x_2) = x_1^2 + x_2^2$.

For $m \geq 2$, the m-th symmetric power L-function associated to f is defined as

$$L(sym^m f, s) := \prod_{p-\text{prime}} \prod_{j=0}^{m} \left(1 - {\alpha_p}^{m-j} {\beta_p}^j p^{-s}\right)^{-1} = \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_f(n^m)}{n^s} = \sum_{n=1}^{\infty} \frac{\lambda_{sym^m f}(n)}{n^s}, \tag{5.7}$$

where $\lambda_{sym^m f}(n)$ is a multiplicative arithmetical function whose value at prime p is given by

$$\lambda_{sym^m f}(p) = \lambda_f(p^m) = \sum_{j=0}^{m} {\alpha_p}^{m-j} {\beta_p}^j. \tag{5.8}$$

Remark 5.1. From Deligne's bound [31], we have $|\lambda_{sym^m f}(n)| \leq d_{m+1}(n) \ll_\epsilon n^\epsilon$.

Finally, we define the twist of m-th symmetric power L-function by a Dirichlet character χ of modulo N as follows:

$$L(sym^m f \times \chi, s) = \sum_{n \geq 1} \frac{\lambda_{sym^m f}(n)\chi(n)}{n^s}. \tag{5.9}$$

The L-functions $L(sym^m f, s)$ and $L(sym^m f \times \chi, s)$ have analytic continuation to the whole complex plane and they satisfy a certain functional equation (see [31]) under $s \to 1-s$.

Remark 5.2. For a Hecke eigenform f, Shimura [65] established the automorphy of $L(sym^2 f, s)$ and exhibited the explicit description of analytic continuation and functional equation of $L(sym^2 f, s)$ under $s \to 1-s$. For $m = 3, 4$, Cogdell and Michel [16] have proved the automorphy of $L(sym^m f, s)$ the analytic continuation and functional equation of $L(sym^m f, s)$ for all $m \geq 3$ under $s \to 1-s$. For more general $m \geq 5$, Newton and his collaborators [55, 56] established the automorphy of $L(sym^m f, s)$.

We assume the following convention:

$$\begin{cases} L(sym^0 f, s) &= \zeta(s), \qquad L(sym^0 f \times \chi, s) = L(\chi, s), \\ L(sym^1 f, s) &= L(f, s), \qquad L(sym^1 f \times \chi, s) = L(f \times \chi, s). \end{cases}$$

This chapter is organized as follows. In the next section, we state the main results of this chapter. Next, we provide some key ingredients which are necessary to prove our results. Finally, we give the proof of our results.

5.2 Statement of results

To state our results, we need to set up a few notations.

A primitive integral positive definite binary quadratic form of a fixed negative discriminant D is defined as $\mathcal{Q}(x_1, x_2) := ax_1^2 + bx_1x_2 + cx_2^2$, where $(x_1, x_2) \in \mathbb{Z}^2$, $a, b, c \in \mathbb{Z}$ and $D = b^2 - 4ac < 0$ with $(a, b, c) = 1$. A primitive and positive definite binary quadratic form is said to be a reduced form if $|b| \leq a \leq c$. Throughout this section, by a reduced form $\mathcal{Q}(x_1, x_2)$, we mean a primitive integral positive definite reduced binary quadratic form with fixed negative discriminant D. Two reduced forms $\mathcal{Q}_1(x_1, x_2)$ and $\mathcal{Q}_2(x_1, x_2)$ are said to be equivalent if there are integers p, q, r and s with $ps - qr = \pm 1$ such that $\mathcal{Q}_1(x_1, x_2) = \mathcal{Q}_2(px_1 + qx_2, rx_1 + sx_2)$. For a fixed discriminant $D < 0$, $\mathcal{S}_D$ denotes the set of inequivalent reduced forms of a fixed discriminant D. It is well-known that for a given discriminant D, $\mathcal{S}_D$ is a finite set. Let $h(D)$ denotes the number of elements of discriminant D in $\mathcal{S}_D$, i.e., $h(D) := \#\mathcal{S}_D$.

We define the following generating function $\theta_D(\tau)$ associated to a reduced form $\mathcal{Q}$:

$$\theta_D(\tau) := \sum_{(x_1,x_2)\in\mathbb{Z}^2} q^{(\mathcal{Q}(x_1,x_2))} = \sum_{n=0}^{\infty} r_D(n)q^n, \qquad q = e^{2\pi i\tau},$$

where $r_D(n) := \#\{(x_1, x_2) \in \mathbb{Z}^2 | n = \mathcal{Q}(x_1, x_2) \text{ for some } \mathcal{Q} \in S_D\}$. Next, we define the character sum $r(n; D)$ as follows:

$$r(n; D) = w_D \sum_{d|n} \chi_D(d), \quad \text{where} \quad w_D = \begin{cases} 6 & \text{if} \quad D = -3, \\ 4 & \text{if} \quad D = -4, \\ 2 & \text{if} \quad D < -4, \end{cases} \tag{5.10}$$

where χ_D is a Dirichlet character modulo $|D|$, is given by Jacobi symbol $\chi_D(d) := \left(\frac{D}{d}\right)$.

Remark 5.3. The formula for $r(n; D)$ depends only on the discriminant D ($= b^2 - 4ac < 0$) and not on the choice of a, b and c.

Remark 5.4. From Weil's bound [31, p. 280], it is easy to deduce that $r_D(n) \ll n^\epsilon$, for any arbitrarily small positive real number $\epsilon > 0$.

Here, the formula $r(n; D)$ given in (5.10) agrees with $r_D(n)$. Thus, we have

$$r_D(n) = r(n; D) = w_D \sum_{d|n} \chi_D(d). \tag{5.11}$$

For more details on quadratic forms and related topics, we refer to [17, Chapter 2].

The main aim of this chapter is to study the asymptotic behaviour of the following sums:

$$S_1(D; X) = \sideset{}{^\flat}\sum_{\substack{\mathcal{Q}(\underline{x}) \leq X \\ \mathcal{Q}(\underline{x}) \in \mathcal{S}_D, \underline{x} \in \mathbb{Z}^2}} \lambda_{w, f \times f}^{\ell}(\mathcal{Q}(\underline{x})) \tag{5.12}$$

and

$$S_2(D; X) = \sideset{}{^\flat}\sum_{\substack{\mathcal{Q}(\underline{x}) \leq X \\ \mathcal{Q}(\underline{x}) \in \mathcal{S}_D, \underline{x} \in \mathbb{Z}^2}} \lambda_{w_1, f \times f}^{\ell_1}(\mathcal{Q}(\underline{x})) \cdots \lambda_{w_r, f \times f}^{\ell_r}(\mathcal{Q}(\underline{x})), \tag{5.13}$$

where the sum runs over the square-free integers represented by reduced forms of a fixed discriminant D.

More precisely, we prove the following results.

Theorem 5.1. *[7, Theorem 2.1] Let $f \in S_k(SL_2(\mathbb{Z}))$ be a normalized Hecke eigenform. Then, for any $\epsilon > 0$, we have the following estimate for the sum $S_1(D; X)$:*

$$S_1(D;X) = XP_M(\log X) + O\left(X^{1-\frac{1}{2\eta}+\epsilon}\right), \tag{5.14}$$

where $P_m(x)$ is a polynomial of degree m in variable x, $M = \left(\binom{2\ell}{\ell} - \binom{2\ell}{\ell-1}\right) w^\ell - 1$ and

$$\eta = w^\ell \begin{cases} \frac{9}{28}\left(\binom{2\ell}{\ell} - \binom{2\ell}{\ell-1}\right) + \frac{5}{4}\left(\binom{2\ell}{\ell-1} - \binom{2\ell}{\ell-2}\right) \\ +\frac{1}{2}\sum_{r=0}^{\ell-2}(2\ell - 2r + 1)\left(\binom{2\ell}{r} - \binom{2\ell}{r-1}\right). \end{cases}$$

In the next theorem, we consider a more general divisor problem associated to Fourier coefficients of Hecke eigenform and improve an estimate obtained by Hua [29] when the indices are given by sum of two squares.

Theorem 5.2. *[7, Theorem 2.2] Let $f \in S_k(SL_2(\mathbb{Z}))$ be a normalized Hecke eigenform. Then, for any $\epsilon > 0$, we have the following estimate for the sum $S_2(D; X)$:*

$$S_2(D;X) = XP_{M_1}(\log X) + O\left(X^{1-\frac{1}{2\eta_1}+\epsilon}\right), \tag{5.15}$$

where $P_m(x)$ is a polynomial of degree m, $\omega = \prod_{i=1}^{r} w_i^{\ell_i}$, $\theta = \sum_{i=1}^{r} \ell_i$,

$$M_1 = \left(\binom{2\theta}{\theta} - \binom{2\theta}{\theta-1}\right)\omega - 1 \text{ and } \eta_1 = \omega \begin{cases} \frac{9}{28}\left(\binom{2\theta}{\theta} - \binom{2\theta}{\theta-1}\right) + \frac{5}{4}\left(\binom{2\theta}{\theta-1} - \binom{2\theta}{\theta-2}\right) \\ +\frac{1}{2}\sum_{r=0}^{\theta-2}(2\theta - 2r + 1)\left(\binom{2\theta}{r} - \binom{2\theta}{r-1}\right). \end{cases}$$

Remark 5.5. Here, we obtain a better estimate as $\eta_1/\omega < 2^{2\theta}$, which improves the result of Hua [29].

5.3 Preparatory results

In this section, we recall some necessary results for L-functions and their Euler product decomposition which are required to prove Theorem 5.1 and Theorem 5.2.

We first express the sums defined in (5.12) and (5.13) in terms of the known arithmetical functions as follows:

$$S_1(D;X) = \sideset{}{^\flat}\sum_{\substack{\mathcal{Q}(\underline{x})\leq X\\ \mathcal{Q}\in\mathcal{S}_D,\underline{x}\in\mathbb{Z}^2}} \lambda^{\ell}_{w,f\times f}(\mathcal{Q}(\underline{x})) = \sideset{}{^\flat}\sum_{n\leq X} \lambda^{\ell}_{w,f\times f}(n) r_D(n) \tag{5.16}$$

and

$$S_2(D;X) = \sideset{}{^\flat}\sum_{\substack{\mathcal{Q}(\underline{x})\leq X\\ \mathcal{Q}\in\mathcal{S}_D,\underline{x}\in\mathbb{Z}^2}} \lambda^{\ell_1}_{w_1,f\times f}(\mathcal{Q}(\underline{x}))\cdots\lambda^{\ell_r}_{w_r,f\times f}(\mathcal{Q}(\underline{x})) = \sideset{}{^\flat}\sum_{n\leq X} \lambda^{\ell}_{w,f\times f}(n)\ldots\lambda^{\ell}_{w,f\times f}(n) r_D(n). \tag{5.17}$$

Now, we define two Dirichlet series that will be used to obtain the approximate behaviour of the sum $S_1(D;X)$ and $S_2(D;X)$ defined in (5.12) and (5.13), respectively.

$$L_1(s) := \sideset{}{^\flat}\sum_{n\geq 1} \frac{\lambda^{\ell}_{w,f\times f}(n) r^*_D(n)}{n^s}, \tag{5.18}$$

and

$$L_2(s) := \sideset{}{^\flat}\sum_{n\geq 1} \frac{\lambda^{\ell_1}_{w_1,f\times f}(n)\ldots\lambda^{\ell_r}_{w_r,f\times f}(n) r^*_D(n)}{n^s}, \tag{5.19}$$

where $r^*_D(n) = \sum_{d|n} \chi_D(d)$. The Dirichlet series $L_1(s)$ and $L_2(s)$ converge absolutely and uniformly for $\Re(s) > 1$. Next, we decompose $L_1(s)$ and $L_2(s)$ in terms of the known L-functions and using the analytic properties of these well-known L-functions, we obtain the desired estimates. Now, we state the decomposition of $L_1(s)$ and $L_2(s)$ in terms of well-known L-functions.

Proposition 5.3. *We have the following decomposition of $L_1(s)$ and $L_2(s)$:*

$$L_1(s) = L_{11}(s) \times G_1(s), \quad L_2(s) = L_{21}(s) \times G_2(s),$$

where

$$\begin{aligned} L_{11}(s) = \zeta(s)^{\left(\binom{2\ell}{\ell}-\binom{2\ell}{\ell-1}\right)} L(\chi_D, s)^{\left(\binom{2\ell}{\ell}-\binom{2\ell}{\ell-1}\right)} \\ \times \prod_{n=0}^{\ell-1} \left(L(sym^{2\ell-2n} f, s)^{\left(\binom{2\ell}{n}-\binom{2\ell}{n-1}\right)} L(sym^{2\ell-2n,s} f \times \chi_D, s)^{\left(\binom{2\ell}{n}-\binom{2\ell}{n-1}\right)} \right), \end{aligned} \tag{5.20}$$

and

$$\begin{aligned} L_{21}(s) = \zeta(s)^{\omega\times\left(\binom{2\theta}{\theta}-\binom{2\theta}{\theta-1}\right)} L(\chi_D, s)^{\omega\times\left(\binom{2\theta}{\theta}-\binom{2\theta}{\theta-1}\right)} \\ \times \prod_{n=0}^{\theta-1} \left(L(sym^{2\theta-2n} f, s)^{\omega\times\left(\binom{2\theta}{n}-\binom{2\theta}{n-1}\right)} L(sym^{2\theta-2n} f \times \chi_D, s)^{\omega\times\left(\binom{2\theta}{n}-\binom{2\theta}{n-1}\right)} \right). \end{aligned} \tag{5.21}$$

Here $\omega = \prod_{i=1}^{r} w_i^{\ell_i}, \theta = \sum_{i=1}^{r} \ell_i$, χ_D is a Dirichlet character modulo $|D|$ and $G_r(s); r = 1, 2$ is a Dirichlet series given in terms of Euler product which converges absolutely and uniformly for $\Re(s) > \frac{1}{2}$ and for $r = 1, 2$, $G_r(s) \neq 0$ for $\Re(s) = 1$.

To prove Proposition 5.3 we use the following Lemma.

Lemma 5.1. *[48, p. 4, Lemma 2.2] Let $\ell \in \mathbb{N}$. For each j with $0 \le j \le \ell$, we define*

$$A_{\ell,j} := \binom{\ell}{\frac{\ell-j}{2}} - \binom{\ell}{\frac{\ell-j}{2} - 1} \quad \text{if} \quad j \equiv \ell \pmod 2. \quad \textit{Then} \quad x^\ell = \sum_{j=0}^{\ell} A_{\ell,j} T_{\ell-2j}(x),$$

where $T_m(x) = U_m(2x)$ and $U_m(x)$ is the m-th Chebyshev polynomial of second kind.

Proof of Proposition 5.3: The arithmetical functions $r_D^*(n) = \sum_{d|n} \chi_D(d)$ and $\lambda_{f\times f}(n)$ are multiplicative. This implies the following Euler product of the function $L_1(s)$ in the region $\Re(s) > 1$:

$$\begin{aligned} L_1(s) &= \sum_{n\geq 1}^{\flat} \frac{\lambda_{w,f\times f}^{\ell}(n) r_D^*(n)}{n^s} = \prod_p \left(1 + \frac{(\lambda_{w,f\times f}(p))^{\ell} r_D^*(p)}{p^s}\right) \\ &= \prod_p \left(1 + \frac{w^{\ell}\lambda_f^{2\ell}(p)(1+\chi_D(p))}{p^s}\right). \end{aligned} \tag{5.22}$$

Now, we obtained the following relation from Lemma 5.1

$$\lambda_f^{2\ell}(p) = \sum_{n=0}^{\ell} \left(\binom{\ell}{n} - \binom{\ell}{n-1}\right) \lambda_{\mathrm{sym}^{(2\ell-2n)}f}(p). \tag{5.23}$$

This yields,

$$\begin{aligned} \lambda_f^{2\ell}(p)(1+\chi_D(p)) = \sum_{n=0}^{\ell} \left(\binom{\ell}{n} - \binom{\ell}{n-1}\right) \lambda_{\mathrm{sym}^{(2\ell-2n)}f}(p)(1+\chi_D(p)) = \\ \sum_{n=0}^{\ell} \left(\binom{\ell}{n} - \binom{\ell}{n-1}\right) \lambda_{\mathrm{sym}^{(2\ell-2n)}f}(p) + \sum_{n=0}^{\ell} \left(\binom{\ell}{n} - \binom{\ell}{n-1}\right) \lambda_{\mathrm{sym}^{(2\ell-2n)}f}(p)\chi_D(p). \end{aligned} \tag{5.24}$$

Now following the argument as in [67, p. 6], for $\Re(s) > 1$, we express the function

$$\begin{aligned} L_{11}(s) = \zeta(s)^{w^{\ell}\left(\binom{2\ell}{\ell}-\binom{2\ell}{\ell-1}\right)} L(\chi_D, s)^{w^{\ell}\left(\binom{2\ell}{\ell}-\binom{2\ell}{\ell-1}\right)} \\ \times \prod_{n=0}^{\ell-1} \left(L(sym^{2\ell-2n}f, s)^{w^{\ell}\left(\binom{2\ell}{n}-\binom{2\ell}{n-1}\right)} L(sym^{2\ell-2n}f \times \chi_D, s)^{w^{\ell}\left(\binom{2\ell}{n}-\binom{2\ell}{n-1}\right)} \right) \end{aligned}$$

as an Euler product of the form

$$\prod_p \left(1 + \frac{A(p)}{p^s} + \frac{A(p^2)}{p^{2s}} + \cdots\right), \quad \text{where} \quad A(p) = -(\lambda_{w,f\times f}(p))^\ell r_D^*(p).$$

Moreover, for each prime p, we define the sequence $\{B(p^r)\}$ as follows: $B(p) = 0$ and for each $r \geq 2$, $B(p^r) = A(p^r) + A(p^{r-1})(\lambda_{w,f\times f}(p))^\ell r_D^*(p)$. It is easy to see that $B(n) \ll n^\epsilon$ for any positive ϵ. Associated to the sequence $\{B(p^r)\}$, we define an Euler product for $G_1(s)$ given by

$$G_1(s) = \prod_p \left(1 + \frac{B(p)}{p^s} + \frac{B(p^2)}{p^{2s}} + \cdots\right)$$

with $B(p^2) = A(p^2) - (\lambda_{w,f\times f}(p))^{2\ell}(r_D^*(p))^2$. Now this gives the required decomposition of $L_1(s)$, i.e.,

$$L_1(s) = L_{11}(s)G_1(s).$$

This completes the proof. The decomposition for $L_2(s)$ can be obtained in the same way.

Lemma 5.2. *For any $\epsilon > 0$, we have*

$$\begin{aligned} \zeta(\sigma + it) &\ll_\epsilon (1 + |t|)^{\max\left\{\frac{13}{42}(1-\sigma),0\right\}+\epsilon} \\ \text{and} \qquad L(\chi, \sigma + it) &\ll_{\epsilon,N} (1 + |t|)^{\max\left\{\frac{1}{3}(1-\sigma),0\right\}+\epsilon} \end{aligned} \tag{5.25}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1$ and $|t| \geq 1$.

Proof. The bound for $\zeta(\sigma + it)$ follows from [11, p. 2, Theorem 5] and the bound for $L(\chi, \sigma + it)$ follows from [28, p. 1, eq. 1.1] and Phragmen - Lindelöf convexity principle. □

Lemma 5.3. *[57, p. 2, Corollary 1.2] For any arbitrarily small $\epsilon > 0$, we have*

$$L(sym^2 f, \sigma + it) \ll_\epsilon (1 + |t|)^{\max\left\{\frac{5}{4}(1-\sigma),0\right\}+\epsilon} \tag{5.26}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1$ and $|t| \geq 1$.

Lemma 5.4. *[31, p. 100] Let $L(f,s)$ be an L-function of degree $m \geq 2$, i.e.,*

$$L(f,s) = \sum_{n\geq 1} \frac{\lambda_f(n)}{n^s} = \prod_{p-\text{prime}} \prod_{j=1}^{m} \left(1 - \frac{\alpha_{p,f,j}}{p^s}\right)^{-1}, \tag{5.27}$$

where $\alpha_{p,f,j}$ $(1 \leq j \leq m)$ are the local parameter of $L(f,s)$ at prime p and $\lambda_f(n) = O(n^\epsilon)$ for any $\epsilon > 0$. We assume that the series and Euler product converge absolutely for $\Re(s) > 1$ and $L(f,s)$ is an analytic function except possibly for pole at $s = 1$ of order r and satisfies a functional equation for $s \to 1 - s$. Then for any $\epsilon > 0$, we have

$$\left(\frac{s-1}{s+1}\right)^r L(f, \sigma + it) \ll_{f,\epsilon} (1 + |t|)^{\frac{m}{2}(1-\sigma)+\epsilon} \tag{5.28}$$

uniformly for $0 \leq \sigma \leq 1$ and $|t| \geq 1$, where $s = \sigma + it$. For $T \geq 1$, we have

$$\int_T^{2T} |L(f, \sigma + it)|^2 \, dt \ll_{f,\epsilon} T^{m(1-\sigma)+\epsilon} \tag{5.29}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1$ and $|T| \geq 1$.

5.4 Proof of results

Proof of Theorem 5.1 To obtain an estimate for the sum $S_1(D; X)$ defined in (5.12), we introduce a compactly supported smooth function $w(x)$ for $1 \leq Y \leq \frac{X}{2}$

such that

$$w(x) = \begin{cases} 1 \text{ if } x \in [2Y, X], \\ 0 \text{ if } x < Y \text{ and } x > X + Y \end{cases}$$

and $w^{(r)}(x) \ll_r Y^{-r}$ for all $r \geq 0$.

For an arithmetical function $h(n)$ [33, p. 60], we have

$$\sum_{n \leq X} h(n) = \sum_{n=0}^{\infty} h(n)w(n) + O\left(\sum_{n<2Y} |h(n)|\right) + O\left(\sum_{X<n<X+Y} |h(n)|\right). \qquad (5.30)$$

Applying Mellin inverse transform we have

$$\sum_{n=0}^{\infty} h(n)w(n) = \frac{1}{2\pi i}\int_{(b)} \tilde{w}(s)\left(\sum_{n\geq 1}\frac{h(n)}{n^s}\right) ds,$$

where b is a positive real number bigger than the abscissa of absolute convergence of the series $\sum_{n\geq 1}\frac{h(n)}{n^s}$ and $\tilde{w}(s)$ is the Mellin transform of w given by

$$\tilde{w}(s) = \int_0^{\infty} w(x)x^s \frac{dx}{x}.$$

Moreover, using integration by parts, we obtain

$$\tilde{w}(s) = \frac{1}{s(s+1)\cdots(s+m-1)}\int_0^{\infty} w^{(m)}(x)x^{s+m-1}dx \ll \frac{Y}{X^{1-\sigma}}\left(\frac{X}{|s|Y}\right)^m \qquad (5.31)$$

for $m \geq 1$, where $\sigma = \Re(s)$. By Cauchy's residue Theorem, we have

$$\sum_{n=0}^{\infty} h(n)w(n) = \operatorname*{Res}_{s=1}\left(\tilde{w}(s)\sum_{n\geq 1}\frac{h(n)}{n^s}\right) + \frac{1}{2\pi i}\int_{(1/2+\epsilon)} \tilde{w}(s)\left(\sum_{n\geq 1}\frac{h(n)}{n^s}\right) ds. \qquad (5.32)$$

As the convexity bound of $\sum_{n\geq 1}\frac{h(n)}{n^s}$ at $\Re(s) = 1/2+\epsilon$ is known [33, Section 2], the contribution from the integral over $|s| \geq T = \frac{X^{1+\epsilon}}{Y}$ on the right hand side of (5.32) is negligibly small (i.e., $O(X^{-A})$ for any $A > 0$) if we choose sufficiently large positive integer m. Therefore, it is sufficient to find an upper bound of the following integral with $T = \frac{X^{1+\epsilon}}{Y}$:

$$\frac{1}{2\pi i}\int_{1/2+\epsilon-iT}^{1/2+\epsilon+iT} \tilde{w}(s)\left(\sum_{n\geq 1}\frac{h(n)}{n^s}\right) ds. \tag{5.33}$$

For details, we refer to [33, Section 3].

Substituting $h(n) = \lambda^{\ell}_{w,f\times f}(n)r^*_D(n)$ in (5.30) gives the following:

$$\begin{aligned}\sideset{}{^\flat}\sum_{n\leq X} \lambda^{\ell}_{w,f\times f}(n)r^*_D(n) = \sideset{}{^\flat}\sum_{n\geq 1} \lambda^{\ell}_{w,f\times f}(n)r^*_D(n)w(n) + O\left(\sideset{}{^\flat}\sum_{n<2Y} |\lambda^{\ell}_{w,f\times f}(n)r^*_D(n)|\right)\\ + O\left(\sideset{}{^\flat}\sum_{X<n<X+Y} |\lambda^{\ell}_{w,f\times f}(n)r^*_D(n)|\right).\end{aligned}$$

Since $\lambda^{\ell}_{w,f\times f}(n)r^*_D(n) \ll n^{\epsilon}$, therefore, by using Mellin inverse transform, we obtain

$$\sideset{}{^\flat}\sum_{n\geq 1} \lambda^{\ell}_{w,f\times f}(n)r^*_D(n)w(n) = \frac{1}{2\pi i}\int_{(1+\epsilon)} \tilde{w}(s)L_1(s)ds \tag{5.34}$$

for some arbitrarily small $\epsilon > 0$. Moreover,

$$O\left(\sideset{}{^\flat}\sum_{n<2Y} |\lambda^{\ell}_{w,f\times f}(n)r^*_D(n)|\right) + O\left(\sideset{}{^\flat}\sum_{X<n<X+Y} |\lambda^{\ell}_{w,f\times f}(n)r^*_D(n)|\right) \ll Y^{1+\epsilon}. \tag{5.35}$$

We now shift the line of integration in (5.34) from $\Re(s) = 1+\epsilon$ to $\Re(s) = \frac{1}{2}+\epsilon$ and then apply the Cauchy's residue theorem to obtain

$$\begin{aligned}\sideset{}{^\flat}\sum_{n\geq 1} \lambda^{\ell}_{w,f\times f}(n)r^*_D(n)w(n) &= \frac{1}{2\pi i}\int_{(1+\epsilon)} \tilde{w}(s)L_1(s)ds\\ &= \operatorname*{Res}_{s=1}\left(\tilde{w}(s)L_1(s)\right) + \frac{1}{2\pi i}\int_{(1/2+\epsilon)} \tilde{w}(s)L_1(s)ds.\end{aligned} \tag{5.36}$$

Since $\tilde{w}(s) \ll \frac{Y}{X^{1-\sigma}}\left(\frac{X}{|s|Y}\right)^m$ for any $m \geq 0$, the contribution for the integral over

$|s| \geq T = \frac{X^{1+\epsilon}}{Y}$ on the right hand side of (5.36) is negligibly small, i.e., $O(X^{-A})$ for any large $A > 0$ if one chooses sufficiently large $m > 0$. Moreover, we have

$$\frac{1}{2\pi i}\int_{(1/2+\epsilon)} \tilde{w}(s)L_1(s)ds = \frac{1}{2\pi i}\int_{1/2+\epsilon-iT}^{1/2+\epsilon+iT} \tilde{w}(s)L_1(s)ds + O(X^{-A})$$
$$\ll \int_{-T}^{T} |\tilde{w}(1/2+\epsilon+it)||L_{11}(1/2+\epsilon+it)G_1(1/2+\epsilon+it)|dt + O(X^{-A}),$$
$$\ll \int_0^T \frac{X^{\frac{1}{2}+\epsilon}}{|\frac{1}{2}+\epsilon+it|}|L_{11}(1/2+\epsilon+it)|dt + O(X^{-A}),$$
$$\ll \left(\int_0^1 + \int_1^T\right) \frac{X^{\frac{1}{2}+\epsilon}}{|\frac{1}{2}+\epsilon+it|}|L_{11}(1/2+\epsilon+it)|dt + O(X^{-A}),$$
$$\ll X^{\frac{1}{2}+\epsilon} + X^{\frac{1}{2}+\epsilon}\int_1^T \frac{|L_{11}(1/2+\epsilon+it)|}{|t|}dt + O(X^{-A}),$$

where estimates in the previous lines are obtained by substituting the bound for $\tilde{w}(s)$ when $m = 1$, and substituting the decomposition $L_1(s) = L_{11}(s)G_1(s)$. Note that in view of Proposition 5.3, the series $G_1(s)$ is absolutely convergent for $\Re(s) > \frac{1}{2}$. Now, we apply dyadic division method, substitute $L_{11}(s)$ from (5.20) and then apply Cauchy-Schwarz inequality to get ($\sigma_0 = 1/2+\epsilon$)

$$\int_1^T \frac{|L_{11}(1/2+\epsilon+it)|}{|t|}dt \ll \log T \max_{2\leq T_1\leq T}\left(\frac{1}{T_1}\int_{T_1/2}^{T_1}|L_{11}(1/2+\epsilon+it)|dt\right)$$
$$\ll \log T \max_{2\leq T_1\leq T}\left\{\begin{array}{l} \frac{1}{T_1}\left(\max\limits_{\frac{T_1}{2}\leq t\leq T_1}\left(|(\zeta(\sigma_0+it)L(\chi_D,\sigma_0+it)|^{w^\ell\left(\binom{2\ell}{\ell}-\binom{2\ell}{\ell-1}\right)}\right)\right) \\ \times\left(\max\limits_{\frac{T_1}{2}\leq t\leq T_1}\left(|(L(sym^2 f,\sigma_0+it)L(sym^2 f\times\chi_D,\sigma_0+it)|^{w^\ell\left(\binom{2\ell}{\ell-1}-\binom{2\ell}{\ell-2}\right)}\right)\right) \\ \times\left(\int_{\frac{T_1}{2}}^{T_1}\left|\prod\limits_{n=0}^{\ell-2}|L(sym^{2\ell-2n}f,\sigma_0+it)|^{\left(\binom{2\ell}{n}-\binom{2\ell}{n-1}\right)}\right|^2 dt\right)^{\frac{1}{2}} \\ \times\left(\int_{\frac{T_1}{2}}^{T_1}\left|\prod\limits_{n=0}^{\ell-2}|L(sym^{2\ell-2n}f\times\chi_D,\sigma_0+it)|^{\left(\binom{2\ell}{n}-\binom{2\ell}{n-1}\right)}\right|^2 dt\right)^{\frac{1}{2}} \end{array}\right.$$
$$\ll T^{w^\ell\left\{\frac{9}{28}\left(\binom{2\ell}{\ell}-\binom{2\ell}{\ell-1}\right)+\frac{5}{4}\left(\binom{2\ell}{\ell-1}-\binom{2\ell}{\ell-2}\right)+\frac{1}{2}\sum_{r=0}^{\ell-2}(2\ell-2r+1)\left(\binom{2\ell}{r}-\binom{2\ell}{r-1}\right)\right\}+\epsilon} = T^{\eta+\epsilon}.$$

which is obtained by substituting the convexity/sub-convexity bound and integral estimate of associated L-functions with $\left(\binom{n}{r} - \binom{n}{r-1}\right) = \frac{n-2r+1}{r}\binom{n}{r-1}$ (for any $1 \leq r \leq n$). Thus, we have

$$S_1(D, X) = \operatorname*{Res}_{s=1}\left(\tilde{w}(s)L_1(s)\right) + X^{\frac{1}{2}+\epsilon}T^{\eta-1+\epsilon} + Y^{1+\epsilon} + O(X^{-A}), \tag{5.37}$$

where

$$\eta = w^{\ell}\left\{\tfrac{9}{28}\left(\tbinom{2\ell}{\ell} - \tbinom{2\ell}{\ell-1}\right) + \tfrac{5}{4}\left(\tbinom{2\ell}{\ell-1} - \tbinom{2\ell}{\ell-2}\right) + \tfrac{1}{2}\sum_{r=0}^{\ell-2}(2\ell-2r+1)\left(\binom{2\ell}{r} - \binom{2\ell}{r-1}\right)\right\}.$$

We know that the Dirichlet series $L_1(s)$ has a pole of order $\left(\binom{2\ell}{\ell} - \binom{2\ell}{\ell-1}\right) w^{\ell}$ at $s = 1$. Now, we substitute $T = \frac{X^{1+\epsilon}}{Y}$ and choose $Y = X^{\frac{\eta-1/2}{\eta}}$ to get the required result.

Proof of Theorem 5.2 The proof follows from exactly same arguments as in the proof of Theorem 5.1.

5.5 Conclusion

In this chapter, we have observed the oscillatory behaviour of product of L-function through the asymptotic behaviour of higher moments of generalised divisor function associated to Fourier coefficients of Rankin-Selberg L-functions associated to normalised Hecke eigenforms. We have extended the result of Zhai. He has studied summatory function over the sequence supported at sum of two squares while we have examined the summatory function over a sparse sequence obtained from binary quadrtaic form of a fixed negative discriminant utilising character sum, Deligne's bound, Weil's bound. We have obtained a better estimate which improves the result of Hua.

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