Chapter 4

The Analytical Solution of The Riemann problem for Magnetogasdynamic system. *

"What is mathematics? It is only a systematic effort of solving puzzles posed by nature"

–Shakuntala Devi

4.1 Introduction

In this chapter, we have considered the one-dimensional hyperbolic system of conservation laws, which governs the unsteady flow of an inviscid isentropic compressible

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magnetogasdynamic fluid with the logarithmic equation of state (see [46], [121],)

$$\begin{cases} (\rho)_t + (\rho\nu)_x = 0, \\ (\rho\nu)_t + (\rho\nu^2 + p(\rho) + \frac{B^2}{2\mu})_x = 0, \end{cases} \tag{4.1}$$

where ρ and ν are density and velocity of the gas. The parameters $\mu > 0$ and $B \geq 0$, represent magnetic permeability and transverse magnetic field, respectively. t and x are independent variables, which denote time and space, respectively. Here p and B are defined as

$$p(\rho) = c_1 \ln \rho \quad \text{and} \quad B(\rho) = c_2\rho, \tag{4.2}$$

where constants $c_1 > 0$ and $c_2 > 0$. Here, we assume the RP for the homogeneous hyperbolic system with piecewise discontinuous initial condition given as,

$$(\rho, \nu)(x, 0) = \begin{cases} (\rho_-, \nu_-), & x < 0, \\ (\rho_+, \nu_+), & x > 0, \end{cases} \tag{4.3}$$

where $\rho_\pm$, and $\nu_\pm$, are given constant states.

In astrophysics, the concept was presented to explain the unique features of molecular clouds that could not be explained by isothermal distribution. [122], [123] and [124] have studied the logotropic dark fluid utilising the logarithmic equation of state as a unification of dark energy and dark matter. The Logotropic model, which is based on the Euler equations with the logarithmic EoS, exhibits more fascinating cosmological characteristics than the various modified forms of the CGM (Chaplygin gas model).

It is important to note that the magnetic field model is more fascinating and physically significant. Although the Riemann problem, in this instance, is exceedingly difficult (see [125]). Brenier [126] focused on studying the one-dimensional Riemann problems and finding the solutions. It is commonly known that there is a solution to the Riemann problem for a hyperbolic 1-D system of conservation laws, for initial data with minor variation. (see [5], [14], [125], [127]). This kind of result does not exist for weakly hyperbolic systems. It follows that it is not strange that some of them could have non - classical solutions. It is used, introducing new singular solutions known as delta-shocks, to resolve the Cauchy problem in this 'non-classical' scenario [128]. The delta shock wave is one of these non-classical solutions. A novel kind of discontinuity that is a generalisation of a normal shock wave is the delta shock wave. In the form of a weighted Dirac delta function, which employs the discontinuity as its support, at least one of the state variables may experience an exceptionally high concentration on this discontinuity. Here is a brief description of some of the consequences of delta shock waves. Construction of delta shock waves and vacuum states for zero pressure Euler equations have been studied for isentropic [42], non-isentropic [49], and isothermal[129] fluids using a vanishing pressure approach. It's an interesting problem to analyse the RP for conservative hyperbolic system. The presence of the delta shock wave in the Riemann solutions of the pressureless gas dynamics model was also investigated in [42, 130].

We consider the system (4.1) with (4.2), which is quasilinear system, and the characteristics fields associated with the characteristics of a system (4.1) are genuinely nonlinear. That means solutions of systems (4.1) and (4.2) have only shock and rarefaction waves. This study is very interesting amongst researchers of the mathematics and physics area, and a lot of current interest in the study of shock, rarefaction, and delta shock waves in the RP solution is seen. ([131], [49], [132], [46], [133]) have examined the cavitation and concentration processes in the Riemann solution for the isentropic and non-isentropic pressureless gasdynamics with various equations of state. Nilsson et al. [128] have investigated the conservation laws of mass, momentum and energy for zero pressure gasdynamics flow and obtained the delta shock waves in the solution.

The Riemann solutions for the homogeneous and inhomogeneous hyperbolic system have been studied by many authors (see [134], [135], [136]). [13] and [137] suggested a p-system with chaplygin gas model $p = -E/\rho$, where $(E > 0)$, to incorporate the lifting force on aeroplane wings into gas dynamics. The nonlinear hyperbolic conservation rules have a system known as the p-system as their prototype. It is the most elementary non trivial system and can be found inside almost all significant physical systems. [126] explored the solution of the aforementioned model with concentration for a specific initial value lying in a specific domain plane. Recently, Sen [138] has examined the limiting result of Riemann solutions to isentropic Euler system for logarithmic EoS with source term (also see [121], [139], [140]), and Chen [42] and Yang [53] analyzed the vanishing pressure of hyperbolic models. For a dusty gas (see [141], [142]), the Riemann solutions, such as shock waves, rarefaction waves, and contact discontinuities, have been formally derived and their characteristics were examined. The existence and uniqueness of Riemann solutions in magnetogasdynamics have obtained constructively using the characteristic technique, and it has been investigated as to how the elementary waves interact (see [143], [144]). The analysis of the Riemann solutions to the homogeneous hyperbolic model in the presence of a magnetic field and a logarithmic EoS have served as the main source of inspiration for the problem considered here. It's application area which is extensive in aerodynamics, cosmology, engineering, and astrophysics. The practical and theoretical applications of the Riemann solutions to the mathematical analysis of hyperbolic systems gave a substantial role enhancement. Determining the global structure of the Riemann solution, we discussed the Riemann solutions for systems (4.1) and (4.3) which have no self-similar solution. Recently, the magnetic field density functional theory is studied by S. Riemann [145] and "the first magnetic fields" by Widrow et al. [146]. In a planar magnetic field, [147] discovered electron states in quantum rings with structural aberrations. Due to the highly non-linear nature and complexity of the magnetogasdynamics controlling system, several simplified models of magnetogasdynamics have been studied in [148], [149], [150]. Currently, the authors in [151] and [152] have discussed the solution to the Riemann problem for one-dimensional van der Waals Gas and dusty gas dynamics. Brenier [126] firstly studied the 1-D Riemann problem for the isentropic Chaplygin gas Euler equations and obtained solutions with concentration when initial data belongs to a specific domain in the phase plane. Furthermore, Zhiqiang Shao [153] and Chun Shen [121] have derived the global solution to the Riemann problem for the isentropic Chaplygin gas magnetogasdynamics equations and discussed the formation of delta shocks and vacuum states as pressure and magnetic field vanish. The above study manifests that the limit of two shock waves can yield the delta shock wave, and the limit of two rarefaction waves can cause the vacuum state. Moreover, Meina Sun [46] analyzed the Riemann problem for the isentropic Euler system with the logarithmic equation of state and derived the analytical solutions explicitly for different cases. The concentration and cavitation phenomena are observed and analyzed in the vanishing pressure limit in the Riemann solutions. Motivated by the above affirmative discussion, the stability of the Riemann solutions for the hyperbolic system with the logarithmic equation of state and magnetic field is introduced in the present chapter. Also, the formation of vacuum states and delta shock waves as magnetic field and pressure vanish have been discussed.

In this chapter, we have analyzed the stability of the solution of the Riemann problem for the hyperbolic system with the logarithmic EoS and magnetic field by using the vanishing pressure and magnetic field limit technique. As per our knowledge, the stability of the Riemann solution for the hyperbolic system with the logarithmic EoS and magnetic field by using vanishing pressure and magnetic field limit technique have not been investigated by any researcher yet. This chapter is organized into the following sections: Section 4.2 deals with the Riemann problem of the quasi-linear hyperbolic system (4.4) describing one dimensional magnetogasdynamic system with logarithmic EoS. In section 4.3, we have discussed the solution of the transport equations. In section 4.4 and 4.5, we have investigated the limiting behaviour of Riemann solutions. Lastly, the conclusions of this study is discussed in section 4.6.

4.2 The Riemann problem for magnetogasdynamic system

The quasi-linear form of the system (4.1) with (4.3) is

$$PW_t + QW_x = 0, \tag{4.4}$$

where

$$W = \begin{bmatrix} \rho \\ \nu \end{bmatrix}, P = \begin{bmatrix} 1 & 0 \\ \nu & \rho \end{bmatrix}, \text{ and } Q = \begin{bmatrix} \nu & \rho \\ \nu^2 + \frac{c_1}{\rho} + \frac{c_2^2\rho}{\mu} & 2\rho\nu \end{bmatrix}.$$

Here, λ_1 and λ_2, the eigenvalues of system (4.1) and (4.2) can directly be obtained from (4.4), as

$$\lambda_1 = \nu - \sqrt{\frac{c_1}{\rho} + \frac{c_2^2\rho}{\mu}}, \quad \lambda_2 = \nu + \sqrt{\frac{c_1}{\rho} + \frac{c_2^2\rho}{\mu}}. \tag{4.5}$$

With respect to both eigenvalues, the right eigenvectors are

$$r_1 = \left(-\frac{\rho}{\sqrt{\frac{c_1}{\rho} + \frac{c_2^2\rho}{\mu}}}, 1\right)^T, \quad r_2 = \left(\frac{\rho}{\sqrt{\frac{c_1}{\rho} + \frac{c_2^2\rho}{\mu}}}, 1\right)^T. \tag{4.6}$$

Since for $c_1, c_2 > 0$, $\nabla\lambda_i.r_i \neq 0$, $i = 1, 2$. Here, $\nabla = (\frac{\partial}{\partial\rho}, \frac{\partial}{\partial\nu})$, which shows that the characteristics roots λ_1 and λ_2 are genuinely non linear. In this instance, the waves are therefore either rarefaction waves or shock waves, which are symbolised by R and S, respectively. The Riemann invariants for these characteristic fields are

$$\begin{cases} w = \nu - \int \frac{\sqrt{\frac{c_1}{\rho} + \frac{c_2^2\rho}{\mu}}}{\rho} d\rho, \\ z = \nu + \int \frac{\sqrt{\frac{c_1}{\rho} + \frac{c_2^2\rho}{\mu}}}{\rho} d\rho. \end{cases} \tag{4.7}$$

Since the system (4.1) and (4.2) with Riemann initial condition (4.3) are invariant under the new stretching coordinates: $(x, t) \to (\kappa x, \kappa t)$ (κ is constant), we have the self-similar solution

$$(\rho, \nu)(x, t) = (\rho, \nu)(\chi), \quad \chi = \frac{x}{t}.$$

The system (4.1), (4.2), and (4.3) is reduced to the boundary value problem as:

$$\begin{cases} -\chi(\rho)_\chi + (\rho\nu)_\chi = 0, \\ -\chi(\rho\nu)_\chi + (\rho\nu^2 + c_1 \ln\rho + \frac{(c_2\rho)^2}{2\mu})_\chi = 0, \end{cases} \tag{4.8}$$

with

$$(\rho, \nu)(\pm\infty) = (\rho_\pm, \nu_\pm).$$

The equation (4.8) can be written in the following form to obtain a smooth solution

$$\begin{bmatrix} \nu - \chi & \rho \\ -\chi\nu + \frac{c_1}{\rho} + \frac{c_2^2\rho}{\mu} + \nu^2 & -\chi\rho + 2\rho\nu \end{bmatrix} \begin{bmatrix} \rho_\chi \\ \nu_\chi \end{bmatrix} = 0. \tag{4.9}$$

On solving, it gives either general solution

$$(\rho, \nu)(\chi) = constant, \quad (\rho > 0),$$

or rarefaction wave solutions written as

$$R_1(\rho_-, \nu_-) : \begin{cases} \chi = \lambda_1 = \nu - \sqrt{\frac{c_1}{\rho} + \frac{c_2^2 \rho}{\mu}}, \\ \nu = \nu_- - \int_{\rho_-}^{\rho} \frac{\sqrt{\frac{c_1}{\rho} + \frac{c_2^2 \rho}{\mu}}}{\rho} d\rho, \\ \rho < \rho_-, \end{cases} \tag{4.10}$$

and

$$R_2(\rho_-, \nu_-) : \begin{cases} \chi = \lambda_2 = \nu + \sqrt{\frac{c_1}{\rho} + \frac{c_2^2 \rho}{\mu}}, \\ \nu = \nu_- + \int_{\rho_-}^{\rho} \frac{\sqrt{\frac{c_1}{\rho} + \frac{c_2^2 \rho}{\mu}}}{\rho} d\rho, \\ \rho > \rho_-. \end{cases} \tag{4.11}$$

Theorem 4.1. *The curve $R_1(\rho_-, \nu_-)$ is monotonically decreasing and convex.*

Proof. From (4.10), the curve $R_1(\rho_-, \nu_-)$ of 1-rarefaction wave is given by

$$\nu = \nu_- - \int_{\rho_-}^{\rho} \frac{\sqrt{\frac{c_1}{\rho} + \frac{c_2^2 \rho}{\mu}}}{\rho} d\rho. \tag{4.12}$$

Differentiating above equation w.r.t. ρ, we get

$$\frac{d\nu}{d\rho} = -\frac{\sqrt{\frac{c_1}{\rho} + \frac{c_2^2 \rho}{\mu}}}{\rho} < 0, \text{ and } \frac{d^2\nu}{d\rho^2} = \frac{\frac{3c_1}{\rho} + \frac{c_2^2 \rho}{\mu}}{2\rho^2 \sqrt{\frac{c_1}{\rho} + \frac{c_2^2 \rho}{\mu}}} > 0.$$

From the above, we can conclude that the curve is monotonically decreasing and convex.

Similarly, we can prove, $\frac{d\nu}{d\rho} > 0$ and $\frac{d^2\nu}{d\rho^2} < 0$, that is the curve $R_2(\rho_-, \nu_-)$ is monotonically increasing and concave in the plane-(ρ, ν) $(\rho > 0)$.

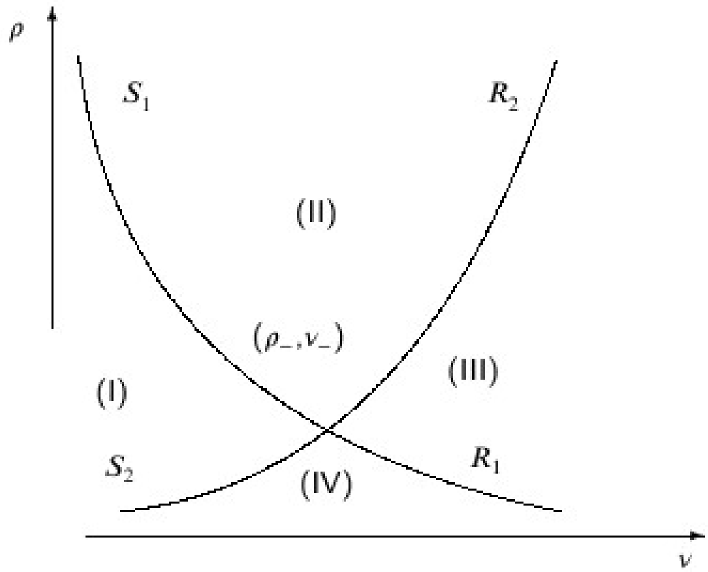

FIGURE 4.1: The (ρ, ν) plane for the system (4.1).

The corresponding R-H conditions are defined as

$$\begin{cases} -\sigma(t)[\rho] + [\rho\nu] = 0, \\ -\sigma(t)[\rho\nu] + [\rho\nu^2 + c_1 \ln\rho + \frac{(c_2\rho)^2}{2\mu}] = 0, \end{cases} \tag{4.13}$$

where $\sigma(t)$ is the velocity of discontinuity and $[\rho] = \rho - \rho_-$. On solving (4.13), we obtain two shocks S_1 and S_2

$$S_1(\rho_-, \nu_-): \begin{cases} \sigma_1(t) = \nu_- - \rho\sqrt{\frac{1}{\rho\rho_-}\left(\frac{c_1(ln\rho - ln\rho_-)}{(\rho-\rho_-)} + \frac{c_2^2(\rho+\rho_-)}{2\mu}\right)}, \\ \nu = \nu_- - \sqrt{\frac{1}{\rho\rho_-}\left(\frac{c_1(ln\rho - ln\rho_-)}{(\rho-\rho_-)} + \frac{c_2^2(\rho+\rho_-)}{2\mu}\right)}(\rho - \rho_-), \\ \rho_- < \rho,. \end{cases} \tag{4.14}$$

$$S_2(\rho_-,\nu_-): \begin{cases} \sigma_2(t) = \nu_- + \rho\sqrt{\frac{1}{\rho\rho_-}\left(\frac{c_1(ln\rho - ln\rho_-)}{(\rho-\rho_-)} + \frac{c_2^2(\rho+\rho_-)}{2\mu}\right)}, \\ \nu = \nu_- + \sqrt{\frac{1}{\rho\rho_-}\left(\frac{c_1(ln\rho - ln\rho_-)}{(\rho-\rho_-)} + \frac{c_2^2(\rho+\rho_-)}{2\mu}\right)}(\rho-\rho_-), \\ \rho_- > \rho. \end{cases} \tag{4.15}$$

Differentiating (4.14)$_2$ with respect to ρ, we obtain

$$\frac{d\nu}{d\rho} = -\frac{1}{2\sqrt{\frac{1}{\rho\rho_-}\left(\frac{c_1(ln\rho-ln\rho_-)}{(\rho-\rho_-)}+\frac{c_2^2(\rho+\rho_-)}{2\mu}\right)}}\left(\frac{c_1}{\rho^2\rho_-} + \frac{c_1(\ln\rho - \ln\rho_-)}{\rho^2(\rho-\rho_-)} + \frac{c_2^2}{2\mu\rho^2\rho_-}(2\rho^2+\rho\rho_-+\rho_-^2)\right) < 0,$$

for $\rho > \rho_-$ (that is, S_1). and

$$\begin{aligned}\frac{d^2\nu}{d\rho^2} = &\frac{c_1^2 P}{\rho^3\rho_-(\rho-\rho_-)^3}\left(\rho_-(4\rho-3\rho_-)\left(\ln\frac{\rho}{\rho_-}\right)^2 + 2(\rho-\rho_-)(\rho-2\rho_-)\ln\frac{\rho}{\rho_-} + (\rho-\rho_-)^2\right)\\ &+ \frac{c_1c_2^2P}{2\mu\rho^3\rho_-(\rho-\rho_-)}\left((\rho-\rho_-)(6\rho^2-4\rho_-^2-2\rho\rho_-) + (8\rho^2\rho_- + 2\rho\rho_-^2 - 6\rho_-^3 - 4\rho^3)\ln\frac{\rho}{\rho_-}\right)\\ &+ \frac{c_2^4P}{4\mu^2\rho^3}(5\rho\rho_- + 3\rho_-^2),\end{aligned}$$

where

$$P = \frac{1}{4\rho\rho_-\left(\frac{1}{\rho\rho_-}\left(\frac{c_1(ln\rho-ln\rho_-)}{(\rho-\rho_-)} + \frac{c_2^2(\rho+\rho_-)}{2\mu}\right)\right)^{3/2}}.$$

For $(\rho > 0)$, in (ρ,ν)-plane, we can say that the curve S_1 (1-shock) is monotonic decreasing. Similarly, from (4.15)$_2$ for $(\rho_- > \rho)$, we find $\nu_\rho > 0$, which implies that the curve S_2 (2-shock) is monotonic increasing in (ρ,ν)-plane. We can derive easily from (4.14) and (4.15) that $\lim_{\rho\to+\infty}\nu = -\infty$ for curve S_1 (1-shock) and $\lim_{\rho\to 0^+}\nu = -\infty$ for curve S_2 (2-shock) respectively.

We conclude from the above two equations, 1-shock curve (2-shock curve) are concave (respectively, convex) is similar to 1-rarefaction (2-rarefaction) wave curve. There are four possible states, which consist of 1-shock curve $S_1(\rho_-,u_-)$, 1-rarefaction curve $R_1(\rho_-,u_-)$, 2-shock curve $S_2(\rho_-,u_-)$ and the 2-rarefaction curve $R_2(\rho_-,u_-)$.

Hence, for the left state (ρ_-, u_-), the curves are separated into four parts, as shown in fig-4.1 as I, II, III and IV. Now, the construction of unique Riemann solution of the system (4.1), connected with the state (ρ_-, u_-) and (ρ_+, u_+) given by $(\rho_+, \nu_+) \in I(\rho_-, \nu_-) : S_1 + S_2$, $(\rho_+, \nu_+) \in II(\rho_-, \nu_-) : S_1 + R_2$, $(\rho_+, \nu_+) \in III(\rho_-, \nu_-) : R_1 + R_2$, $(\rho_+, \nu_+) \in IV(\rho_-, \nu_-) : R_1 + S_2$.

4.3 Riemann solution of the transport equations

The transport equations for the system (4.1) can be written as (see [130], [133])

$$\begin{cases} (\rho)_t + (\rho\nu)_x = 0, \\ (\rho\nu)_t + (\rho\nu^2)_x = 0, \end{cases} \tag{4.16}$$

using the same initial condition as (4.3).

System (4.16) has two equal eigenvalues $\lambda_1 = \lambda_2 = \nu$, and corresponding to these eigenvalues there exists one right eigenvector as

$$\overrightarrow{u} = (u, 0)^T.$$

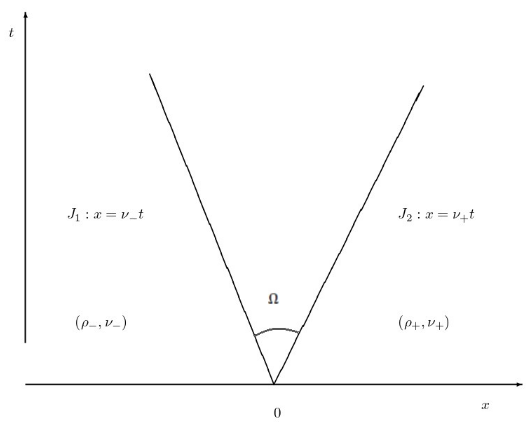

FIGURE 4.2: Characteristics overlapping domain.

Since $\nabla\lambda.\overrightarrow{u} = 0$, it follows that the characteristic fields corresponding to each characteristic root are linearly degenerate. The self-similar solutions are given as

$$(\rho,\nu)(x,t) = (\rho,\nu)(\chi), \quad \chi = \frac{x}{t}.$$

Using the system (4.16), the boundary value problem of the following differential equations can be expressed as:

$$\begin{cases} -\chi(\rho)_\chi + (\rho\nu)_\chi = 0, \\ -\chi(\rho\nu)_\chi + (\rho\nu^2)_\chi = 0. \end{cases} \tag{4.17}$$

with $(\rho,\nu)(\pm\infty) = (\rho_\pm,\nu_\pm)$.

The system (4.17) can be written as:

$$\begin{bmatrix} \nu-\chi & \rho \\ 0 & \rho(\nu-\chi) \end{bmatrix} \begin{bmatrix} \rho_\chi \\ \nu_\chi \end{bmatrix} = 0, \tag{4.18}$$

which ensures either the singular solution

$$\begin{cases} \rho = 0, \\ \nu = \chi, \end{cases} \tag{4.19}$$

or the general solution (constant)

$$(\rho,\nu)(\chi) = \text{constant} \;\; (\rho \neq 0).$$

That is known as vacuum state, where ν is any smooth function.

The R-H condition holds at $(\chi = \sigma(t))$,

$$\begin{cases} -\sigma(t)[\rho] + [\rho\nu] = 0, \\ -\sigma(t)[\rho\nu] + [\rho\nu^2] = 0, \end{cases} \tag{4.20}$$

where $[\rho] = \rho_+ - \rho_-$ is jump of ρ across the discontinuity. Considering the equation (4.20) we have

$$J : \chi = \sigma(t) = \nu_-(= \lambda_-) = \nu_+(= \lambda_+), \tag{4.21}$$

which is a contact discontinuity. Only the characteristics of the two-sided solution in the (x, t) plane can be seen on the slip line.

From the Riemann problems (4.16) and (4.3), contact jumps, vacuum, or delta shock waves can be used to connect two steady states $(\rho_\pm, \nu_\pm)$. Three possible cases arise.

Case-1: $\nu_- > \nu_+$, as shown in the figure 4.2, the characteristic curves starting from the origin overlap in the range of Ω. Therefore, the singularity should be displayed in Ω. Here Ω is a region where ν is a continuous function. Bounded jumps do not meet the R-H condition, so the singularity is not a finite-amplitude jump. To put it another way, there is no solution that is both solutions are piecewise smooth and bounded. Motivated by [130], we look for a solution with a delta distribution in the jump.

Case-2: $\nu_- = \nu_+$, the possibility of a contact discontinuity between particular states $(\rho_\pm, \nu_\pm)$ is straight forward to comprehend.

Case-3: $\nu_- < \nu_+$, there is no characteristics which passes through the region $\nu_- < \frac{x}{t} < \nu_+$, and a vacuum is created in this region. The conclusion can be stated as follows:

$$(\rho,\nu)(\chi)=\begin{cases}(\rho_-,\nu_-), & -\infty<\chi<\nu_-,\\ (0,\chi), & \nu_-\leq\chi\leq\nu_+,\\ (\rho_+,\nu_+), & \nu_+<\chi<+\infty.\end{cases} \tag{4.22}$$

The smooth curve $\Gamma=\{(x(\alpha),t(\alpha)):m_1<\alpha<m_2\}$, is the foundation for the two-dimensional weighted delta function $\omega(\alpha)\delta_\Gamma$, which is described as:

$$\langle\omega(\alpha)\delta_\Gamma,\varphi\rangle=\int_{m_1}^{m_2}\omega(\alpha)\varphi(x(\alpha),t(\alpha))d\alpha, \tag{4.23}$$

for $\varphi\in C_0^\infty(R\times R^+)$.

Consider the solutions of (4.16) with (4.3) given as

$$(\rho,\nu)(x,t)=\begin{cases}(\rho_-,\nu_-), & x<\sigma t,\\ (\omega(t)\delta(x-\sigma t),\sigma), & x=\sigma t,\\ (\rho_+,\nu_+), & x>\sigma t.\end{cases} \tag{4.24}$$

The generalized Rankine-Hugoniot conditions using the formula

$$\begin{cases}\frac{dx(t)}{dt}=\sigma(t),\\ \frac{d\omega(t)}{dt}=\sigma(t)[\rho]-[\rho\nu],\\ \frac{d(\omega(t)\sigma)}{dt}=\sigma(t)[\rho\nu]-[\rho\nu^2],\end{cases} \tag{4.25}$$

and the initial condition $(x(0),\omega(0))=(0,0)$, ensuring that the solutions are unique, the entropy condition: $\nu_+<\sigma<\nu_-$, must be satisfied for the delta-shocks.

Solving the equation (4.25) with the given initial data, we obtained that when $\rho_- = \rho_+$,

$$\sigma = \frac{(\nu_- + \nu_+)}{2}, x(t) = \frac{(\nu_- + \nu_+)}{2}t, \omega(t) = (\rho_-\nu_- - \rho_+\nu_+)t;$$

when $\rho_- \neq \rho_+$,

$$\sigma = \frac{\sqrt{\rho_-}\nu_- + \sqrt{\rho_+}\nu_+}{\sqrt{\rho_-} + \sqrt{\rho_+}}, x(t) = \frac{\sqrt{\rho_-}\nu_- + \sqrt{\rho_+}\nu_+}{\sqrt{\rho_-} + \sqrt{\rho_+}}t, \omega(t) = \sqrt{\rho_-\rho_+}(\nu_- - \nu_+)t.$$

4.4 Delta-shocks

We consider the formation of delta-shock waves in the Riemann solutions of model (4.1) and (4.2) for the case $(\rho_+, \nu_+) \in I(\rho_-, \nu_-)$ with $\nu_- > \nu_+$ as pressureless and vanishing magnetic field.

4.4.1 Limiting behavior of Riemann solution($c_1, c_2 \to 0$)

When $(\rho_+, \nu_+) \in I(\rho_-, \nu_-)$, for fixed $c_1, c_2 > 0$, let us consider the intermediate state (ρ_*, ν_*) connected with (ρ_-, ν_-) by a S_1 (1-shock) and (ρ_+, ν_+) by a S_2 (2-shock) with shock speed σ_1 and σ_2 respectively, as follows

$$S_1(\rho_-, \nu_-) : \begin{cases} \sigma_1(t) = \nu_- - \rho_*\sqrt{\frac{1}{\rho_*\rho_-}\left(\frac{c_1(ln\rho_* - ln\rho_-)}{(\rho_* - \rho_-)} + \frac{c_2^2(\rho_* + \rho_-)}{2\mu}\right)}, \\ \nu_* = \nu_- - \sqrt{\frac{1}{\rho_*\rho_-}\left(\frac{c_1(ln\rho_* - ln\rho_-)}{(\rho_* - \rho_-)} + \frac{c_2^2(\rho_* + \rho_-)}{2\mu}\right)}(\rho_* - \rho_-), \\ \rho_- < \rho_*, \end{cases} \tag{4.26}$$

$$S_2(\rho_-,\nu_-):\begin{cases}\sigma_2(t)=\nu_*+\rho_+\sqrt{\frac{1}{\rho_*\rho_+}\left(\frac{c_1(ln\rho_+-ln\rho_*)}{(\rho_+-\rho_*)}+\frac{c_2^2(\rho_*+\rho_+)}{2\mu}\right)},\\ \nu_+=\nu_*+\sqrt{\frac{1}{\rho_*\rho_+}\left(\frac{c_1(ln\rho_+-ln\rho_*)}{(\rho_+-\rho_*)}+\frac{c_2^2(\rho_++\rho_*)}{2\mu}\right)}(\rho_+-\rho_*),\\ \rho_*>\rho_+.\end{cases} \tag{4.27}$$

From (4.26) and (4.27), we have

$$\begin{aligned}\nu_--\nu_+=&\sqrt{\left(\frac{1}{\rho_-}-\frac{1}{\rho_*}\right)\left(c_1(ln\rho_*-ln\rho_-)+\frac{c_2^2(\rho_*^2-\rho_-^2)}{2\mu}\right)}\\ &+\sqrt{\left(\frac{1}{\rho_*}-\frac{1}{\rho_+}\right)\left(c_1(ln\rho_+-ln\rho_*)+\frac{c_2^2(\rho_+^2-\rho_*^2)}{2\mu}\right)},\rho_*>\rho_\pm.\end{aligned} \tag{4.28}$$

We must prove the following lemmas before exploring the limiting behavior of the Riemann solutions to (4.1) and (4.2).

Lemma 4.2. *It exists that*

$$\lim_{c_1,c_2\to 0}\rho_*=+\infty$$

Proof. Let $\lim_{c_1,c_2\to 0}\inf\rho_*=m_1$ and $\lim_{c_1,c_2\to 0}\sup\rho_*=m_2$.

Firstly, we show that $m_1=m_2$. If $m_1<m_2$, then by the continuity of $\rho_*(c_1,c_2)$, there exists a sequence $\{(c_1,c_2)(n)\}_{n=1}^{\infty}\subset(0,1)$ such that

$$\lim_{n\to\infty}(c_1,c_2)(n)=0 \quad \textit{and} \quad \lim_{n\to\infty}\rho_*((c_1,c_2)(n))=c,$$

for some $c \in (m_1, m_2)$. Substituting this sequence into (4.28) and taking the limit, we have

$$\nu_- - \nu_+ = \lim_{n\to\infty} \left(\left(\frac{1}{\rho_-} - \frac{1}{\rho_*((c_1,c_2)(n))} \right) \left(c_1(ln\rho_*((c_1,c_2)(n)) - ln\rho_-) + \frac{c_2^2(\rho_*^2((c_1,c_2)(n)) - \rho_-^2)}{2\mu} \right) \right)^{\frac{1}{2}} + \lim_{n\to\infty} \left(\left(\frac{1}{\rho_*((c_1,c_2)(n))} - \frac{1}{\rho_+} \right) \times \left(c_1(ln\rho_+ - ln\rho_*((c_1,c_2)(n))) + \frac{c_2^2(\rho_+^2 - \rho_*^2((c_1,c_2)(n)))}{2\mu} \right) \right)^{\frac{1}{2}} = 0, \tag{4.29}$$

which is in contradiction with $\nu_- > \nu_+$. Hence we have $m_1 = m_2$.

Now we show that $\lim_{c_1,c_2\to 0} \rho_* = +\infty$. If $m_1 = m_2 \in (0, +\infty)$, then $\lim_{c_1,c_2\to 0} \rho_* = m_1$. We can also get the contradiction by taking the limit of (4.28) by following the same procedure as used above. Therefore $m_1 = m_2 = 0$ or $m_1 = m_2 = +\infty$. However, due to the entropy condition $\rho_* > \max\{\rho_-, \rho_+\}$:

$$\lim_{c_1,c_2\to 0} \rho_*(c_1, c_2) = +\infty.$$

Lemma 4.3. *Let*

$$\lim_{c_1,c_2\to 0} (c_2^2\rho_*^2 + 2\mu c_1 \ln \rho_*) = \frac{2\mu\rho_-\rho_+(\nu_- - \nu_+)^2}{(\sqrt{\rho_-} + \sqrt{\rho_+})^2}.$$

Lemma 4.4.

$$\lim_{c_1,c_2\to 0} \nu_* = \lim_{c_1,c_2\to 0} \sigma_1 = \lim_{c_1,c_2\to 0} \sigma_2 = \frac{\sqrt{\rho_-}\nu_- + \sqrt{\rho_+}\nu_+}{\sqrt{\rho_-} + \sqrt{\rho_+}} = \sigma, \tag{4.30}$$

$$\lim_{c_1,c_2\to 0}\int_{\sigma_1 t}^{\sigma_2 t}\rho_* dx = (\sigma[\rho]-[\rho\nu])t = \sqrt{\rho_-\rho_+}(\nu_- - \nu_+)t = \omega(t). \tag{4.31a}$$

$$\lim_{c_1,c_2\to 0}\int_{\sigma_1}^{\sigma_2}\rho_*\nu_* dx = \sigma[\rho\nu]-[\rho\nu^2]. \tag{4.31b}$$

Proof. Suppose that $c_1, c_2 \to 0$ in (4.26) and using lemma 4.3, we have

$$\begin{aligned}\lim_{c_1,c_2\to 0}\nu_* &= \nu_- - \lim_{c_1,c_2\to 0}\sqrt{\left(\frac{1}{\rho_-}-\frac{1}{\rho_*}\right)\left(c_1(ln\rho_* - ln\rho_-)+\frac{c_2^2(\rho_*^2-\rho_-^2)}{2\mu}\right)}\\ &= \nu_- - \frac{\sqrt{\rho_+}(\nu_- - \nu_+)}{\sqrt{\rho_-}+\sqrt{\rho_+}} = \frac{\sqrt{\rho_-}\nu_- + \sqrt{\rho_+}\nu_+}{\sqrt{\rho_-}+\sqrt{\rho_+}} = \sigma.\end{aligned} \tag{4.32}$$

From equation $(4.26)_1$ and using lemma 4.3, we have

$$\begin{aligned}\lim_{c_1,c_2\to 0}\sigma_1(t) &= \nu_- - \lim_{c_1,c_2\to 0}\sqrt{\frac{\rho_*^2}{\rho_*\rho_-}\left(\frac{c_1(ln\rho_* - ln\rho_-)}{(\rho_* - \rho_-)}+\frac{c_2^2\rho_*\rho_-}{2\mu}\left(\frac{1}{\rho_*}+\frac{1}{\rho_-}\right)\right)}\\ &= \nu_- - \frac{\sqrt{\rho_+}(\nu_- - \nu_+)}{\sqrt{\rho_-}+\sqrt{\rho_+}} = \frac{\sqrt{\rho_-}\nu_- + \sqrt{\rho_+}\nu_+}{\sqrt{\rho_-}+\sqrt{\rho_+}} = \sigma.\end{aligned} \tag{4.33}$$

Similarly, using equation (4.27) and (4.32) we get

$$\lim_{c_1,c_2\to 0}\sigma_2 = \lim_{c_1,c_2\to 0}\nu_* = \lim_{c_1,c_2\to 0}\sqrt{\frac{\rho_+^2}{\rho_*\rho_+}\left(\frac{c_1(ln\rho_+ - ln\rho_*)}{(\rho_+ - \rho_*)}+\frac{c_2^2(\rho_* + \rho_+)}{2\mu}\right)} = \sigma. \tag{4.34}$$

From (4.33) and (4.34), we see that the two shocks S_1 and S_2 coincide at $(c_1, c_2 \to 0)$. Its velocity is the same as the velocity of the delta shock waves in the transport equations using the same Riemann initial condition $(\rho_\pm, \nu_\pm)$.

For S_1 and S_2, by the R-H conditions (4.13), we get

$$\begin{cases} \sigma_1(\rho_* - \rho_-) = \rho_*\nu_* - \rho_-\nu_-, \\ \sigma_2(\rho_+ - \rho_*) = \rho_+\nu_+ - \rho_*\nu_*. \end{cases} \tag{4.35}$$

From (4.33) and (4.34) it concludes that

$$\lim_{c_1,c_2\to 0}(\sigma_1 - \sigma_2)\rho_* = \lim_{c_1,c_2\to 0}(\rho_+\nu_+ - \rho_-\nu_- + \sigma_1\rho_- - \sigma_2\rho_+) = [\rho\nu] - \sigma[\rho]. \tag{4.36}$$

That is

$$\lim_{c_1,c_2\to 0}\int_{\sigma_1 t}^{\sigma_2 t} \rho_* dx = (\sigma[\rho] - [\rho\nu])t = \sqrt{\rho_-\rho_+}(\nu_- - \nu_+)t = \omega(t). \tag{4.37}$$

Now from $(4.13)_2$, one also has

$$\begin{cases} \sigma_1(\rho_*\nu_* - \rho_-\nu_-) = \rho_*\nu_*^2 + c_1\ln\rho_* + \frac{(c_2\rho_*)^2}{2\mu} - \rho_-\nu_-^2 - c_1\ln\rho_- - \frac{(c_2\rho_-)^2}{2\mu}, \\ \sigma_2(\rho_+\nu_+ - \rho_*\nu_*) = \rho_+\nu_+^2 + c_1\ln\rho_+ + \frac{(c_2\rho_+)^2}{2\mu} - \rho_*\nu_*^2 - c_1\ln\rho_* - \frac{(c_2\rho_*)^2}{2\mu}, \end{cases}$$

which gives

$$(\sigma_2 - \sigma_1)\rho_*\nu_* = -\sigma_1\rho_-\nu_- + \sigma_2\rho_+\nu_+ + \rho_-\nu_-^2 + c_1\ln\rho_- + \frac{(c_2\rho_-)^2}{2\mu} - \rho_+\nu_+^2 - c_1\ln\rho_+ - \frac{(c_2\rho_+)^2}{2\mu}.$$

Using (4.33) and (4.34) it concludes that

$$\lim_{c_1,c_2\to 0}\int_{\sigma_1}^{\sigma_2} \rho_*\nu_* dx = \lim_{c_1,c_2\to 0}(\sigma_2 - \sigma_1)\rho_*\nu_* = \sigma[\rho\nu] - [\rho\nu^2].$$

Hence, it is proved.

4.4.2 Formation of Delta-shock

Here are the following results: This very well represents the limits for the case of $(\rho_+, \nu_+) \in I(\rho_-, \nu_-)$.

Theorem 4.5. *Let us suppose that* $\nu_- > \nu_+$ *and* $(\rho_+, \nu_+) \in I(\rho_-, \nu_-)$. *For any fixed* $c_1, c_2 > 0$*, assuming that* (ρ, ν) *is a solution containing the two shocks* S_1 *and* S_2 *of system (4.1) and (4.2) and having the Riemann initial data (4.3), in the case of* $c_1, c_2 > 0$*,* (ρ, ν) *converges in the sense of distribution. The limit functions* ρ *and* $\rho\nu$ *are the sum of the step function and the delta(*δ*)-measure of weights* $\omega_1 = (\sigma[\rho] - [\rho\nu])t$ *and* $\omega_2 = (\sigma[\rho\nu] - [\rho\nu^2])t$ *respectively, that constructed a delta-shock solution of (4.16) with the same initial data* $(\rho_\pm, \nu_\pm)$.

Proof. Put $\chi = \frac{x}{t}$ and $c_1, c_2 > 0$, the Riemann solution to the magnetogasdynamics with logarithmic equation of state (4.1)-(4.2) can be written as

$$(\rho, \nu)(\chi) = \begin{cases} (\rho_-, \nu_-), & \chi < \sigma_1, \\ (\rho_*, \nu_*), & \sigma_1 < \chi < \sigma_2, \\ (\rho_+, \nu_+), & \chi > \sigma_2. \end{cases} \tag{4.38}$$

The above equation satisfies the following weak formulations

$$\int_{-\infty}^{+\infty} \rho(\chi)\varphi(\chi)d\chi + \int_{-\infty}^{+\infty} (\chi - \nu(\chi))\rho(\chi)\varphi'(\chi)d\chi = 0, \tag{4.39}$$

and

$$\begin{aligned} \int_{-\infty}^{+\infty} \rho(\chi)\nu(\chi)\varphi(\chi)d\chi - \int_{-\infty}^{+\infty} \left(c_1 \ln \rho(\chi) + \frac{c_2^2(\rho(\chi))^2}{2\mu}\right) \varphi'(\chi)d\chi \\ + \int_{-\infty}^{+\infty} (\chi - \nu(\chi))\rho(\chi)\nu(\chi)\varphi'(\chi)d\chi = 0, \end{aligned} \tag{4.40}$$

for the test function $\varphi \in C_0^\infty(-\infty, +\infty)$.

For (III) term of integral (4.40), we obtain

$$\int_{-\infty}^{+\infty}(\chi - \nu(\chi))\rho(\chi)\nu(\chi)\varphi'(\chi)d\chi = \left(\int_{-\infty}^{\sigma_1} + \int_{\sigma_1}^{\sigma_2} + \int_{\sigma_2}^{+\infty}\right)(\chi - \nu(\chi))\rho(\chi)\nu(\chi)\varphi'(\chi)d\chi. \quad (4.41)$$

Now, from the (I) and (III) of the right side of (4.41), we have

$$\begin{aligned}
&\int_{-\infty}^{\sigma_1}(\chi - \nu(\chi))\rho(\chi)\nu(\chi)\varphi'(\chi)d\chi + \int_{\sigma_2}^{+\infty}(\chi - \nu(\chi))\rho(\chi)\nu(\chi)\varphi'(\chi)d\chi \\
&\quad = \rho_-\nu_-\sigma_1\varphi(\sigma_1) - \rho_-\nu_-^2\varphi(\sigma_1) - \rho_-\nu_-\int_{-\infty}^{\sigma_1}\varphi(\chi)d\chi \\
&\quad - \rho_+\nu_+\sigma_2\varphi(\sigma_2) + \rho_+\nu_+^2\varphi(\sigma_2) - \rho_+\nu_+\int_{\sigma_2}^{+\infty}\varphi(\chi)d\chi.
\end{aligned} \quad (4.42)$$

Taking the limit of both sides as $c_1, c_2 \to 0$ in (4.42) we have

$$\begin{aligned}
&\lim_{c_1,c_2\to 0}\left(\int_{-\infty}^{\sigma_1} + \int_{\sigma_2}^{+\infty}\right)(\chi - \nu(\chi))\rho(\chi)\nu(\chi)\varphi'(\chi)d\chi \\
&= ([\rho\nu^2] - \sigma[\rho\nu])\varphi(\sigma) - \int_{-\infty}^{+\infty}(\rho_0\nu_0)(\chi - \sigma)\varphi(\chi)d\chi,
\end{aligned} \quad (4.43)$$

where $(\rho_0\nu_0)(\chi - \sigma) = \frac{1}{2}(\rho_-\nu_- + \rho_+\nu_+ + [\rho\nu]H(\chi - \sigma))$ and H is Heaviside function. Solving II part of right side of (4.41) we have

$$\begin{aligned}
&\int_{\sigma_1}^{\sigma_2}(\chi - \nu(\chi))\rho(\chi)\nu(\chi)\varphi'(\chi)d\chi = \int_{\sigma_1}^{\sigma_2}(\chi - \nu_*)\rho_*\nu_*\varphi'(\chi)d\chi \\
&= -\left(\frac{\varphi(\sigma_2) - \varphi(\sigma_1)}{\sigma_2 - \sigma_1}\nu_* - \frac{\sigma_2\varphi(\sigma_2) - \sigma_1\varphi(\sigma_1)}{\sigma_2 - \sigma_1} + \frac{1}{\sigma_2 - \sigma_1}\int_{\sigma_1}^{\sigma_2}\varphi(\chi)d\chi\right) \\
&\times \nu_*\rho_*(\sigma_2 - \sigma_1).
\end{aligned} \quad (4.44)$$

Taking the limit $(c_1, c_2 \to 0)$ in (4.44), and using the equation (4.36) and lemma 4.4, we obtain

$$\lim_{c_1,c_2\to 0}\int_{\sigma_1}^{\sigma_2}(\chi - \nu(\chi))\rho(\chi)\nu(\chi)\varphi'(\chi)d\chi = \sigma([\rho\nu] - \sigma[\rho])(\sigma\varphi'(\sigma) - \sigma\varphi'(\sigma) - \varphi(\sigma) + \varphi(\sigma)) = 0. \quad (4.45)$$

Similarly, II integral of (4.40) we can write as

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\left(c_1\ln\rho(\chi)+\frac{c_2^2(\rho(\chi))^2}{2\mu}\right)\varphi'(\chi)d\chi\\
&\quad=\left(\int_{-\infty}^{\sigma_1}+\int_{\sigma_1}^{\sigma_2}+\int_{\sigma_2}^{+\infty}\right)\left(c_1\ln\rho(\chi)+\frac{c_2^2(\rho(\chi))^2}{2\mu}\right)\varphi'(\chi)d\chi\\
&\quad=\int_{-\infty}^{\sigma_1}\left(c_1\ln\rho_-+\frac{c_2^2\rho_-^2}{2\mu}\right)\varphi'(\chi)d\chi+\int_{\sigma_1}^{\sigma_2}\left(c_1\ln\rho_*+\frac{c_2^2\rho_*^2}{2\mu}\right)\varphi'(\chi)d\chi\\
&\qquad+\int_{\sigma_2}^{+\infty}\left(c_1\ln\rho_++\frac{c_2^2\rho_+^2}{2\mu}\right)\varphi'(\chi)d\chi\\
&\quad=\left(c_1\ln\rho_-+\frac{c_2^2\rho_-^2}{2\mu}\right)\varphi(\sigma_1)+c_1\ln\rho_*(\varphi(\sigma_2)-\varphi(\sigma_1))\\
&\qquad+\frac{c_2^2\rho_*^2}{2\mu}(\varphi(\sigma_2)-\varphi(\sigma_1))-\left(c_1\ln\rho_++\frac{c_2^2\rho_+^2}{2\mu}\right)\varphi(\sigma_2).
\end{aligned}\tag{4.46}$$

Taking the limit $(c_1, c_2 \to 0)$ in (4.46) and using lemma (4.2-4.4), we obtain

$$\lim_{c_1,c_2\to 0}\int_{-\infty}^{+\infty}\left(c_1\ln\rho(\chi)+\frac{c_2^2(\rho(\chi))^2}{2\mu}\right)\varphi'(\chi)d\chi=0.\tag{4.47}$$

Summarizing (4.43),(4.45)and (4.47) gives to

$$\lim_{c_1,c_2\to 0}\int_{-\infty}^{+\infty}((\rho\nu)(\chi)-(\rho_0\nu_0)(\chi-\sigma))\varphi(\chi)d\chi=(\sigma[\rho\nu]-[\rho\nu^2])\varphi(\sigma),\tag{4.48}$$

that is true for each $\varphi\in C_0^\infty(-\infty,+\infty)$. As we did before, taking limit of the last term in (4.39) we obtain

$$\begin{aligned}
&\lim_{c_1,c_2\to 0}\int_{-\infty}^{+\infty}(\chi-\nu(\chi))\rho(\chi)\varphi'(\chi)d\chi\\
&\qquad=([\rho\nu]-\sigma[\rho])\varphi(\sigma)-\int_{-\infty}^{\sigma}\rho_-\varphi(\chi)d\chi-\int_{\sigma}^{+\infty}\rho_+\varphi(\chi)d\chi\\
&\qquad=([\rho\nu]-\sigma[\rho])\varphi(\sigma)-\int_{-\infty}^{+\infty}\rho_0(\chi-\sigma)\varphi(\chi)d\chi,
\end{aligned}\tag{4.49}$$

where $\rho_0(\chi) = \rho_- + [\rho]H(\chi)$. Again from (4.39), we have

$$\lim_{c_1,c_2\to 0} \int_{-\infty}^{+\infty} (\rho(\chi) - \rho_0(\chi - \sigma))\varphi(\chi)d\chi = (\sigma[\rho] - [\rho\nu])\varphi(\sigma). \tag{4.50}$$

Finally, we look at the limits of ρ and $\rho\nu$ for $c_1, c_2 \to 0$, by tracking the time dependence of weights of the delta-measure and supposed $\Phi(x,t) \in C_0^\infty((-\infty,+\infty) \times [0,+\infty))$ we have

$$\lim_{c_1,c_2\to 0} \int_0^{+\infty} \int_{-\infty}^{+\infty} \rho(x/t)\Phi(x,t)dxdt = \lim_{c_1,c_2\to 0} \int_0^{+\infty} t \left(\int_{-\infty}^{+\infty} \rho(\chi)\Phi(\chi t,t)d\chi \right) dt. \tag{4.51}$$

Applying (4.50) (t-parameter), this gives

$$\begin{aligned} \int_{-\infty}^{+\infty} \rho(\chi)\Phi(\chi t,t)d\chi &= \int_{-\infty}^{+\infty} \rho_0(\chi - \sigma)\Phi(\chi t,t)d\chi + (\sigma[\rho] - [\rho\nu])\Phi(\sigma t,t) \\ &= \frac{1}{t}\int_{-\infty}^{+\infty} \rho_0(\frac{x}{t} - \sigma)\Phi(x,t)dx + (\sigma[\rho] - [\rho\nu])\Phi(\sigma t,t). \end{aligned} \tag{4.52}$$

Using (4.52) into (4.51), we have

$$\begin{aligned} \lim_{c_1,c_2\to 0} \int_0^{+\infty} \int_{-\infty}^{+\infty} \rho(x/t)\Phi(x,t)dxdt &= \int_0^{+\infty} \int_{-\infty}^{+\infty} \rho_0(x - \sigma t)\Phi(x,t)dxdt \\ &+ \int_0^{+\infty} t(\sigma[\rho] - [\rho\nu])\Phi(\sigma t,t)dt. \end{aligned} \tag{4.53}$$

According to (4.23), the second (right) term of (4.53) is equal to

$$\langle \omega_1(t)\delta_\Gamma, \Phi(.,.), \rangle \quad \text{where} \quad \omega_1(t) = t(\sigma[\rho] - [\rho\nu]).$$

Similarly, from (4.48), we arrive at

$$\lim_{c_1,c_2\to 0}\int_0^{+\infty}\int_{-\infty}^{+\infty}\rho(x/t)\nu(x/t)\Phi(x,t)dxdt = \int_0^{+\infty}\int_{-\infty}^{+\infty}(\rho_0\nu_0)(x-\sigma t)\Phi(x,t)dxdt + \int_0^{+\infty} t(\sigma[\rho\nu]-[\rho\nu^2])\Phi(\sigma t,t)dt. \tag{4.54}$$

The second (right) term of (4.54) equals to

$$\langle \omega_2(t)\delta_\Gamma, \Phi(.,.),\rangle \quad \text{where} \quad \omega_2(t) = t(\sigma[\rho\nu]-[\rho\nu^2]).$$

Hence proved .

4.5 Vacuum states

Here, we study the formation of vacuum state in the Riemann solution to (4.1) with (4.2) as $(\rho_+,\nu_+)\in III(\rho_-,\nu_-)$ with $\rho_\pm > 0$ and $\nu_- < \nu_+$ with pressureless and vanishing magnetic field. It consists of Riemann solutions R_1 and R_2, known as the backward and forward rarefaction wave. (ρ_*,ν_*) as the intermediate states of two steady states $(\rho_\pm,\nu_\pm)$, given as follows

$$R_1(\rho_-,\nu_-):\begin{cases}\chi = \lambda_1 = \nu - \sqrt{\frac{c_1}{\rho}+\frac{c_2^2\rho}{\mu}},\\ \nu = \nu_- - \int_{\rho_-}^{\rho}\frac{\sqrt{\frac{c_1}{s}+\frac{c_2^2 s}{\mu}}}{s}ds,\\ \rho_*\le\rho\le\rho_-,\end{cases} \tag{4.55}$$

and

$$R_2(\rho_-,\nu_-):\begin{cases}\chi=\lambda_2=\nu+\sqrt{\frac{c_1}{\rho}+\frac{c_2^2\rho}{\mu}},\\ \nu=\nu_++\int_{\rho_+}^{\rho}\frac{\sqrt{\frac{c_1}{s}+\frac{c_2^2 s}{\mu}}}{s}ds,\\ \rho_*\le\rho\le\rho_+.\end{cases}\tag{4.56}$$

Subtracting from $(4.56)_2$ and $(4.55)_2$, we get

$$\nu_+-\nu_-=\int_{\rho_*}^{\rho_-}\frac{\sqrt{\frac{c_1}{s}+\frac{c_2^2 s}{\mu}}}{s}ds+\int_{\rho_*}^{\rho_+}\frac{\sqrt{\frac{c_1}{s}+\frac{c_2^2 s}{\mu}}}{s}ds,\quad \rho_*\le\rho_\pm.\tag{4.57}$$

If $\lim_{c_1,c_2\to 0}\rho_*=C\in(0,min(\rho_-,\rho_+)]$, for $\rho_\pm>0$, then by

$$\begin{aligned}\int_{\rho_*}^{\rho_-}\frac{\sqrt{A_1+\frac{A_2}{s}}}{s}ds=&-2\sqrt{A_1+\frac{A_2}{\rho_-}}+\sqrt{A_1}\ln\left(A_2+2A_1\rho_-+2\sqrt{A_1}\sqrt{A_1+\frac{A_2}{\rho_-}}\rho_-\right)\\&+2\sqrt{A_1+\frac{A_2}{\rho_*}}-\sqrt{A_1}\ln\left(A_2+2A_1\rho_*+2\sqrt{A_1}\sqrt{A_1+\frac{A_2}{\rho_*}}\rho_*\right),\end{aligned}\tag{4.58}$$

result in

$$\begin{aligned}0\le&\int_{\rho_*}^{\rho_-}\frac{\sqrt{\frac{c_1}{s}+\frac{c_2^2 s}{\mu}}}{s}ds\le\int_{\rho_*}^{\rho_-}\frac{\sqrt{\frac{c_1}{s}+\frac{c_2^2\rho_-}{\mu}}}{s}ds\\ =&-2\sqrt{\frac{c_1}{\rho_-}+\frac{c_2^2\rho_-}{\mu}}+c_2\sqrt{\frac{\rho_-}{\mu}}\ln\left(c_1+2\frac{c_2^2\rho_-^2}{\mu}+2c_2\sqrt{\frac{\rho_-}{\mu}}\sqrt{\frac{c_1}{\rho_-}+\frac{c_2^2\rho_-}{\mu}}\rho_-\right)\\ &+2\sqrt{\frac{c_1}{\rho_*}+\frac{c_2^2\rho_-}{\mu}}-c_2\sqrt{\frac{\rho_-}{\mu}}\ln\left(c_1+2\frac{c_2^2\rho_*\rho_-}{\mu}+2c_2\sqrt{\frac{\rho_-}{\mu}}\sqrt{\frac{c_1}{\rho_*}+\frac{c_2^2\rho_-}{\mu}}\rho_*\right)\to 0,\end{aligned}\tag{4.59}$$

as $c_1, c_2 \to 0$. Here $A_1 = c_2^2\rho_-/\mu$ and $A_2 = c_1$

Using Squeeze theorem in the multi variable calculus, we have

$$\lim_{c_1,c_2\to 0}\int_{\rho_*}^{\rho_-}\frac{\sqrt{\frac{c_1}{s}+\frac{c_2^2 s}{\mu}}}{s}ds = 0. \tag{4.60}$$

Similarly,

$$\lim_{c_1,c_2\to 0}\int_{\rho_*}^{\rho_+}\frac{\sqrt{\frac{c_1}{s}+\frac{c_2^2 s}{\mu}}}{s}ds = 0. \tag{4.61}$$

Using (4.60) and (4.61) in (4.57), we have $\nu_- - \nu_+ = 0$, which is in contradiction with $\nu_- < \nu_+$. That means, $\lim_{c_1,c_2\to 0}\rho_* = 0$, occurs vacuum. From (4.55), we see that

$$\begin{aligned}\nu_- - \sqrt{\frac{c_1}{\rho}+\frac{c_2^2\rho}{\mu}} \le \lambda_1 = \nu_- - \sqrt{\frac{c_1}{\rho}+\frac{c_2^2\rho}{\mu}} + \int_{\rho}^{\rho_-}\frac{\sqrt{\frac{c_1}{s}+\frac{c_2^2 s}{\mu}}}{s}ds \\ \le \nu_- - \sqrt{\frac{c_1}{\rho}+\frac{c_2^2\rho}{\mu}} + \int_{\rho}^{\rho_-}\frac{\sqrt{\frac{c_1}{s}+\frac{c_2^2\rho_-}{\mu}}}{s}ds,\end{aligned} \tag{4.62}$$

$\rho_* \le \rho \le \rho_-$.

Using the result (4.58), we derive that

$$\begin{aligned}\nu_- - \sqrt{\frac{c_1}{\rho}+\frac{c_2^2\rho}{\mu}} + \int_{\rho}^{\rho_-}\frac{\sqrt{\frac{c_1}{s}+\frac{c_2^2\rho_-}{\mu}}}{s}ds = \nu_- - \sqrt{\frac{c_1}{\rho}+\frac{c_2^2\rho}{\mu}} + \sqrt{\frac{c_1}{\rho}+\frac{c_2^2\rho_-}{\mu}} \\ - \sqrt{\frac{c_2^2\rho_-}{\mu}}\ln\left(\sqrt{\frac{c_1}{\rho}+\frac{c_2^2\rho_-}{\mu}}+\sqrt{\frac{c_2^2\rho_-}{\mu}}\right) - \sqrt{\frac{c_2^2\rho_-}{\mu}}\ln\rho \\ - \sqrt{\frac{c_1}{\rho_-}+\frac{c_2^2\rho_-}{\mu}} + \sqrt{\frac{c_2^2\rho_-}{\mu}}\ln\left(\sqrt{\frac{c_1}{\rho_-}+\frac{c_2^2\rho_-}{\mu}}+\sqrt{\frac{c_2^2\rho_-}{\mu}}\right) + \sqrt{\frac{c_2^2\rho_-}{\mu}}\ln\rho_-.\end{aligned} \tag{4.63}$$

In this case, the uniform boundedness of $\rho(\chi)$ for c_1 and c_2 is as follows

$$\lim_{c_1,c_2\to 0}\left(\nu_- - \sqrt{\frac{c_1}{\rho}+\frac{c_2^2\rho}{\mu}} + \int_{\rho}^{\rho_-}\frac{\sqrt{\frac{c_1}{s}+\frac{c_2^2\rho_-}{\mu}}}{s}ds\right) = \lim_{c_1,c_2\to 0}\left(\nu_- - \sqrt{\frac{c_1}{\rho}+\frac{c_2^2\rho}{\mu}}\right)$$
$$= \nu_-. \tag{4.64}$$

Therefore, by Squeeze theorem in multi variable calculus, we obtained

$$\lim_{c_1,c_2\to 0}\lambda_1 = \nu_-.$$

Similarly, we can obtain

$$\lim_{c_1,c_2\to 0}\lambda_1 = \nu_- and \lim_{c_1,c_2\to 0}\nu(\chi) = \chi,\ \ for\ \chi\in(\nu_-,\nu_+). \tag{4.65}$$

We have established the following theorems using the aforementioned results.

Theorem 4.6. *If $(\rho_+,\nu_+)\in III(\rho_-,\nu_-)$ and $\nu_- < \nu_+$ as $(c_1,c_2\to 0)$, is the limit of Riemann solution of system (4.1) and (4.2) with initial data (4.3), only the Riemann solution of the zero pressure flow transport equation (4.16) with same initial data, including the two contact jumps $\chi = x/t = \nu_\pm$ and the vacuum state in addition to the two steady states.*

Remark 4.7. If $(\rho_+,\nu_+)\in III(\rho_-,\nu_-)$ and $\nu_- < \nu_+$ as $(c_1,c_2\to 0)$, the vacuum occurs and rarefaction waves R_1 and R_2 makes two contact jumps $\nu=\nu_-$ and $\nu=\nu_+$, respectively, connecting the steady states $(\rho_\pm,\nu_\pm)$ with the vacuum $\rho=0$.

Remark 4.8. From the result (4.37), if $(\rho_+,\nu_+)\in I(\rho_-,\nu_-)$ and $\nu_- > \nu_+$ as $(c_1,c_2\to 0)$, the Riemann solution of system (4.1) and (4.2) admits $\delta-$shock solution of transport equation (4.16) and (4.3).

4.6 Conclusions

In this chapter, the stability of the solutions of the Riemann problem for the hyperbolic system with the logarithmic EoS and magnetic field is studied. The formation of vacuum states and delta shock waves solutions for the system is obtained as magnetic field and pressure vanish. The Riemann invariants for these characteristic fields are obtained. Further, we analyzed that the Riemann solution consisting two shocks converges to the delta shock wave solution of transport equations, and the Riemann solutions consisting two rarefaction waves converges to the vacuum state solution which is intermediate state between two-contact discontinuity solution of transport equations. Hence, it is proved that the Riemann solutions of system with logarithmic EoS and magnetic field converges to the solution of the corresponding system as magnetic field and pressure vanish, which confirmed that the solution of the Riemann problem for the hyperbolic system with the logarithmic EoS and magnetic field is stable. It is worth noting that although the logarithmic equation of state and magnetogasdynamics system is more complex than the corresponding gas dynamic system, all the parallel results show similar behaviour. Also, the presence of the logarithmic EoS and magnetic field makes both the shock and rarefaction stronger than they would have been without a magnetic field.

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