Discrete-Time Super-Twisting Fractional-Order Observer With Implicit Euler Method

Xiaogang Xiong, *Member, IEEE*, Rahul Kumar Sharma, *Member, IEEE*, Shyam Kamal, *Member, IEEE*,
Sandip Ghosh, *Member, IEEE*, Yang Bai, *Member, IEEE*, and Yunjiang Lou, *Senior Member, IEEE*

***Abstract*—The work presented in this brief describes the design of a discrete-time super-twisting algorithm based fractional-order observer for a class of non-linear fractional-order systems. The proposed observer is shown to achieve higher performance as compared to the conventional integer-order observers in terms of robustness and convergence time. It generalizes the design of observers for the class of non-linear fractional-order systems. The peaking phenomenon is observed to be less significant in the proposed approach. Chattering is suppressed with the Fractional Adams-Moulton Method, which is an implicit Euler discretization technique. The significance of the proposed observer is illustrated through a simulation example.**

***Index Terms*—Observers, Fractional-Order Systems, Super-Twisting Algorithm (STA), Fractional Adams-Moulton (FAM) Method, Implicit Euler Discretization, Chattering Suppression.**

I. INTRODUCTION

OBSERVER design is an important task required to be performed in feedback control systems when only partial information about the states is known. Among the existing approaches, sliding mode based observers are very popular. Higher-order sliding mode algorithms are often used by researchers in the control community [1]. In this brief, Super-Twisting Algorithm (STA) is used which is a type of second-order Sliding Mode Control (SMC) scheme. It requires only the position information for its implementation. However, its applicability requires systems to have relative degree one with respect to the sliding variable. It is often used in various problems related to robust control and observation for non-linear systems. As compared to observers using first-order SMC approach, the input and output generated by STA based observer are smooth, which makes it suitable for many practical applications. However, when implemented with explicit Euler discretization, the undesired chattering problem persists. The magnitude of the chattering varies proportionally with the gains of the STA as well as the sampling time. One of the possible approaches to minimize chattering is to use the implicit Euler discretization [2], [3], [4]. In this context, full Euler discretized form of STA results only in the standard first-order accuracy of SMC [3].

For fractional-order systems, various works have been reported which design observers in continuous time as well as discrete time. An adaptive observer with fractional-order dynamics is designed for a class of fractional-order systems in [13]. The convergence is proved using Lyapunov's approach. Fractional-order disturbance observer has been designed in [9]. For systems with integer-order dynamics, a fractional-order observer is designed in [8]. Fault detection and state estimation schemes using observer based techniques have also been reported in the literature [15], [19]. The significance of observers has been demonstrated in practical applications [20]. STA has been used for finite-time estimation of derivatives of some class of input signals [6], [11]. For second-order systems, design of differentiator may also serve the purpose of estimating the state. But, such a direct signal based approach does not utilize the mathematical model of the system. If the system model is known, the observer design utilizing the available model is more appropriate for state estimation [10].

Derivatives and integrals of fractional order have been defined in different ways in the literature [14]. The Caputo definition has an inherent assumption that the function should satisfy the differentiability condition. This results into a mathematical restriction being imposed on the permissible classes of functions that can be operated according to such definition [7]. On the other hand, the Riemann-Liouville definition of fractional-order derivative allows operation on functions which posses non-differentiability at some finite points also. In this brief, an observer having fractional-order dynamics is proposed which can either be designed using Riemann-Liouville, Caputo or Grunwald-Letnikov definitions.

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Xiaogang Xiong is with the School of Mechanical Engineering and Automation, Harbin Institute of Technology (Shenzhen), Shenzhen 518055, China (e-mail: louyj@hit.edu.cn).

Rahul Kumar Sharma, Shyam Kamal, and Sandip Ghosh are with the Department of Electrical Engineering, Indian Institute of Technology (BHU), Varanasi 221005, India (e-mail: rahulks.rs.eee16@iitbhu.ac.in; shyamkamal.eee@iitbhu.ac.in; sghosh.eee@iitbhu.ac.in).

Yang Bai is with the Department of Information Science and Engineering, Ritsumeikan University, Kusatsu 525-8577, Japan (e-mail: yangbai@fc.ritsumei.ac.jp).

Yunjiang Lou is with the School of Mechanical Engineering and Automation, Harbin Institute of Technology (Shenzhen), Shenzhen 518055, China (e-mail: louyj@hit.edu.cn).

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Discretization using Fractional Adams-Moulton (FAM) technique is done which can be further used for effective realization of the designed observer on experimental setups.

This brief is composed of the following parts: Section II discusses about the problem formulation, Section III proposes the FAM based discretized form of the proposed fractional-order observer. Section IV describes the outcomes with using simulation and comparisons. Section V concludes.

Notations and definitions: Let $\mathcal{A}$ represent a closed interval of real numbers. Throughout the presentation, the definition $\text{proj}(\mathcal{A}, z) \triangleq \text{argmin}_{\xi\in\mathcal{A}}(\xi - z)^2$ is employed, which is interpreted as a projection onto the set $\mathcal{A}$. Let us define a set $\mathcal{A} = [M, N]$ with $M \leq N \in \mathbb{R}$. Then, the following expression results:

$$\text{proj}([M, N], z) = \max(M, \min(z, N)). \tag{1}$$

The set-valued signum function is defined as:

$$\text{sgn}(z) \triangleq \begin{cases} [-1, 1] & \text{if } z = 0 \\ \{z/|z|\} & \text{if } z \neq 0. \end{cases} \tag{2}$$

Such representation can be found in [3], [5]. For a non-negative scalar, the following relationship holds $K \geq 0$ [5]:

$$y - z \in -K\text{sgn}(y) \iff y = z - \text{proj}_{[-K,K]}(z) \tag{3}$$

In this brief, the general notation D^α denotes fractional-order derivative which can be defined using Riemann-Liouville, Caputo or Grunwald-Letnikov definitions [14].

II. Problem Formulation

Consider the folowing system,

$$\dot{x}_1 = x_2; \tag{4a}$$
$$\dot{x}_2 = f(t, x_1, x_2, u) + g(t, x_1, x_2, u) \tag{4b}$$
$$y = x_1 \tag{4c}$$

where $x := [x_1, x_2] \in \mathbb{R}^2$ is the state, $f(\cdot)$ denotes the nominal part of the system function and is known, while $g(\cdot)$ represents the unknown uncertainties. The functions $f(\cdot)$ and $g(\cdot)$ are assumed to be Lebesgue measurable such that $g(\cdot)$ is uniformly bounded in any compact region of the state space, i.e., $|g(t, x_1, x_2, u)| \leq \mathcal{G}$ with some constant $\mathcal{G}$. The aim is to design a finite-time convergent observer capable of estimating the state x_2 of system (4) given the position information x_1 and the nominal equations of the system. Conventionally, the structure of the observer is as follows [6]:

$$\dot{\hat{x}}_1 = \kappa_1\sqrt{|x_1 - \hat{x}_1|}\text{sgn}(x_1 - \hat{x}_1) + \hat{x}_2 \tag{5a}$$
$$\dot{\hat{x}}_2 \in \kappa_2\text{sgn}(x_1 - \hat{x}_1) + f(t, x_1, x_2, u), \tag{5b}$$

where $\hat{x} := [\hat{x}_1, \hat{x}_2]$ is the estimation of $x = [x_1, x_2]$, κ_1 and κ_2 are positive constant gains. Defining $e_1 := x_1 - \hat{x}_1$ and $e_2 := x_2 - \hat{x}_2$, the resulting form of dynamic error terms of (4) and (5) has the exact form of the super-twisting algorithm (STA):

$$\dot{e}_1 = -\kappa_1\sqrt{|e_1|}\text{sgn}(e_1) + e_2 \tag{6a}$$
$$\dot{e}_2 \in -\kappa_2\text{sgn}(e_1) + g(t, x_1, x_2, u), \tag{6b}$$

where e_2 contains the unknown variable x_2. Tuning the gain values κ_1 and κ_2 carefully using the information about $\mathcal{G}$, the error terms e_1 and e_2 become finite-time convergent to the origin [6].

The problem encountered in the classical observer of integer order represented as (5) is that it assumes the model with integer-order dynamics should be known in advance. However, many physical phenomena can only be modelled by considering fractional-order operators which are generalized integrals and derivatives [14]. Fractional-order models provide more accurate results. In many control problems, the difficulties encountered with integer-order operators have been solved quite comfortably by using fractional-order operators [11], [21]. So, there is a need to design observers for the generalized class of systems. Another problem with the observer (5) is that the information about $\mathcal{G}$ is required to be known in order to design it. Also, estimating the gain values κ_1 and κ_2 are usually difficult.

III. Discrete-Time Fractional-Order Observer

The difficulty with the integer-order model can be overcome by relaxing the order of the derivative operators in (4) as follows:

$$D^\alpha x_1 = x_2; \tag{7a}$$
$$D^\alpha x_2 = f(t, x_1, x_2, u) + g(t, x_1, x_2, u) \tag{7b}$$
$$y = x_1. \tag{7c}$$

Integer-order model (4) can be obtained as a special case of the more generalized fractional-order model (7) so obtained. When $\alpha = 1$, (7) is exactly in the same form of the model with integer-order dynamics (4). With this general fractional-order model (7), the difficulty with observer (5) can be tackled by using fractional-order observer for the nominal system:

$$D^\alpha\hat{x}_1 = \kappa_1\sqrt{|x_1 - \hat{x}_1|}\text{sgn}(x_1 - \hat{x}_1) + \hat{x}_2 \tag{8a}$$
$$D^\alpha\hat{x}_2 \in \kappa_2\text{sgn}(x_1 - \hat{x}_1) + f(t, x_1, \hat{x}_2, u), \tag{8b}$$

where D^α is the derivative of fractional order α with $0 < \alpha \leq 1$. Here, the fractional-order derivative can be defined using Riemann-Liouville, Caputo or Grunwald-Letnikov definitions [14]. Defining the error terms in the same way, the error dynamics becomes:

$$D^\alpha e_1 = -\kappa_1\sqrt{|e_1|}\text{sgn}(e_1) + e_2 \tag{9a}$$
$$D^\alpha e_2 \in -\kappa_2\text{sgn}(e_1) + \Delta(t, x_1, x_2, u) \tag{9b}$$

where $D^\alpha e_1 := D^\alpha x_1 - D^\alpha\hat{x}_1$, $D^\alpha e_2 := D^\alpha x_2 - D^\alpha\hat{x}_2$ and $\Delta(t, x_1, x_2, u) := g(t, x_1, x_2, u) + f(t, x_1, x_2, u) - f(t, x_1, \hat{x}_2, u)$. Numerical integration of (8) is a cumbersome work. The problem of numerical chattering due to the involved non-smooth signum functions in the equations makes the application of the usual discrete-time methods, e.g., explicit-Euler method difficult. Also, (8) is a fractional-order differential equation (FDE) with discontinuity on the right hand side and it may not be treated directly with the same numerical methods as used for the continuous FDEs. Here, a new method for the discretization of (8) is proposed which suppresses the undesired chattering in a significant way. Redefining $e_2 := v + \varphi$ and rewriting (9):

$$D^\alpha e_1 = -\kappa_1|e_1|^{\frac{1}{2}}\text{sgn}(e_1) + v + \varphi \tag{10a}$$

$$D^{\alpha} e_2 = -\kappa_2 \mathrm{sgn}(e_1) + \Delta(t) = D^{\alpha} v + D^{\alpha} \varphi \tag{10b}$$

$$D^{\alpha} v = -\kappa_2 \mathrm{sgn}(e_1); \quad D^{\alpha} \varphi = \Delta(t), \tag{10c}$$

where the uncertain quantity e_2 is composed of the components v and φ of which v is known, φ is the unknown disturbance such that $D^{\alpha}\varphi := \Delta(t)$. Here, (10a) is discretized based on FAM method which belongs to the family of k-step Fractional Linear Multi-step Method (FLMM) [12]:

$$E_{1,k-1} = \omega_k^{(\alpha-1)} e_{1,0} - \sum_{j=0}^{k-1} \omega_{k-j}^{(\alpha)} e_{1,j} \tag{11a}$$

$$e_{1,k} - E_{1,k-1} = h^{\alpha} e_{2,k} - h^{\alpha} \kappa_1 |e_{1,k}|^{\frac{1}{2}} \mathrm{sgn}(e_{1,k}) \tag{11b}$$

where $e_{i,k} := e_i(kh), i \in \{1, 2\}$, $k \in \mathbb{N}$, $h > 0$ is the time step, $E_{1,k-1}$ represents the lag term, $\omega_j^{(\alpha)}$ are the coefficients in the power series expansion of $(1-\xi)^{\alpha}$ in the FAM method, i.e., $(1-\xi)^{\alpha} = \Sigma_{j=0}^{\infty} \omega_j^{(\alpha)} \xi^j$ and $\omega_k^{(\alpha-1)} = \Sigma_{j=0}^{k} \omega_j^{(\alpha)}$. The discretization of (10b) with the same manner is:

$$E_{2,k-1} = \omega_k^{(\alpha-1)} e_{2,0} - \sum_{j=0}^{k-1} \omega_{k-j}^{(\alpha)} e_{2,j} \tag{12a}$$

$$e_{2,k} - E_{2,k-1} = -h^{\alpha} \kappa_2 \mathrm{sgn}(e_{1,k}) + h^{\alpha} \Delta_k \tag{12b}$$

where $e_{2,k} = v_k + \varphi_k$ and $E_{2,k-1}$ represents the lag term. Substituting (12) to (11) leads to,

$$e_{1,k} = E_{1,k-1} + h^{\alpha} E_{2,k-1} + h^{\alpha} u_{1,k} + h^{2\alpha} \Delta_k \tag{13a}$$

$$u_{1,k} \in -(\kappa_1 |e_{1,k}|^{\frac{1}{2}} + h^{\alpha} \kappa_2) \mathrm{sgn}(e_{1,k}) \tag{13b}$$

where $E_{1,k-1}$ is known but $E_{2,k-1}$ is unknown because it is the sum of historical data of the state $e_{2,j}, 0 \le j \le k$ that contains the unknown quantity φ_k shown in (10). Viewing the uncertain terms $E_{2,k-1}$ and Δ_k as disturbance results in the nominal form of (13) expressed as:

$$\tilde{e}_{1,k} = E_{1,k-1} + h^{\alpha} \tilde{u}_{1,k} \tag{14a}$$

$$\tilde{u}_{1,k} \in -(\kappa_1 |\tilde{e}_{1,k}|^{\frac{1}{2}} + h^{\alpha} \kappa_2) \mathrm{sgn}(\tilde{e}_{1,k}) \tag{14b}$$

where $\tilde{e}_{1,k}$ represents the nominal form of $e_{1,k}$. From (14), for $\tilde{e}_{1,k} \neq 0$, one has

$$|\tilde{e}_{1,k}| + h^{\alpha} \kappa_1 |\tilde{e}_{1,k}|^{\frac{1}{2}} + h^{2\alpha} \kappa_2 = |E_{1,k-1}| \tag{15}$$

which means $|\tilde{e}_{1,k}| < |E_{1,k-1}|$. As larger gains result in a shorter time to reach the sliding surface, the gain term $(\kappa_1 |\tilde{e}_{1,k}|^{\frac{1}{2}} + h^{\alpha} \kappa_2)$ in (14) is replaced by $(\kappa_1 |e_{1,k-1}|^{\frac{1}{2}} + h^{\alpha} \kappa_2)$ without solving (15) explicitely:

$$\tilde{e}_{1,k} = E_{1,k-1} + h^{\alpha} \tilde{u}_{1,k} \tag{16a}$$

$$\tilde{u}_{1,k} \in -(\kappa_1 |E_{1,k-1}|^{\frac{1}{2}} + h^{\alpha} \kappa_2) \mathrm{sgn}(\tilde{e}_{1,k}) \tag{16b}$$

Using (3), the equivalence of (16) is obtained as:

$$\tilde{e}_{1,k} = E_{1,k-1} + h^{\alpha} \tilde{u}_{1,k}; \tilde{u}_{1,k} = -\frac{1}{h^{\alpha}} \mathrm{proj}_{C_{1,k}}\big(E_{1,k-1}\big) \tag{17}$$

where proj$(\cdot)$ is defined in (1) and the set $C_{1,k}$ is defined as:

$$C_{1,k} := \big[-D_{1,k}, D_{1,k}\big], D_{1,k} := h^{\alpha} \kappa_1 |E_{1,k-1}|^{\frac{1}{2}} + h^{2\alpha} \kappa_2$$

For (17), the sliding surface is defined as follows.

Definition 1: The sliding surface in discrete form for (16) or its equivalence (17) is defined as $\Sigma_d = \{\tilde{e}_{1,k} \in \mathbb{R} | \tilde{e}_{1,k} = 0\}$.

Equation (17) is guaranteed to reach Σ_d after a finite discrete steps as evident from the following lemma.

Lemma 1: Consider the system (16) or the equivalence (17) with $e_{1,0} := e_1(0)$. After a finite step $k_0 := \lceil |e_{1,0}|/(h^{2\alpha} k_2) \rceil$, (16) or (17) is on Σ_d, i.e., $\tilde{e}_{1,k} = 0$ for $k \ge k_0 + 1$.

Proof: From (17) as well as (1), for $|e_{1,k}| > D_{1,k} > 0$, one has $\tilde{e}_{1,k} = E_{1,k-1} - D_{1,k}$ if $E_{1,k-1} > D_{1,k} > 0$ and $\tilde{e}_{1,k} = E_{1,k-1} + D_{1,k}$ if $E_{1,k-1} < -D_{1,k} < 0$, that is, $\{|\tilde{e}_{1,k}|\}$ is always decreasing. While, for $|E_{1,k-1}| \le D_{1,k}$, one has $\tilde{e}_{1,k} = E_{1,k-1} - E_{1,k-1} = 0$. Let us begin with the case $|e_{1,0}| > D_{1,1}$. From the above three cases, after at most $k_0 := \lceil |e_{1,0}|/(h^{2\alpha} k_2) \rceil \ge 0$ with $\lceil \cdot \rceil$ being the ceiling function, $|e_{1,k}| \le D_{1,k}$ is ensured and $\tilde{e}_{1,k} = 0$ at $k \ge k_0 + 1$ is obtained. ■

From Lemma 1, after (17) has reached sliding surface Σ_d, it results that $\tilde{e}_{1,k} = 0$, $|E_{1,k-1}| \le D_{1,k}$ and $\tilde{u}_{1,k} = -E_{1,k-1}/h^{\alpha}$ for $k \ge k_0 + 1$ from definition (1). So, due to the expression of $\tilde{u}_k$ in (14), the gain of $\tilde{u}_k$ switches between the cases $\tilde{e}_{1,k} \neq 0$ and $\tilde{e}_{1,k} = 0$, that is, (17) changes into:

$$\tilde{e}_{1,k} = E_{1,k-1} + h^{\alpha} \tilde{u}_{1,k}, \tilde{u}_{1,k} = -\frac{1}{h^{\alpha}} \mathrm{proj}_{C_{1,k}}\big(E_{1,k-1}\big) \tag{18a}$$

$$C_{1,k} := \begin{cases} [-D_{1,k}, D_{1,k}] & \text{if } |E_{1,k-1}| > D_{1,k} \\ [-D_{2,k}, D_{2,k}] & \text{else} \end{cases} \tag{18b}$$

where $D_{2,k} := h^{1+\alpha} k_2$. The advantage of switching the gain is that, the overestimation of the gain $D_{1,k}$ only happens before reaching the sliding surface Σ_d.

Considering $u_{1,k}$ in (13) is approximated by the term $\tilde{u}_{1,k}$, the following expression results:

$$e_{1,k} = E_{1,k-1} + h^{\alpha} \tilde{u}_{1,k} + h^{\alpha} E_{2,k-1} + h^{2\alpha} \Delta_k \tag{19a}$$

$$\tilde{u}_{1,k} = -\frac{1}{h^{\alpha}} \mathrm{proj}_{C_{1,k}}\big(E_{1,k-1}\big), \tag{19b}$$

where the terms $E_{2,k-1}$ and Δ_k are viewed as unknown disturbances here. Comparing (19) to (17), it is easy to conclude the following from Lemma 1.

Lemma 2: Consider the system (19) with $e_{1,0} = e_1(t_0)$. Assume that the condition $\sup_k |h^{\alpha} \Delta_k + E_{2,k-1}| < h^{\alpha} \kappa_2$ is satisfied. Then, after at most a finite number of steps $k_0 := \lceil |e_{1,0}|/(h^{2\alpha} \kappa_2) \rceil$, the state of the system (19) remains in the subset: $\mathcal{R}_1 = \{e_{1,k} \in \mathbb{R} | e_{1,k} = h^{\alpha}(E_{k-1} + h^{\alpha} \Delta_k)\}$ for $k > k_0$.

Proof: By using the relation (3), the discrete-time system (19) can be equivalently rewritten as $e_{1,k} \in E_{1,k-1} - D_{1,k} \mathrm{sgn}(\tilde{e}_{1,k}) + h^{\alpha} E_{2,k-1} + h^{2\alpha} \Delta_k$ with $\tilde{e}_{1,k}$ defined in (16) and $D_{1,k}$ given in (18). With Lemma 1, the sliding surface Σ_d is reached for $k > k_0$. As $e_{1,k} = \tilde{e}_{1,k} + h^{\alpha}(E_{k-1} + h^{\alpha} \Delta_k)$, one has $e_{1,k} = h^{\alpha}(E_{k-1} + h^{\alpha} \Delta_k)$. Similar conclusion can be found in [5, Proposition 1]. ■

By inserting the input (19b), the unknown term $E_{2,k-1} + h^{\alpha} \Delta_k$ attenuated by the factor h^{α} determines the estimation accuracy. Because term $e_{2,k}$ can be divided into two separated parts v_k and φ_k, of which v_k can be managed, v_k can be used to improve the estimation accuracy. Suppose the state of (19) is within $\mathcal{R}_1$ after $k \geq k_0$,

$$e_{1,k} = h^{\alpha} E_{2,k-1} + h^{2\alpha} \Delta_k \tag{20a}$$

$$E_{2,k-1} = \omega_k^{(\alpha-1)}(v_0 + \varphi_0) - \sum_{j=0}^{k-1} \omega_{k-j}^{(\alpha)}(v_j + \varphi_j) \tag{20b}$$

where, v_k and φ_k are updated according to (10) as:

$$v_k = \Psi_{k-1} + h^{\alpha} u_{2,k},\ u_{2,k} \in -\kappa_2 \text{sgn}(e_{1,k}) \tag{21a}$$

$$\varphi_k = \Phi_{k-1} + h^{\alpha}\Delta_k,\ \Phi_{k-1} = \omega_k^{(\alpha-1)}\varphi_0 - \sum_{j=0}^{k-1} \omega_{k-j}^{(\alpha)}\varphi_j \tag{21b}$$

$$\Psi_{k-1} := \omega_k^{(\alpha-1)} v_0 - \sum_{j=0}^{k-1} \omega_{k-j}^{(\alpha)} v_j. \tag{21c}$$

The above equations clearly show that $E_{2,k-1} = \Psi_{k-1} + \Phi_{k-1}$ and $e_{1,k} = h^{\alpha}\Psi_{k-1} + h^{\alpha}\varphi_k$ where Ψ_{k-1} is a known input. In order achieve more accurate estimation, the input Ψ_{k-1} in $e_{1,k} = h^{\alpha}\Psi_{k-1} + h^{\alpha}\varphi_k$ is replaced by v_k. Here, the term $u_{2,k}$ compensates the matched disturbance term φ_k:

$$e_{1,k} = h^{\alpha} v_k + h^{\alpha}\varphi_k,\ v_k = \Psi_{k-1} + h^{\alpha} u_{2,k}. \tag{22}$$

As $e_{1,k} = h^{\alpha} v_k + h^{\alpha}\varphi_k$ can be handled for $k \geq k_0$. Suppose $E_{2,k-1}$ is split into two components when $k = k_0 + i$, i.e., $E_{2,k-1} = \bar{E}_{2,k-1} + R_{1,k-1}$ where, $i \geq 1$ is an integer. From (22),

$$\begin{aligned} e_{1,k} &= h^{\alpha} v_k + h^{\alpha}\varphi_k = h^{\alpha}\Psi_{k-1} + h^{2\alpha} u_{2,k} + h^{\alpha}\Phi_{k-1} \\ &+ h^{2\alpha}\Delta_k = h^{\alpha}(\bar{E}_{2,k-1} + R_{1,k-1}) + h^{2\alpha}(u_{2,k} + \Delta_k) \end{aligned} \tag{23a}$$

$$v_k = \Psi_{k-1} + h^{\alpha} u_{2,k};\ u_{2,k} \in -\kappa_2 \text{sgn}(e_{1,k}) \tag{23b}$$

where,

$$\begin{aligned} \bar{E}_{2,k-1} &= \omega_k^{(\alpha-1)}(v_0 + \varphi_0) - \sum_{j=0}^{k_0-1} \omega_{k-j}^{(\alpha)}(v_j + \varphi_j) \\ &= \underbrace{\omega_k^{(\alpha-1)} v_0 - \sum_{j=0}^{k_0-1} \omega_{k-j}^{(\alpha)} v_j}_{=\Psi_{k_0}} + \underbrace{\omega_k^{(\alpha-1)}\varphi_0 - \sum_{j=0}^{k_0-1} \omega_{k-j}^{(\alpha)}\varphi_j}_{=\Phi_{k_0}} \end{aligned} \tag{24a}$$

$$\begin{aligned} R_{1,k-1} &= -\sum_{j=k_0}^{k_0+i-1} \omega_{k-j}^{(\alpha)} v_j - \sum_{j=k_0}^{k_0+i-1} \omega_{k-j}^{(\alpha)}\varphi_j \\ &= -\frac{1}{h^{\alpha}} \sum_{j=k_0}^{k_0+i-1} \omega_{k-j}^{(\alpha)} e_{1,j}, \end{aligned} \tag{24b}$$

where the last expression is obtained because when $k > k_0$, one has $e_{1,k} = h^{\alpha} v_k + h^{\alpha}\varphi_k$ from (22). For $k > k_0$, Ψ_{k_0} and $R_{1,k-1}$ can be calculated separately while $|\Phi_{k_0}|$ is unknown but bounded because $k_0 := \text{argmin}_k\{\tilde{e}_{1,k} = 0\}$ is a finite number according to Lemma 1. Then, by considering Φ_{k_0} and Δ_k as unknown disturbances, let us see the following nominal form of (23):

$$\tilde{e}_{1,k} = h^{\alpha}(\Psi_{k_0} + R_{1,k-1}) + h^{2\alpha}\tilde{u}_{2,k} \tag{25a}$$

$$v_k = \Psi_{k-1} + h^{\alpha}\tilde{u}_{2,k},\ \tilde{u}_{2,k} = -\kappa_2 \text{sgn}(\tilde{e}_{1,k}) \tag{25b}$$

Comparing (23b) to (14b), from (19b), the explicit calculation of $\tilde{u}_{2,k}$ becomes easy:

$$\tilde{e}_{1,k} = h^{\alpha}(\Psi_{k_0} + R_{1,k-1} + h^{\alpha}\tilde{u}_{2,k}),\ v_k = \Psi_{k-1} + h^{\alpha}\tilde{u}_{2,k} \tag{26a}$$

$$\tilde{u}_{2,k} = -\frac{1}{h^{2\alpha}} \text{proj}_{C_{2,k}}\left(h^{\alpha}\Psi_{k_0} + h^{\alpha} R_{1,k-1}\right), \tag{26b}$$

where, $C_{2,k} := [-D_{2,k}, D_{2,k}]$ with $D_{2,k} = h^{2\alpha}\kappa_2$. Using $\tilde{u}_{2,k}$ from the nominal equations (26) in the perturbed equations (22), the following equations are obtained:

$$e_{1,k} = h^{\alpha} v_k + h^{\alpha}\varphi_k;\ v_k = \Psi_{k-1} + h^{\alpha}\tilde{u}_{2,k}; \tag{27a}$$

$$\tilde{u}_{2,k} = -\frac{1}{h^{2\alpha}} \text{proj}_{C_{2,k}}\left(h^{\alpha}\Psi_{k_0} + h^{\alpha} R_{1,k-1}\right) \tag{27b}$$

which have the characteristics.

Theorem 1: Take (27) with the initial state $e_{1,0} := e_1(0)$. Assume that the quantity $e_{1,k}$ lies in the subset $\mathcal{R}_1$ for $k \geq k_0$ and $\sup_k |\Delta_k| < \mu$ with some $0 < \mu < \kappa_2$. Then, $e_{1,k}$ enters the neighborhood of the origin: $\mathcal{R}_2 = \{e_k \in \mathbb{R} | e_{1,k} = h^{\alpha} e_{2,k}, |e_{2,k}| < \mu h^{\alpha}\}$ as $k \to \infty$.

Proof: Because $e_{1,k}$ is lying within the subset $\mathcal{R}_1$ for some $k \geq k_0$, from (26a), $e_{1,k} = h^{\alpha} v_k + h^{\alpha}\varphi_k, k = k_0, k_0 + 1, k_0 + 2, \ldots,$. From the equations (20b)-(24), due to $k > k_0$, one can define Φ_{k_0}, Ψ_{k_0} and $R_{1,k-1}$. Then, the error dynamics $e_{1,k-1} = hv_{k-1} + h\varphi_{k-1}$ becomes,

$$e_{1,k} = h^{\alpha}(\Phi_{k_0} + \Psi_{k_0} + R_{1,k-1}) + h^{2\alpha}(\tilde{u}_{2,k} + \Delta_k) \tag{28}$$

where, $\tilde{u}_{2,k}$ is given as in (26b). From Lemma 1 and Lemma 2, one can see that after finite discrete steps, $\tilde{e}_{1,k}$ remains on Σ_d with $\tilde{e}_{1,k-1} = 0$, and $\tilde{u}_k = \text{proj}_{C_k}(E_{1,k-1}) = E_{1,k-1}$. In the same way, from the step k_0, after finite steps, $\tilde{u}_{2,k} = -\text{proj}_{C_{2,k}}(h^{\alpha}\Psi_{k_0} + h^{\alpha} R_{1,k-1})/h^{2\alpha} = -(\Psi_{k_0} + R_{1,k-1})/h^{\alpha}$. Substituting it into (28) leads to $e_{1,k} = h^{\alpha}\Phi_{k_0} + h^{2\alpha}\Delta_k$. As iteration continues, i.e., $k \to \infty$, the multiplier $\omega_{k-j}^{(\alpha)} \to 0$ in Φ_{k_0}, then one has $\Phi_{k_0} \to 0$ and $e_{1,k} = h^{2\alpha}\Delta_k$ as $k \to \infty$. As $e_{1,k} = h^{\alpha}(v_k + \varphi_k) = h^{\alpha} e_{2,k}$ for $k \geq k_0$, $|e_{2,k}| = h^{\alpha}|\Delta_k| < \mu h^{\alpha}$ is obtained as $k \to \infty$. ■

The results from Lemma 2 to Theorem 1 are valid after step $k \geq k_0$, with v_k and $\tilde{u}_{2,k}$ (27). But, v_k is obtained from $k = 0$ to $k_0 - 1$, depending on $\tilde{u}_{2,k}$ in (27) which is undefined. In order to maintain consistency with Lemma 2, consider defining $\tilde{u}_{2,k}$ and v_k for $k > k_0$ and $k \leq k_0$:

$$e_{1,k} = E_{1,k-1} + h^{\alpha} E_{2,k-1} + h^{\alpha} u_k + h^{2\alpha}\Delta_k \tag{29a}$$

$$u_k = \tilde{u}_{1,k} + v_k,\quad \tilde{u}_{1,k} = -\frac{1}{h} \text{proj}_{C_{1,k}}\left(E_{1,k-1}\right) \tag{29b}$$

$$v_k = \Psi_k + h^{\alpha}\tilde{u}_{2,k}, \tilde{u}_{2,k} = -\frac{1}{h^{2\alpha}} \text{proj}_{C_{2,k}}\left(\bar{E}_{1,k-1}\right) \tag{29c}$$

where, $\bar{E}_{1,k-1} := \Psi_{k-1}$ if $\tilde{e}_{1,k} \neq 0$ and $\bar{E}_{1,k-1} := h^{\alpha}\Psi_{k_0} + h^{\alpha} R_{1,k-1}$ if $\tilde{e}_{1,k} = 0$. Using the proposed integration scheme of the STA with fractional-order dynamics (29) in the observer (8):

$$\hat{x}_{1,k} = \hat{X}_{1,k-1} - h^{\alpha}\tilde{u}_{1,k} + h^{\alpha}\hat{x}_{2,k}, \tag{30a}$$

$$\hat{x}_{2,k} = \hat{X}_{2,k-1} - h^{\alpha}\tilde{u}_{2,k} + h^{\alpha} f(t, x_{1,k}, \hat{x}_{2,k}, u), \tag{30b}$$

$$\hat{X}_{i,k-1} = \omega_k^{(\alpha-1)}\hat{x}_{i,0} - \sum_{j=0}^{k-1} \omega_{k-j}^{(\alpha)}\hat{x}_{i,j} \tag{30c}$$

where, $\tilde{u}_{1,k}$ is defined in (19b), $\tilde{u}_{2,k}$ is defined in (27b), $E_{1,k-1}$ is the same as define defined in (29). The discrete-time observer (30) is obtained according to (29) by comparing (10) with (8).

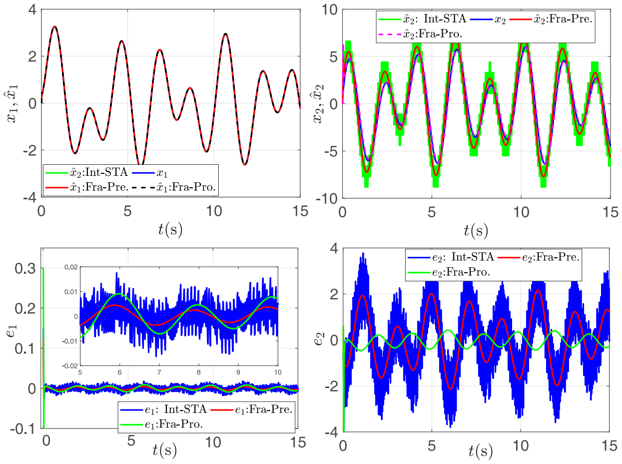

Fig. 1. Comparisons with different observer implementations for estimating x_2 for the second-order system (32). The control input is $u = 0.1\cos(2t) + 0.1\sqrt{10}\cos(\sqrt{10}t)$. The parameters are set as $\alpha = 0.8, \kappa_1 = 1.5\sqrt{L}, \kappa_2 = 1.1L, L = 50$ and $h = 0.01$ sec. The other parameters such as L, R, C are the same as in [11].

IV. ILLUSTRATIVE EXAMPLE

The example of a circuit system consisting of fractional-order elements is considered [16]. The fractional-order models have been proved to describe more accurate behaviour than the corresponding integer-order models [17]. One such circuit system can be described by the following commensurate fractional-order model:

$$\frac{D^{\alpha} v_0}{dt} = \frac{1}{C}\left(i_L - \frac{v_0}{R}\right), \frac{D^{\alpha} i_L}{dt} = \frac{1}{L}(uV_{in} - v_0) \tag{31}$$

where i_L is the current through the fractional-order inductor, v_C is the voltage across the fractional-order capacitor and V_{in} is DC voltage input. L, C and R are fractional-order inductance, fractional-order capacitance and resistance, respectively. The above model can be represented in the following canonical form [18]:

$$D^{\alpha} x_1 = x_2; \quad D^{\alpha} x_2 = u\frac{V_{in}}{LC} - \frac{x_2}{RC} - \frac{x_1}{LC} \tag{32}$$

where $x_1 := v_0$ and $x_2 := i_L/C - v_0/(RC)$ are considered as the states of the system. Using the information of the avalable state x_1 and the order α, the objective is to estimate the state x_2 independent of the control input u.

Three algorithms are implemented to obtain the estimation of the state $\hat{x}_2$. The comparison results are shown in Fig. 1. The proposed scheme (30) is denoted as "Fra-Pro.", the conventional integer-order STA based observer (5) implemented with explicit Euler method is denoted by "Int-STA" and the authors' previous mixed integer and fractional-order STA is denoted by "Fra-Pre." [11]. The benchmark values of x_1 and x_2 are obtained by integrating the system model (31) with the method in [22]. As observed from Fig. 1, the proposed scheme (30) has the maximum estimation accuracy for the state x_2 of the system. Also, the chattering is significantly suppressed with the proposed scheme as compared with the other techniques.

V. CONCLUSION

With the proposed discretization scheme, the designed fractional-order observer estimates the unknown state exactly in finite time. The simulation results of the proposed methodology has been obtained by considering an example of a circuit system consisting of elements with fractional-order dynamics. The results show satisfactory performance with the desired properties as compared to the classical integer-order observers. An analysis using Lyapunov theory remains to be explored.

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