Chapter 2

Weak sharp minima for interval-valued functions and its primal-dual characterizations using generalized Hukuhara subdifferentiability

2.1 Introduction

In this chapter, the concept of weak sharp minima (WSM) for interval-valued functions is defined. The notion of WSM is a generalization of the notion of sharp minima [166] to include the possibility of a non-unique solution set. Sharp minima are also referred to as strongly unique local minima in the work of Cromme [52]. The terminology weak sharp minima was introduced by Ferris in [62], where it is extensively developed. After that, Burke and Ferris [36] gave the necessary and sufficient conditions for WSM and proved the finite termination at WSM of algorithms. Further, Burke and Deng [34] generalized this work to the normed linear space setting and presented two characterizations, namely primal and dual, using the concept of subdifferentiability and support function. Thereafter, the concept of WSM has been studied by many researchers in different aspects, for instance, see [129, 144, 155, 182, 182, 204] and references therein.

2.2 Motivation

WSM plays an important role in the sensitivity analysis and convergence analysis of conventional optimization problems, see [34–36, 126, 143, 145, 201]. It is also seen that many algorithms exhibit finite termination at WSM [36,63,154,192,204]. In [204], Zhou and Wang presented the concept of WSM by using conjugate functions and established the finite termination property for convex programming and variational inequality problem, respectively. Subsequently, Matsushita and Xu [154] solved the convex optimization problem by the proximal point algorithm in a finite number of steps under the assumption that the solution set is a set of WSM. Wang et al. [192] studied the finite termination of sequences generated by inexact proximal point algorithms for finding zeroes of a maximal monotone (set-valued) operator T on a Hilbert space. Motivated by these properties and the wide applications of WSM in conventional optimization, in this chapter, we attempt to propose and mathematically characterize the notion of WSM for convex IVFs.

2.3 Contributions

The major contributions in this chapter are as follows:

- The notion of support function in $I(\mathbb{R})^n$ is defined.
- The concept of gH-subdifferentiabilty is presented for IVFs.
- Some necessary properties such as nonemptiness, convexity, closedness, and boundedness of gH-subdifferential set are derived.
- WSM for IVF is proposed.
- Two characterizations, namely primal and dual, of WSM are given for IVFs.

- Applications of the proposed study are given.

2.4 Support function in $I(\mathbb{R})^n$

In this section, we attempt to extend the conventional notion of support functions for subsets of $I(\mathbb{R})^n$. The derived concepts of support function are used later in Section 2.6 to derive dual characterizations of WSM for convex IVFs.

Definition 2.1 (Support function of a subset of $I(\mathbb{R})^n$). *Let $\boldsymbol{S}$ be a nonempty subset of $I(\mathbb{R})^n$. Then, the support function of $\boldsymbol{S}$ at $x \in \mathbb{R}^n$, denoted by $\boldsymbol{\psi}^*_{\boldsymbol{S}}(x)$, is defined by*

$$\boldsymbol{\psi}^*_{\boldsymbol{S}}(x) = \sup_{\widehat{\boldsymbol{A}} \in \boldsymbol{S}} x^\top \odot \widehat{\boldsymbol{A}}.$$

Lemma 2.1 *Let $\boldsymbol{S}_1, \boldsymbol{S}_2$ be two nonempty subsets of $I(\mathbb{R})^n$ such that $\boldsymbol{S}_1 \subseteq \boldsymbol{S}_2$. Then, for any $x \in X \subseteq \mathbb{R}^n$,*

$$\boldsymbol{\psi}^*_{\mathbf{S_1}}(x) \preceq \boldsymbol{\psi}^*_{\mathbf{S_2}}(x).$$

Proof: For any $\widehat{\mathbf{B}} = \left([\underline{b}_1, \overline{b}_1], [\underline{b}_2, \overline{b}_2], \ldots, [\underline{b}_n, \overline{b}_n]\right) \in \mathbf{S}_2$ and $x \in X$, we have $x^\top \odot \widehat{\mathbf{B}} \preceq \boldsymbol{\psi}^*_{\mathbf{S_2}}(x)$. Given $\mathbf{S}_1 \subseteq \mathbf{S}_2$, i.e., for any $\widehat{\mathbf{D}} = \left([\underline{d}_1, \overline{d}_1], [\underline{d}_2, \overline{d}_2], \ldots, [\underline{d}_n, \overline{d}_n]\right) \in \mathbf{S}_1$, we have $\widehat{\mathbf{D}} \in \mathbf{S}_2$. Therefore,

$$x^\top \odot \widehat{\mathbf{D}} \preceq \boldsymbol{\psi}^*_{\mathbf{S_2}}(x).$$

Since $\widehat{\mathbf{D}}$ is arbitrary, we get

$$\boldsymbol{\psi}^*_{\mathbf{S_1}}(x) = \sup_{\widehat{\mathbf{E}} \in \mathbf{S}_1} x^\top \odot \widehat{\mathbf{E}} \preceq \boldsymbol{\psi}^*_{\mathbf{S_2}}(x).$$

□

Theorem 2.1 *Let $\mathscr{K}$ be a nonempty closed convex cone in $X \subseteq \mathbb{R}^n$. Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be two nonempty subsets of $I(\mathbb{R})^n$. Then,*

$$\boldsymbol{\psi}^*_{\boldsymbol{P}}(x) \preceq \boldsymbol{\psi}^*_{\boldsymbol{Q}}(x) \text{ for all } x \in \mathscr{K} \quad \text{if and only if} \quad \boldsymbol{\psi}^*_{\boldsymbol{P}}(x) \preceq \boldsymbol{\psi}^*_{\boldsymbol{Q} \oplus \mathscr{K}^\circ}(x) \text{ for all } x \in X,$$

where $\mathscr{K}^o$ is the polar cone of $\mathscr{K}$.

Proof: Let $\boldsymbol{\psi}^*_{\mathbf{P}}(x) \preceq \boldsymbol{\psi}^*_{\mathbf{Q}}(x)$ for all $x \in \mathscr{K}$. Consider $x \in \mathscr{K}$. Clearly $\mathbf{Q} \subseteq \mathbf{Q} \oplus K^o$. Then, by Lemma 2.1, we have

$$\begin{aligned}\boldsymbol{\psi}^*_{\mathbf{Q}}(x) &\preceq \boldsymbol{\psi}^*_{\mathbf{Q}\oplus\mathscr{K}^o}(x)\\ &\preceq \boldsymbol{\psi}^*_{\mathbf{Q}}(x) \oplus \psi^*_{\mathscr{K}^o}(x) \text{ by (ii) of Lemma 1.9 and Definition 2.1}\\ &\preceq \boldsymbol{\psi}^*_{\mathbf{Q}}(x) \text{ because } \psi^*_{\mathscr{K}^o}(x) = 0.\end{aligned}$$

Therefore,

$$\boldsymbol{\psi}^*_{\mathbf{Q}}(x) = \boldsymbol{\psi}^*_{\mathbf{Q}\oplus\mathscr{K}^o}(x) \text{ for all } x \in \mathscr{K}. \tag{2.1}$$

Also, by hypothesis, we have $\boldsymbol{\psi}^*_{\mathbf{P}}(x) \preceq \boldsymbol{\psi}^*_{\mathbf{Q}}(x)$ for all $x \in \mathscr{K}$, and hence

$$\psi^*_{\mathbf{P}}(x) \preceq \boldsymbol{\psi}^*_{\mathbf{Q}\oplus\mathscr{K}^o}(x) \text{ for all } x \in \mathscr{K}. \tag{2.2}$$

Suppose now if $x \notin \mathscr{K}$, then there exists $z \in \mathscr{K}^o$ such that $\langle z, x\rangle > 0$. Thus, for any $\widehat{\mathbf{A}} \in \mathbf{Q}$ and $\lambda \geq 0$, $\widehat{\mathbf{A}} \oplus \lambda z \in \mathbf{Q} \oplus \mathscr{K}^o$. Also, $x^\top \odot (\widehat{\mathbf{A}} \oplus \lambda z)$

$$\begin{aligned}&= \left[\min\left\{\sum_{i=1}^n x_i(\underline{a}_i + \lambda z_i), \sum_{i=1}^n x_i(\overline{a}_i + \lambda z_i)\right\}, \max\left\{\sum_{i=1}^n x_i(\underline{a}_i + \lambda z_i), \sum_{i=1}^n x_i(\overline{a}_i + \lambda z_i)\right\}\right]\\ &= \left[\min\left\{\sum_{i=1}^n x_i\underline{a}_i + \lambda x^\top z, \sum_{i=1}^n x_i\overline{a}_i + \lambda x^\top z\right\}, \max\left\{\sum_{i=1}^n x_i\underline{a}_i + \lambda x^\top z, \sum_{i=1}^n x_i\overline{a}_i + \lambda x^\top z\right\}\right].\end{aligned}$$

Note that as $\lambda \to +\infty$, $\lambda x^\top z \to +\infty$, and therefore $x^\top \odot (\widehat{\mathbf{A}} \oplus \lambda z) \to +\infty$, which implies

$$\boldsymbol{\psi}^*_{\mathbf{Q}\oplus\mathscr{K}^o}(x) = [+\infty, +\infty].$$

Thus,

$$\boldsymbol{\psi}^*_{\mathbf{P}}(x) \preceq \boldsymbol{\psi}^*_{\mathbf{Q}\oplus\mathscr{K}^o}(x) \text{ for all } x \in X\backslash\mathscr{K}. \tag{2.3}$$

Therefore, from (2.2) and (2.3), we have

$$\boldsymbol{\psi}^*_{\mathbf{P}}(x) \preceq \boldsymbol{\psi}^*_{\mathbf{Q}\oplus\mathscr{K}^o}(x) \text{ for all } x \in X.$$

Proof of the converse part follows from (2.1). This completes the proof. □

Lemma 2.2 *Let P be nonempty a subset of $\mathbb{R}^n$ and $\boldsymbol{Q}$ be a nonempty closed convex subset of $I(\mathbb{R})^n$. Then, for any $x \in \mathbb{R}^n$,*

$$\psi^*_P(x) \preceq \boldsymbol{\psi}^*_{\boldsymbol{Q}}(x) \textit{ if and only if } P \subseteq \boldsymbol{Q}.$$

Proof: Let $\psi^*_P(x) \preceq \boldsymbol{\psi}^*_{\mathbf{Q}}(x)$ for $x \in \mathbb{R}^n$. Therefore, for any $p \in P$ and $x \in \mathbb{R}^n$, we have

$$\begin{aligned} &\langle x, p\rangle \preceq \sup_{\widehat{\mathbf{Q}}_i \in \mathbf{Q}} x^\top \odot \widehat{\mathbf{Q}}_i, \text{ where } \widehat{\mathbf{Q}}_i = \left(\left[\underline{q}_{i1}, \overline{q}_{i1}\right], \left[\underline{q}_{i2}, \overline{q}_{i2}\right], \ldots, \left[\underline{q}_{in}, \overline{q}_{in}\right]\right) \\ \implies \quad &\langle x, p\rangle \preceq \sup_{\widehat{\mathbf{Q}}_i \in \mathbf{Q}} \left[\min\left\{\sum_{j=1}^n x_j \underline{q}_{ij}, \sum_{j=1}^n x_j \overline{q}_{ij}\right\}, \max\left\{\sum_{j=1}^n x_j \underline{q}_{ij}, \sum_{j=1}^n x_j \overline{q}_{ij}\right\}\right]. \end{aligned}$$

We now consider the following two possible cases.

- Case 1. Let $\sum_{j=1}^n x_j \underline{q}_{ij} \leq \sum_{j=1}^n x_j \overline{q}_{ij}$. In this case, we have

$$\langle x, p\rangle \preceq \sup_{\widehat{\mathbf{Q}}_i \in \mathbf{Q}} \left[\sum_{j=1}^n x_j \underline{q}_{ij}, \sum_{j=1}^n x_j \overline{q}_{ij}\right]. \tag{2.4}$$

Next, define two sets S_1 and S_2 such that $S_1 = \left\{\underline{Q}_1, \underline{Q}_2, \ldots, \underline{Q}_n, \ldots\right\}$ and $S_2 = \left\{\overline{Q}_1, \overline{Q}_2, \ldots, \overline{Q}_n, \ldots\right\}$, where $\underline{Q}_i = \left(\underline{q}_{i1}, \underline{q}_{i2}, \ldots, \underline{q}_{in}\right) \in \mathbb{R}^n$ and $\overline{Q}_i = (\overline{q}_{i1}, \overline{q}_{i2}, \ldots, \overline{q}_{in}) \in \mathbb{R}^n$. Therefore, (2.4) along with Remark 1.4 gives,

$$\langle x, p\rangle \leq \sup_{\underline{Q}_i \in S_1} \left\langle x, \underline{Q}_i\right\rangle \tag{2.5}$$

$$\text{and } \langle x, p\rangle \leq \sup_{\overline{Q}_i \in S_2} \left\langle x, \overline{Q}_i \right\rangle. \tag{2.6}$$

Thus, from (2.5) and Lemma 1.11, we have $p \in S_1$, i.e., $p = \underline{Q}_m$ for some m. To show that $p \in \mathbf{Q}$, we have to show that $p = \overline{Q}_m$ as well.

Note that

$$\langle x, p\rangle = \left\langle x, \underline{Q}_m \right\rangle \leq \left\langle x, \overline{Q}_m \right\rangle \text{ for all } x \in \mathbb{R}^n \text{ because } \sum_{j=1}^{n} x_j \underline{q}_{ij} \leq \sum_{j=1}^{n} x_j \overline{q}_{ij}$$
$$\implies \langle x, p\rangle \leq \sup \left\langle x, \overline{Q}_m \right\rangle \text{ for all } x \in \mathbb{R}^n$$
$$\implies \langle x, p\rangle \leq \psi^*_{S'}(x) \text{ for all } x \in \mathbb{R}^n, \text{ where } S' = \left\{\overline{Q}_m\right\}. \tag{2.7}$$

Thus, from equation (2.7) and Lemma 1.11, we have $p \in S'$, i.e., $p = \overline{Q}_m$. Hence, $p \in \mathbf{Q}$. Since p is arbitrary, $P \subseteq \mathbf{Q}$.

- Case 2. Let $\sum_{j=1}^{n} x_j \overline{q}_{ij} \leq \sum_{j=1}^{n} x_j \underline{q}_{ij}$. By following similar steps as in Case 1, in this case also, we get $P \subseteq \mathbf{Q}$.

Proof of the converse part follows from Lemma 2.1. □

Lemma 2.3 *For $x \in \mathbb{R}^n$ and $\widehat{\boldsymbol{A}} = (\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_n) \in \boldsymbol{S} \subseteq \mathbb{R}^n$, we have*

$$x^\top \odot \widehat{\boldsymbol{A}} \preceq \|x\| \odot \left[\|\widehat{\boldsymbol{A}}\|_{I(\mathbb{R})^n}, \|\widehat{\boldsymbol{A}}\|_{I(\mathbb{R})^n}\right].$$

Proof: Note that $x^\top \odot \widehat{\mathbf{A}}$

$$= \left[\min\left\{\sum_{i=1}^{n} x_i \underline{a}_i, \sum_{i=1}^{n} x_i \overline{a}_i\right\}, \max\left\{\sum_{i=1}^{n} x_i \underline{a}_i, \sum_{i=1}^{n} x_i \overline{a}_i\right\}\right]$$
$$\preceq \left[\min\left\{\sum_{i=1}^{n} |x_i| \|\mathbf{A}_i\|_{I(\mathbb{R})}, \sum_{i=1}^{n} |x_i| \|\mathbf{A}_i\|_{I(\mathbb{R})}\right\}, \max\left\{\sum_{i=1}^{n} |x_i| \|\mathbf{A}_i\|_{I(\mathbb{R})}, \sum_{i=1}^{n} |x_i| \|\mathbf{A}_i\|_{I(\mathbb{R})}\right\}\right]$$

$$= \left[\min\left\{\|x\|\|\widehat{\mathbf{A}}\|_{I(\mathbb{R})^n}, \|x\|\|\widehat{\mathbf{A}}\|_{I(\mathbb{R})^n}\right\}, \max\left\{\|x\|\|\widehat{\mathbf{A}}\|_{I(\mathbb{R})^n}, \|x\|\|\widehat{\mathbf{A}}\|_{I(\mathbb{R})^n}\right\}\right]$$
$$\preceq \|x\| \odot \left[\|\widehat{\mathbf{A}}\|_{I(\mathbb{R})^n}, \|\widehat{\mathbf{A}}\|_{I(\mathbb{R})^n}\right].$$

□

Lemma 2.4 *The support function of a nonempty set $\boldsymbol{S} \subseteq I(\mathbb{R})^n$ is finite everywhere if and only if $\boldsymbol{S}$ is bounded.*

Proof: Suppose that $\mathbf{S}$ is bounded, i.e., we have $M > 0$ such that $\|\widehat{\mathbf{A}}\|_{I(\mathbb{R})^n} \leq M$ for all $\widehat{\mathbf{A}} = (\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n) \in \mathbf{S}$ with $\mathbf{A}_i = [\underline{a}_i, \overline{a}_i]$ for each $i = 1, 2, \ldots, n$. By Lemma 2.3 and $\|\widehat{\mathbf{A}}\|_{I(\mathbb{R})^n} \leq M$, for any $x \in \mathbb{R}^n$, we have

$$x^\top \odot \widehat{\mathbf{A}} \preceq \|x\| \odot \left[\|\widehat{\mathbf{A}}\|_{I(\mathbb{R})^n}, \|\widehat{\mathbf{A}}\|_{I(\mathbb{R})^n}\right] \preceq \|x\| \odot [M, M] \preceq \|x\|M.$$

Since $\widehat{\mathbf{A}} \in \mathbf{S}$ is arbitrary chosen, therefore

$$\boldsymbol{\psi}^*_{\mathbf{S}}(x) \;=\; \sup_{\widehat{\mathbf{A}} \in \mathbf{S}} x^\top \odot \widehat{\mathbf{A}} \preceq \|x\|M.$$

Hence, $\boldsymbol{\psi}^*_{\mathbf{S}}(x)$ is finite everywhere.

Conversely, let $\boldsymbol{\psi}^*_{\mathbf{S}}(x)$ is finite for every $x \in \mathbb{R}^n$. Therefore, there exists an $M > 0$ such that $\boldsymbol{\psi}^*_{\mathbf{S}}(x) \preceq M$, which implies that for any $x \in \mathbb{R}^n$ and $\widehat{\mathbf{A}} \in \mathbf{S}$, we have

$$x^\top \odot \widehat{\mathbf{A}} = \left[\min\left\{\sum_{i=1}^n x_i \underline{a}_i, \sum_{i=1}^n x_i \overline{a}_i\right\}, \max\left\{\sum_{i=1}^n x_i \underline{a}_i, \sum_{i=1}^n x_i \overline{a}_i\right\}\right] \preceq M$$
$$\implies \sum_{i=1}^n x_i \underline{a}_i \leq M \text{ and } \sum_{i=1}^n x_i \overline{a}_i \leq M.$$

Take $\sum_{i=1}^n x_i \underline{a}_i \leq M$, then by Remark 1.1, we have

$$\langle x, \underline{a} \rangle \leq M, \text{ where } \underline{a} = (\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n) \in \mathbb{R}^n. \tag{2.8}$$

If $\underline{a} \neq 0$, choose $x = \frac{\underline{a}}{\|\underline{a}\|}$, then (2.8) gives

$$\begin{aligned} & \left\langle \frac{\underline{a}}{\|\underline{a}\|}, \underline{a} \right\rangle \leq M \\ \implies \quad & \|\underline{a}\| \leq M, \text{ where } \underline{a} = (\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n) \in \mathbb{R}^n \\ \implies \quad & |\underline{a}_i| \leq M \text{ for each } i = 1, 2, \ldots, n. \end{aligned}$$

Similarly, when we take $\sum_{i=1}^{n} x_i \overline{a}_i \leq M$, we get $|\overline{a}_i| \leq M$ for each $i = 1, 2, \ldots, n$. Therefore, we have

$$\mathbf{A}_i = [\underline{a}_i, \overline{a}_i] \preceq M \text{ for each } i = 1, 2, \ldots, n \implies \widehat{\mathbf{A}} \preceq M.$$

Since $\widehat{\mathbf{A}} \in \mathbf{S}$ was arbitrary chosen, therefore we have $\widehat{\mathbf{A}} \preceq M$ for all $\widehat{\mathbf{A}} \in \mathbf{S}$. Hence, $\mathbf{S}$ is bounded. □

2.5 gH-subdifferentiability of IVFs

In this section, we develop gH-subdifferential calculus for convex IVFs that are used later to find the dual characterization of WSM for convex IVFs.

Definition 2.2 (gH-subdifferentiability). *Let $\boldsymbol{F} : X \subseteq \mathbb{R}^n \to \overline{I(\mathbb{R})}$ be a proper convex IVF and $\bar{x} \in \text{dom}(\boldsymbol{F})$. Then, gH-subdifferential of $\boldsymbol{F}$ at $\bar{x}$, denoted by $\boldsymbol{\partial F}(\bar{x})$ is defined by*

$$\boldsymbol{\partial F}(\bar{x}) = \left\{ \widehat{\boldsymbol{G}} \in I(\mathbb{R})^n : (x - \bar{x})^\top \odot \widehat{\boldsymbol{G}} \preceq \boldsymbol{F}(x) \ominus_{gH} \boldsymbol{F}(\bar{x}) \text{ for all } x \in X \right\}. \tag{2.9}$$

The elements of (2.9) are known as gH-subgradients of $\boldsymbol{F}$ at $\bar{x}$. Further, if $\boldsymbol{\partial F}(\bar{x}) \neq \emptyset$, we say that $\boldsymbol{F}$ is gH-subdifferentiable at $\bar{x}$.

Example 2.1 *Consider* $\boldsymbol{F} : \mathbb{R} \to I(\mathbb{R})$ *be a convex IVF such that* $\boldsymbol{F}(x) = |x| \odot \boldsymbol{A}$, *where* $\boldsymbol{0} \preceq \boldsymbol{A}$. *Let us find gH-subdifferential set of* $\boldsymbol{F}$ *at* 0.

$$\begin{aligned} \boldsymbol{\partial F}(0) &= \left\{\boldsymbol{G} \in I(\mathbb{R}) : (x-0) \odot \boldsymbol{G} \preceq \boldsymbol{F}(x) \ominus_{gH} \boldsymbol{F}(0) \text{ for all } x \in \mathbb{R}\right\} \\ &= \left\{\boldsymbol{G} \in I(\mathbb{R}) : x \odot \boldsymbol{G} \preceq |x| \odot \boldsymbol{A} \text{ for all } x \in \mathbb{R}\right\} \end{aligned} \tag{2.10}$$

- *Case 1.* $x \leq 0$. *In this case, for all* $x \in \mathbb{R}$, (2.10) *gives,*

$$x \odot \boldsymbol{G} \preceq (-x) \odot \boldsymbol{A} \implies (-1) \odot \boldsymbol{A} \preceq \boldsymbol{G}.$$

- *Case 2.* $x > 0$. *In this case, for all* $x \in \mathbb{R}$, (2.10) *gives,*

$$x \odot \boldsymbol{G} \preceq x \odot \boldsymbol{A} \implies \boldsymbol{G} \preceq \boldsymbol{A}.$$

Hence, from Case 1 and Case 2, we have $\boldsymbol{\partial F}(0) = \{\boldsymbol{G} \in I(\mathbb{R}) : (-1) \odot \boldsymbol{A} \preceq \boldsymbol{G} \preceq \boldsymbol{A}\}$.

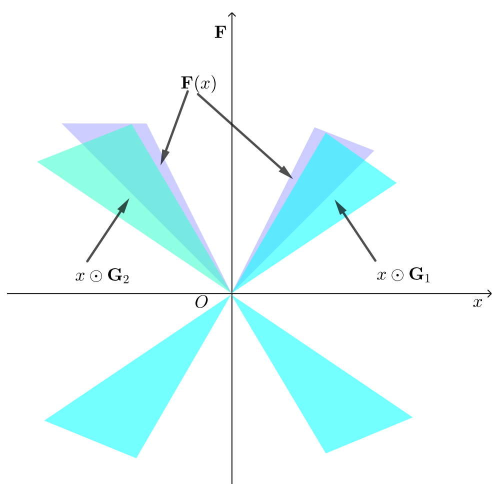

Figure 2.1: Graphs of $\mathbf{F}(x) = |x| \odot \mathbf{A}$, $x \odot \mathbf{G}_1$ and $x \odot \mathbf{G}_2$, where $\mathbf{A} = [1, 2]$ and $\mathbf{G}_1$, $\mathbf{G}_2$ are two possible subgradients of $\mathbf{F}$ at 0

In Figure 2.1, the IVF $\mathbf{F}$, with $\mathbf{A} = \left[\frac{1}{4}, 1\right]$, is drawn by light purple region and $x \odot \mathbf{G}_1$ and $x \odot \mathbf{G}_2$, with two possible gH-subgradients $\mathbf{G}_1$ and $\mathbf{G}_2$ at 0, are shown by cyan color.

Lemma 2.5 *Let X be a nonempty convex subset of $\mathbb{R}^n$ and $\boldsymbol{F}: X \to \overline{I(\mathbb{R})}$ be a proper convex IVF. Then, for any $\bar{x} \in \text{dom}(\boldsymbol{F})$ and $h \in \mathbb{R}^n$ such that $\bar{x} + h \in X$, the gH-subdifferential set of $\boldsymbol{F}$ at $\bar{x}$ is*

$$\boldsymbol{\partial F}(\bar{x}) = \left\{ \widehat{\boldsymbol{G}} \in I(\mathbb{R})^n : h^\top \odot \widehat{\boldsymbol{G}} \preceq \boldsymbol{F}_{\mathscr{D}}(\bar{x})(h) \right\},$$

where $\boldsymbol{F}_{\mathscr{D}}(\bar{x})(h)$ is gH-directional derivative of $\boldsymbol{F}$ at $\bar{x}$ in the direction of h.

Proof: Suppose $\widehat{\mathbf{G}} \in \boldsymbol{\partial}\mathbf{F}(\bar{x})$. Then, by Definition 2.2, we have

$$(x - \bar{x})^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \text{ for all } x \in X. \tag{2.11}$$

By taking $x = \bar{x} + \lambda h$ with $\lambda > 0$ and $h \in \mathbb{R}^n$ in (2.11), we get

$$\begin{aligned} & h^\top \odot \widehat{\mathbf{G}} \preceq \frac{\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})}{\lambda} \\ \implies \quad & h^\top \odot \widehat{\mathbf{G}} \preceq \lim_{\lambda \to 0} \frac{\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})}{\lambda} \\ \implies \quad & h^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{F}_{\mathscr{D}}(\bar{x})(h). \end{aligned}$$

Next, if we take any $\widehat{\mathbf{G}} \in I(\mathbb{R})^n$ such that $h^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{F}_{\mathscr{D}}(\bar{x})(h)$ for all $h \in \mathbb{R}^n$. Then, by a similar reasoning as above it can be seen that $\widehat{\mathbf{G}} \in \boldsymbol{\partial}\mathbf{F}(\bar{x})$.

□

Theorem 2.2 *Let X be a nonempty convex subset of $\mathbb{R}^n$ and $\boldsymbol{F}: X \to \overline{I(\mathbb{R})}$ be a proper convex IVF with $\boldsymbol{F}(x) = [\underline{F}(x), \overline{F}(x)]$, where $\underline{F}$, $\overline{F} : X \to \overline{\mathbb{R}}$ are extended real-valued functions. Then, for any $\bar{x} \in \text{dom}(\boldsymbol{F})$, $\boldsymbol{\partial F}(\bar{x})$ is closed and convex.*

Proof: We first prove the closedness of $\boldsymbol{\partial}\mathbf{F}(\bar{x})$. Let $\left\{\widehat{\mathbf{G}}_k\right\}$ be a sequence in $\boldsymbol{\partial}\mathbf{F}(\bar{x})$, which converges to $\widehat{\mathbf{G}} \in I(\mathbb{R})^n$, where $\widehat{\mathbf{G}}_k = (\mathbf{G}_{k1}, \mathbf{G}_{k2}, \ldots, \mathbf{G}_{kn})$ and $\widehat{\mathbf{G}} = (\mathbf{G}_1, \mathbf{G}_2, \ldots,$

$\mathbf{G}_n$). Since $\widehat{\mathbf{G}}_k \in \boldsymbol{\partial}\mathbf{F}(\bar{x})$, for all $h \in \mathbb{R}^n$ such that $\bar{x} + h \in X$, we have

$$\begin{aligned}
&h^\top \odot \widehat{\mathbf{G}}_k \preceq \mathbf{F}(\bar{x}+h) \ominus_{gH} \mathbf{F}(\bar{x}),\\
\implies &\min\left\{\sum_{i=1}^n h_i \underline{g}_{ki}, \sum_{i=1}^n h_i \overline{g}_{ki}\right\} \leq \min\{\underline{F}(\bar{x}+h) - \underline{F}(\bar{x}), \overline{F}(\bar{x}+h) - \overline{F}(\bar{x})\}\\
\text{and } &\max\left\{\sum_{i=1}^n h_i \underline{g}_{ki}, \sum_{i=1}^n h_i \overline{g}_{ki}\right\} \leq \max\{\underline{F}(\bar{x}+h) - \underline{F}(\bar{x}), \overline{F}(\bar{x}+h) - \overline{F}(\bar{x})\}.
\end{aligned} \tag{2.12}$$

Since the sequence $\left\{\widehat{\mathbf{G}}_k\right\}$ converges to $\widehat{\mathbf{G}}$, in view of Remark 1.6, the sequences $\{\underline{g}_{ki}\}$ and $\{\overline{g}_{ki}\}$ converge to $\underline{g}_i$ and $\overline{g}_i$, respectively, for each $i = 1, 2, \ldots, n$. Thus,

$$\sum_{i=1}^n h_i \underline{g}_{ki} \to \sum_{i=1}^n h_i \underline{g}_i \text{ and } \sum_{i=1}^n h_i \overline{g}_{ki} \to \sum_{i=1}^n h_i \overline{g}_i \text{ as } k \to \infty. \tag{2.13}$$

Therefore, in view of (2.12) and (2.13), we have

$$\begin{aligned}
\left(\min\left\{\sum_{i=1}^n h_i \underline{g}_{ki}, \sum_{i=1}^n h_i \overline{g}_{ki}\right\}\right) &\to \left(\min\left\{\sum_{i=1}^n h_i \underline{g}_i, \sum_{i=1}^n h_i \overline{g}_i\right\}\right)\\
&\leq \min\left\{\underline{F}(\bar{x}+h) - \underline{F}(\bar{x}), \overline{F}(\bar{x}+h) - \overline{F}(\bar{x})\right\}
\end{aligned}$$

and

$$\begin{aligned}
\left(\max\left\{\sum_{i=1}^n h_i \underline{g}_{ki}, \sum_{i=1}^n h_i \overline{g}_{ki}\right\}\right) &\to \left(\max\left\{\sum_{i=1}^n h_i \underline{g}_i, \sum_{i=1}^n h_i \overline{g}_i\right\}\right)\\
&\leq \max\left\{\underline{F}(\bar{x}+h) - \underline{F}(\bar{x}), \overline{F}(\bar{x}+h) - \overline{F}(\bar{x})\right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left[\min\left\{\sum_{i=1}^n h_i \underline{g}_i, \sum_{i=1}^n h_i \overline{g}_i\right\}, \max\left\{\sum_{i=1}^n h_i \underline{g}_i, \sum_{i=1}^n h_i \overline{g}_i\right\}\right] \preceq \mathbf{F}(\bar{x}+h) \ominus_{gH} \mathbf{F}(\bar{x})\\
\implies \quad &h^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{F}(\bar{x}+h) \ominus_{gH} \mathbf{F}(\bar{x}) \text{ for all } h \in X.
\end{aligned}$$

Therefore, $\widehat{\mathbf{G}} \in \boldsymbol{\partial}\mathbf{F}(\bar{x})$, and hence $\boldsymbol{\partial}\mathbf{F}(\bar{x})$ is closed.

To prove the convexity of $\boldsymbol{\partial}\mathbf{F}(\bar{x})$, let $\widehat{\mathbf{H}} = (\mathbf{H}_1, \mathbf{H}_2, \ldots, \mathbf{H}_n)$ and $\widehat{\mathbf{K}} = (\mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_n)$ be any two elements of $\boldsymbol{\partial}\mathbf{F}(\bar{x})$ with $\mathbf{H}_i = [\underline{h}_i, \overline{h}_i]$ and $\mathbf{K}_i = [\underline{k}_i, \overline{k}_i]$ for each $i = 1, 2, \ldots, n$. Then, for all $\lambda_1, \lambda_2 \geq 0$, with $\lambda_1 + \lambda_2 = 1$ and for any $d \in \mathbb{R}^n$, we have

$$\begin{aligned} d^\top \odot \left(\lambda_1 \odot \widehat{\mathbf{H}} \oplus \lambda_2 \odot \widehat{\mathbf{K}}\right) &= \Bigg[\min\left\{\sum_{i=1}^n d_i(\lambda_1 \underline{h}_i + \lambda_2 \underline{k}_i), \sum_{i=1}^n d_i(\lambda_1 \overline{h}_i + \lambda_2 \overline{k}_i)\right\}, \\ &\quad \max\left\{\sum_{i=1}^n d_i(\lambda_1 \underline{h}_i + \lambda_2 \underline{k}_i), \sum_{i=1}^n d_i(\lambda_1 \overline{h}_i + \lambda_2 \overline{k}_i)\right\}\Bigg]. \end{aligned}$$

- Case 1. Let $\min\left\{\sum_{i=1}^n d_i(\lambda_1 \underline{h}_i + \lambda_2 \underline{k}_i), \sum_{i=1}^n d_i(\lambda_1 \overline{h}_i + \lambda_2 \overline{k}_i)\right\} = \sum_{i=1}^n d_i(\lambda_1 \underline{h}_i + \lambda_2 \underline{k}_i)$. Then, $d^\top \odot \left(\lambda_1 \odot \widehat{\mathbf{H}} \oplus \lambda_2 \odot \widehat{\mathbf{K}}\right)$

$$\begin{aligned} &= \left[\sum_{i=1}^n d_i(\lambda_1 \underline{h}_i + \lambda_2 \underline{k}_i), \sum_{i=1}^n d_i(\lambda_1 \overline{h}_i + \lambda_2 \overline{k}_i)\right] \\ &= \left[\sum_{i=1}^n \lambda_1 d_i \underline{h}_i, \sum_{i=1}^n \lambda_1 d_i \overline{h}_i\right] \oplus \left[\sum_{i=1}^n \lambda_2 d_i \underline{k}_i, \sum_{i=1}^n \lambda_2 d_i \overline{k}_i\right] \\ &= \lambda_1 \odot d^\top \odot \widehat{\mathbf{H}} \oplus \lambda_2 \odot d^\top \odot \widehat{\mathbf{K}} \\ &\preceq \lambda_1 \odot \mathbf{F}_{\mathscr{D}}(\bar{x})(d) \oplus \lambda_2 \odot \mathbf{F}_{\mathscr{D}}(\bar{x})(d) \text{ by Lemma 2.5} \\ &= \mathbf{F}_{\mathscr{D}}(\bar{x})(d) \text{ for any } d \in \mathbb{R}^n. \end{aligned}$$

  Hence, $d^\top \odot (\lambda_1 \odot \widehat{\mathbf{H}} \oplus \lambda_2 \odot \widehat{\mathbf{K}}) \preceq \mathbf{F}_{\mathscr{D}}(\bar{x})(d)$ for any $d \in \mathbb{R}^n$. Therefore, by Lemma 2.5, $\lambda_1 \odot \widehat{\mathbf{H}} \oplus \lambda_2 \odot \widehat{\mathbf{K}} \in \boldsymbol{\partial}\mathbf{F}(\bar{x})$.

- Case 2. Let $\min\left\{\sum_{i=1}^n d_i(\lambda_1 \underline{h}_i + \lambda_2 \underline{k}_i), \sum_{i=1}^n d_i(\lambda_1 \overline{h}_i + \lambda_2 \overline{k}_i)\right\} = \sum_{i=1}^n d_i(\lambda_1 \overline{h}_i + \lambda_2 \overline{k}_i)$. Then, $d^\top \odot (\lambda_1 \odot \widehat{\mathbf{H}} \oplus \lambda_2 \odot \widehat{\mathbf{K}})$

$$= \left[\sum_{i=1}^n d_i(\lambda_1 \overline{h}_i + \lambda_2 \overline{k}_i), \sum_{i=1}^n d_i(\lambda_1 \underline{h}_i + \lambda_2 \underline{k}_i)\right]$$

$$
\begin{aligned}
&= \left[\sum_{i=1}^{n} \lambda_1 d_i \overline{h}_i, \sum_{i=1}^{n} \lambda_1 d_i \underline{h}_i\right] \oplus \left[\sum_{i=1}^{n} \lambda_2 d_i \overline{k}_i, \sum_{i=1}^{n} \lambda_2 d_i \underline{k}_i\right] \\
&= \lambda_1 \odot d^\top \odot \widehat{\mathbf{H}} \oplus \lambda_2 \odot d^\top \odot \widehat{\mathbf{K}} \\
&\preceq \lambda_1 \odot \mathbf{F}_{\mathscr{D}}(\bar{x})(d) \oplus \lambda_2 \odot \mathbf{F}_{\mathscr{D}}(\bar{x})(d) \text{ by Lemma 2.5} \\
&= \mathbf{F}_{\mathscr{D}}(\bar{x})(d) \text{ for any } d \in X.
\end{aligned}
$$

Hence, $d^\top \odot \left(\lambda_1 \odot \widehat{\mathbf{H}} \oplus \lambda_2 \odot \widehat{\mathbf{K}}\right) \preceq \mathbf{F}_{\mathscr{D}}(\bar{x})(d)$ for any $d \in X$. Therefore, by Lemma 2.5, $\lambda_1 \odot \widehat{\mathbf{H}} \oplus \lambda_2 \odot \widehat{\mathbf{K}} \in \boldsymbol{\partial}\mathbf{F}(\bar{x})$.

Thus, for any $\bar{x} \in \operatorname{dom}(\mathbf{F}), \boldsymbol{\partial}\mathbf{F}(\bar{x})$ is convex. □

Theorem 2.3 *Let X be a nonempty convex subset of $\mathbb{R}^n$ and let $\boldsymbol{F} : X \to I(\mathbb{R})$ be a gH-differentiable convex IVF at $\bar{x} \in X$. Then,*

$$\boldsymbol{\partial}\boldsymbol{F}(\bar{x}) = \left\{\nabla \boldsymbol{F}(\bar{x})\right\}.$$

Proof: Let $\widehat{\mathbf{G}} \in \boldsymbol{\partial}\mathbf{F}(\bar{x})$. Since $\mathbf{F}$ is gH-differentiable at $\bar{x}$, with the help of Lemma 1.4 and Lemma 2.5, we get

$$
\begin{aligned}
& h^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{L}_{\bar{x}}(h) \text{ for all } h \in \mathbb{R}^n \\
\implies & h^\top \odot \widehat{\mathbf{G}} \preceq \sum_{i=1}^{n} h_i \odot D_i\mathbf{F}(\bar{x}) \text{ by Theorem 1.2.}
\end{aligned}
\tag{2.14}
$$

Replacing h by $-h$ in (2.14), we obtain

$$
\begin{aligned}
& (-h)^\top \odot \widehat{\mathbf{G}} \preceq \sum_{i=1}^{n} (-h_i) \odot D_i\mathbf{F}(\bar{x}) \\
\implies & \sum_{i=1}^{n} h_i \odot D_i\mathbf{F}(\bar{x}) \preceq h^\top \odot \widehat{\mathbf{G}} \text{ for all } h \in \mathbb{R}^n.
\end{aligned}
\tag{2.15}
$$

Thus, (2.14) and (2.15), simultaneously give

$$\sum_{i=1}^{n} h_i \odot D_i\mathbf{F}(\bar{x}) = h^\top \odot \widehat{\mathbf{G}} \text{ for all } h \in \mathbb{R}^n. \tag{2.16}$$

Therefore, for each $i \in \{1, 2, \ldots, n\}$, by choosing $h = e_i$ in (2.16), we have $D_i\mathbf{F}(\bar{x}) = \mathbf{G}_i$. Hence,$\nabla\mathbf{F}(\bar{x}) = \widehat{\mathbf{G}}$. Since $\widehat{\mathbf{G}} \in \boldsymbol{\partial}\mathbf{F}(\bar{x})$ is arbitrary, $\boldsymbol{\partial}\mathbf{F}(\bar{x}) = \big\{\nabla\mathbf{F}(\bar{x})\big\}$. □

Lemma 2.6 *Let X be a nonempty convex subset of $\mathbb{R}^n$ and $\boldsymbol{F} : X \to \overline{I(\mathbb{R})}$ be a proper convex IVF with $\boldsymbol{F}(x) = [\underline{F}(x), \overline{F}(x)]$, where $\underline{F}$, $\overline{F} : X \to \overline{\mathbb{R}}$ are extended real-valued functions. Then, the subdifferential set of $\boldsymbol{F}$ at $\bar{x} \in \operatorname{int}(\operatorname{dom}(\boldsymbol{F}))$ can be obtained by the subdifferential sets of $\underline{F}$ and $\overline{F}$ at $\bar{x}$ and vice-versa.*

Proof: Since $\mathbf{F}$ is proper convex, with the help of Lemma 1.3, we note that $\underline{F}$ and $\overline{F}$ are also convex. Therefore, by the property of real-valued proper convex functions, the subdifferential sets of $\underline{F}$ and $\overline{F}$ at $\bar{x} \in \operatorname{int}(\operatorname{dom}(\mathbf{F}))$ are nonempty (see [19]). Let $\underline{g} = (\underline{g}_1, \underline{g}_2, \ldots, \underline{g}_n) \in \partial\underline{F}(\bar{x})$ and $\overline{g} = (\overline{g}_1, \overline{g}_2, \ldots, \overline{g}_n) \in \partial\overline{F}(\bar{x})$. Then, by Definition 2.2 of gH-subdifferentiability, for any $h \in \mathbb{R}^n$ such that $\bar{x} + h \in X$, we have

$$h^\top \odot \underline{g} \leq \underline{F}(\bar{x} + h) - \underline{F}(\bar{x}) \text{ and } h^\top \odot \overline{g} \leq \overline{F}(\bar{x} + h) - \overline{F}(\bar{x}). \tag{2.17}$$

Note that $\mathbf{F}(\bar{x} + h) \ominus_{gH} \mathbf{F}(\bar{x})$

$$\begin{aligned} =& \big[\min\{\underline{F}(\bar{x} + h) - \underline{F}(\bar{x}), \overline{F}(\bar{x} + h) - \overline{F}(\bar{x})\}, \\ & \qquad \max\{\underline{F}(\bar{x} + h) - \underline{F}(\bar{x}), \overline{F}(\bar{x} + h) - \overline{F}(\bar{x})\}\big] \\ \Longrightarrow & \big[\min\{h^\top \odot \underline{g}, h^\top \odot \overline{g}\}, \max\{h^\top \odot \underline{g}, h^\top \odot \overline{g}\}\big] \preceq \mathbf{F}(\bar{x} + h) \ominus_{gH} \mathbf{F}(\bar{x}) \text{ by } (2.17) \\ \Longrightarrow & h^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{F}(\bar{x} + h) \ominus_{gH} \mathbf{F}(\bar{x}), \text{ where } \widehat{\mathbf{G}} = (\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_n) \text{ with } \mathbf{G}_i = [\underline{g}_i, \overline{g}_i] \\ \Longrightarrow & \widehat{\mathbf{G}} \in \boldsymbol{\partial}\mathbf{F}(\bar{x}). \end{aligned}$$

Thus, for any $\underline{g} \in \partial\underline{F}(\bar{x})$ and $\overline{g} \in \partial\overline{F}(\bar{x})$, we have the corresponding $\widehat{\mathbf{G}} \in \boldsymbol{\partial}\mathbf{F}(\bar{x})$.

To prove the converse part, for any $\bar{x} \in \text{int}(\text{dom}(\mathbf{F}))$, take $\widehat{\mathbf{G}} = (\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_n) \in \boldsymbol{\partial}\mathbf{F}(\bar{x})$ with $\mathbf{G}_i = [\underline{g}_i, \overline{g}_i]$, $i = 1, 2, \ldots, n$. Then, by Definition 2.2, we have

$$\begin{aligned} & h^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{F}(\bar{x} + h) \ominus_{gH} \mathbf{F}(\bar{x}) \text{ for all } h \in \mathbb{R}^n \text{ such that } \bar{x} + h \in X \\ \implies & \left[\min\left\{\sum_{i=1}^n h_i \underline{g}_i, \sum_{i=1}^n h_i \overline{g}_i\right\}, \max\left\{\sum_{i=1}^n h_i \underline{g}_i, \sum_{i=1}^n h_i \overline{g}_i\right\} \right] \preceq \mathbf{F}(\bar{x} + h) \ominus_{gH} \mathbf{F}(\bar{x}). \end{aligned}$$

Therefore,

$$\min\left\{\sum_{i=1}^n h_i \underline{g}_i, \sum_{i=1}^n h_i \overline{g}_i\right\} \leq \min\left\{\underline{F}(\bar{x}+h) - \underline{F}(\bar{x}), \overline{F}(\bar{x}+h) - \overline{F}(\bar{x})\right\} \tag{2.18}$$

and

$$\max\left\{\sum_{i=1}^n h_i \underline{g}_i, \sum_{i=1}^n h_i \overline{g}_i\right\} \leq \max\left\{\underline{F}(\bar{x}+h) - \underline{F}(\bar{x}), \overline{F}(\bar{x}+h) - \overline{F}(\bar{x})\right\}. \tag{2.19}$$

We now consider the following two possible cases.

- Case 1. Let $\min\left\{\sum_{i=1}^n h_i \underline{g}_i, \sum_{i=1}^n h_i \overline{g}_i\right\} = \sum_{i=1}^n h_i \underline{g}_i$ and $\min\left\{\underline{F}(\bar{x}+h) - \underline{F}(\bar{x}), \overline{F}(\bar{x}+h) - \overline{F}(\bar{x})\right\}$ $= \underline{F}(\bar{x}+h) - \underline{F}(\bar{x})$. In this case, by (2.18) and (2.19), we have

$$\begin{aligned} & \sum_{i=1}^n h_i \underline{g}_i \leq \underline{F}(\bar{x}+h) - \underline{F}(\bar{x}) \text{ and } \sum_{i=1}^n h_i \overline{g}_i \leq \overline{F}(\bar{x}+h) - \overline{F}(\bar{x}) \\ \implies & h^\top \odot \underline{g} \leq \underline{F}(\bar{x}+h) - \underline{F}(\bar{x}) \text{ and } h^\top \odot \overline{g} \leq \overline{F}(\bar{x}+h) - \overline{F}(\bar{x}), \\ & \text{where } \underline{g} = (\underline{g}_1, \underline{g}_2, \ldots, \underline{g}_n) \in \mathbb{R}^n \text{ and } \overline{g} = (\overline{g}_1, \overline{g}_2, \ldots, \overline{g}_n) \in \mathbb{R}^n. \end{aligned}$$

  Thus, we get $\underline{g} \in \partial \underline{F}(\bar{x})$ and $\overline{g} \in \partial \overline{F}(\bar{x})$, which are required.

- Case 2. Let $\min\left\{\sum_{i=1}^n h_i \underline{g}_i, \sum_{i=1}^n h_i \overline{g}_i\right\} = \sum_{i=1}^n h_i \overline{g}_i$ and $\min\left\{\underline{F}(\bar{x}+h) - \underline{F}(\bar{x}), \overline{F}(\bar{x}+h) - \right.$

$\overline{F}(\bar{x})\Big\} = \underline{F}(\bar{x}+h) - \underline{F}(\bar{x})$. In this case, by (2.18) and (2.19), we get

$$\sum_{i=1}^{n} h_i \overline{g}_i \leq \underline{F}(\bar{x}+h) - \underline{F}(\bar{x}) \text{ and } \sum_{i=1}^{n} h_i \underline{g}_i \leq \overline{F}(\bar{x}+h) - \overline{F}(\bar{x})$$

$$\implies h^\top \odot \overline{g} \leq \underline{F}(\bar{x}+h) - \underline{F}(\bar{x}) \text{ and } h^\top \odot \underline{g} \leq \overline{F}(\bar{x}+h) - \overline{F}(\bar{x}),$$

$$\text{where } \overline{g} = (\overline{g}_1, \overline{g}_2, \ldots, \overline{g}_n) \in \mathbb{R}^n \text{ and } \underline{g} = (\underline{g}_1, \underline{g}_2, \ldots, \underline{g}_n) \in \mathbb{R}^n.$$

Thus, in this case, we get $\overline{g} \in \partial \underline{F}(\bar{x})$ and $\underline{g} \in \partial \overline{F}(\bar{x})$.

From Case 1 and Case 2, it is clear that for any $\widehat{\mathbf{G}} \in \boldsymbol{\partial}\mathbf{F}(\bar{x})$, we can obtain the subgradients of $\overline{F}$ and $\underline{F}$ at $\bar{x}$. This completes the proof for the converse part. □

Remark 2.1 *By Lemma 2.6, it is easy to note that for any proper convex IVF* $\boldsymbol{F}(x) = [\underline{F}(x), \overline{F}(x)]$ *and* $\bar{x} \in \text{int}(\text{dom}(\boldsymbol{F}))$, $\boldsymbol{\partial F}(\bar{x})$ *is nonempty.*

Theorem 2.4 *Let* X *be a nonempty convex subset of* $\mathbb{R}^n$ *and* $\boldsymbol{F} : X \to \overline{I(\mathbb{R})}$ *be a proper convex IVF with* $\boldsymbol{F}(x) = [\underline{F}(x), \overline{F}(x)]$*, where* $\underline{F}$*,* $\overline{F} : X \to \overline{\mathbb{R}}$ *are extended real-valued functions. Then, at any* $\bar{x} \in \text{int}(\text{dom}(\boldsymbol{F}))$*,*

$$\boldsymbol{F}_{\mathscr{D}}(\bar{x})(h) = \boldsymbol{\psi}^*_{\boldsymbol{\partial F}(\bar{x})}(h) \text{ for all } h \in \mathbb{R}^n \text{ such that } \bar{x}+h \in X,$$

where $\boldsymbol{F}_{\mathscr{D}}(\bar{x})(h)$ *is gH-directional derivative of* $\boldsymbol{F}$ *at* $\bar{x}$ *in the direction of* h*.*

Proof: Note that $\underline{F}(x)$ and $\overline{F}(x)$ are proper convex, and therefore $\partial\underline{F}(\bar{x})$ and $\partial\overline{F}(\bar{x})$ are nonempty. Let $\underline{g} = (\underline{g}_1, \underline{g}_2, \ldots, \underline{g}_n) \in \partial\underline{F}(\bar{x})$ and $\overline{g} = (\overline{g}_1, \overline{g}_2, \ldots, \overline{g}_n) \in \partial\overline{F}(\bar{x})$ for $\bar{x} \in \text{int}(\text{dom}(\mathbf{F}))$. By the property of real-valued convex functions (see [54]), we have

$$\underline{F}_{\mathscr{D}}(\bar{x})(h) = \psi^*_{\partial\underline{F}(\bar{x})}(h) \text{ and } \overline{F}_{\mathscr{D}}(\bar{x})(h) = \psi^*_{\partial\overline{F}(\bar{x})}(h) \text{ for all } h \in \mathbb{R}^n \text{ such that } \bar{x}+h \in X.$$

Due to Theorem 1.3, we get

$$\mathbf{F}_{\mathscr{D}}(\bar{x})(h) = \Big[\min\Big\{\underline{F}_{\mathscr{D}}(\bar{x})(h), \overline{F}_{\mathscr{D}}(\bar{x})(h)\Big\}, \max\Big\{\underline{F}_{\mathscr{D}}(\bar{x})(h), \overline{F}_{\mathscr{D}}(\bar{x})(h)\Big\}\Big]$$

$$= \left[\min \left\{ \psi^*_{\partial \underline{F}(\bar{x})}(h), \psi^*_{\partial \overline{F}(\bar{x})}(h) \right\}, \max \left\{ \psi^*_{\partial \underline{F}(\bar{x})}(h), \psi^*_{\partial \overline{F}(\bar{x})}(h) \right\} \right]. \tag{2.20}$$

We now consider the following two possible cases.

- Case 1. Let $\psi^*_{\partial \underline{F}(\bar{x})}(h) \leq \psi^*_{\partial \overline{F}(\bar{x})}(h)$. In this case, by (2.20), we get

$$\begin{aligned} \mathbf{F}_{\mathscr{D}}(\bar{x})(h) &= \left[\psi^*_{\partial \underline{F}(\bar{x})}(h), \psi^*_{\partial \overline{F}(\bar{x})}(h) \right] = \left[\sup_{\underline{g} \in \partial \underline{F}(\bar{x})} h^\top \odot \underline{g}, \sup_{\overline{g} \in \partial \overline{F}(\bar{x})} h^\top \odot \overline{g} \right] \\ &= \sup_{\underline{g} \in \partial \underline{F},\ \overline{g} \in \partial \overline{F}(\bar{x})} \left[h^\top \odot \underline{g}, h^\top \odot \overline{g} \right] = \sup_{\underline{g} \in \partial \underline{F},\ \overline{g} \in \partial \overline{F}(\bar{x})} \left[\sum_{i=1}^n h_i \underline{g}_i, \sum_{i=1}^n h_i \overline{g}_i \right]. \end{aligned} \tag{2.21}$$

We have seen in Lemma 2.6 that corresponding to every $\underline{g} \in \partial \underline{F}(\bar{x})$ and $\overline{g} \in \partial \overline{F}(\bar{x})$, we get $\widehat{\mathbf{G}} \in \boldsymbol{\partial} \mathbf{F}(\bar{x})$ and vice-versa. Thus, for $\underline{g} = (\underline{g}_1, \underline{g}_2, \ldots, \underline{g}_n) \in \partial \underline{F}(\bar{x})$ and $\overline{g} = (\overline{g}_1, \overline{g}_2, \ldots, \overline{g}_n) \in \partial \overline{F}(\bar{x})$, by (2.21), we obtain

$$\begin{aligned} \mathbf{F}_{\mathscr{D}}(\bar{x})(h) &= \sup_{\widehat{\mathbf{G}} \in \boldsymbol{\partial} \mathbf{F}(\bar{x})} h^\top \odot \widehat{\mathbf{G}}, \text{ where } \widehat{\mathbf{G}} = (\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_n) \text{ with } \mathbf{G}_i = [\underline{g}_i, \overline{g}_i] \\ &= \boldsymbol{\psi}^*_{\boldsymbol{\partial} \mathbf{F}(\bar{x})}(h). \end{aligned}$$

- Case 2. Let $\psi^*_{\partial \overline{F}(\bar{x})}(h) \leq \psi^*_{\partial \underline{F}(\bar{x})}(h)$. In this case, by (2.20), we have

$$\begin{aligned} \mathbf{F}_{\mathscr{D}}(\bar{x})(h) &= \left[\psi^*_{\partial \overline{F}(\bar{x})}(h), \psi^*_{\partial \underline{F}(\bar{x})}(h) \right] \\ &= \left[\sup_{\overline{g} \in \partial \overline{F}(\bar{x})} h^\top \odot \overline{g}, \sup_{\underline{g} \in \partial \underline{F}(\bar{x})} h^\top \odot \underline{g} \right] \\ &= \sup_{\underline{g} \in \partial \underline{F},\ \overline{g} \in \partial \overline{F}(\bar{x})} \left[h^\top \odot \overline{g}, h^\top \odot \underline{g} \right] \\ &= \sup_{\underline{g} \in \partial \underline{F},\ \overline{g} \in \partial \overline{F}(\bar{x})} \left[\sum_{i=1}^n h_i \overline{g}_i, \sum_{i=1}^n h_i \underline{g}_i \right] \\ &= \sup_{\widehat{\mathbf{G}} \in \boldsymbol{\partial} \mathbf{F}(\bar{x})} h^\top \odot \widehat{\mathbf{G}}, \text{ where } \widehat{\mathbf{G}} = (\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_n) \text{ with } \mathbf{G}_i = [\underline{g}_i, \overline{g}_i] \end{aligned}$$

$$=\boldsymbol{\psi}^*_{\boldsymbol{\partial}\mathbf{F}(\bar{x})}(h).$$

Hence, from Case 1 and Case 2, we have

$$\mathbf{F}_{\mathscr{D}}(\bar{x})(h) = \boldsymbol{\psi}^*_{\boldsymbol{\partial}\mathbf{F}(\bar{x})}(h) \text{ for all } h \in \mathbb{R}^n \text{ such that } \bar{x} + h \in X.$$

□

Theorem 2.5 *Let $\boldsymbol{F} : X \to \overline{I(\mathbb{R})}$ be a proper convex IVF and $\bar{x} \in \text{int}(\text{dom}(\boldsymbol{F}))$. Then, the gH-subdifferential set of $\boldsymbol{F}$ at $\bar{x}$ is bounded.*

Proof: Note that by Theorem 1.3, for $\bar{x} \in \text{int}(\text{dom}(\mathbf{F}))$, the directional derivative of $\mathbf{F}$ at $\bar{x}$ exists everywhere. Thus, for all $h \in \mathbb{R}^n$ such that $\bar{x} + h \in X$, we have

$$\begin{aligned} & \mathbf{F}_{\mathscr{D}}(\bar{x})(h) \text{ is finite} \\ \implies \quad & \boldsymbol{\psi}^*_{\boldsymbol{\partial}\mathbf{F}(\bar{x})}(h) \text{ is finite by Theorem 2.4} \\ \implies \quad & \boldsymbol{\partial}\mathbf{F}(\bar{x}) \text{ is bounded by Lemma 2.4.} \end{aligned}$$

Hence, the gH-subdifferential set of $\mathbf{F}$ at $\bar{x} \in \text{int}(\text{dom}(\mathbf{F}))$ is bounded, i.e., for every $\widehat{\mathbf{G}} \in \boldsymbol{\partial}\mathbf{F}(\bar{x})$, there exists an $M > 0$ such that $\|\widehat{\mathbf{G}}\|_{I(\mathbb{R})^n} \leq M$. □

Theorem 2.6 *Let X be a nonempty convex subset of $\mathbb{R}^n$ and $\boldsymbol{F}$ be a convex IVF on X such that $\boldsymbol{F}$ has gH-subgradient at every $x \in X$. Then, $\boldsymbol{F}$ is gH-Lipschitz continuous on X.*

Proof: Since $\mathbf{F}$ has gH-subdradient at every $x \in X$, then there exists a $\widehat{\mathbf{G}} \in I(\mathbb{R})^n$ such that

$$\begin{aligned} & (y - x)^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{F}(y) \ominus_{gH} \mathbf{F}(x) \text{ for all } y \in X \\ \implies \quad & (-1) \odot \left((x - y)^\top \odot \widehat{\mathbf{G}}\right) \preceq \mathbf{F}(y) \ominus_{gH} \mathbf{F}(x) \end{aligned}$$

$$\implies \quad \mathbf{F}(x) \ominus_{gH} \mathbf{F}(y) \preceq (x-y)^\top \odot \widehat{\mathbf{G}}$$

$$\implies \quad \mathbf{F}(x) \ominus_{gH} \mathbf{F}(y) \preceq \|x-y\| \odot \left[\|\widehat{\mathbf{G}}\|_{I(\mathbb{R})^n}, \|\widehat{\mathbf{G}}\|_{I(\mathbb{R})^n}\right] \text{ by Lemma 2.3}$$

$$\implies \quad \|\mathbf{F}(x) \ominus_{gH} \mathbf{F}(y)\|_{I(\mathbb{R})} \leq \|\widehat{\mathbf{G}}\|_{I(\mathbb{R})^n}\|x-y\| \text{ by Lemma 1.5}$$

$$\implies \quad \|\mathbf{F}(x) \ominus_{gH} \mathbf{F}(y)\|_{I(\mathbb{R})} \leq M\|x-y\|, \text{ where } \|\widehat{\mathbf{G}}\|_{I(\mathbb{R})^n} \leq M \text{ by Theorem 2.5.}$$

Thus, $\mathbf{F}$ is gH-Lipschitz continuous on X. □

2.6 Weak sharp minima and its characterizations

In this section, we present the main results—primal and dual characterizations of WSM for a gH-lsc and convex IVF.

Definition 2.3 (WSM for an IVF). *Let $\boldsymbol{F} : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ be a gH-lsc and convex IVF. Let $\bar{S}$ and S be two nonempty closed convex sets such that $\bar{S} \subseteq S \subseteq \mathbb{R}^n$. Further, let $dom(\boldsymbol{F}) \cap S \neq \emptyset$. Then, the set $\bar{S}$ is said to be a set of WSM of $\boldsymbol{F}$ over the set S with modulus $\alpha > 0$ if*

$$\boldsymbol{F}(\overline{x}) \oplus \alpha \operatorname{dist}(x, \bar{S}) \preceq \boldsymbol{F}(x) \text{ for all } \overline{x} \in \bar{S} \text{ and } x \in S.$$

Remark 2.2 *Let $\boldsymbol{F} : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ be a gH-lsc and convex IVF with $\boldsymbol{F}(x) = [\underline{F}(x), \overline{F}(x)]$ for all $x \in \mathbb{R}^n$, where $\underline{F}, \overline{F} : \mathbb{R}^n \to \overline{\mathbb{R}}$ be two extended real-valued functions. Then, $\bar{S}$ is a set of WSM of $\boldsymbol{F}$ over S with modulus $\alpha > 0$ if and only if $\bar{S}$ is a set of WSM of $\underline{F}$ and $\overline{F}$ over S with modulus $\alpha > 0$. The reason is as follows. By Remark 1.5 and Lemma 1.3, it is easy to see that the functions $\underline{F}$ and $\overline{F}$ are lsc and convex. Let $\bar{S}$ be a set of WSM of $\boldsymbol{F}$ over S with modulus $\alpha > 0$. Then,*

$$\boldsymbol{F}(\bar{x}) \oplus \alpha \operatorname{dist}(x, \bar{S}) \preceq \boldsymbol{F}(x) \text{ for all } \bar{x} \in \bar{S} \text{ and } x \in S$$

$$\iff \quad [\underline{F}(\bar{x}) + \alpha \operatorname{dist}(x, \bar{S}), \overline{F}(\bar{x}) + \alpha \operatorname{dist}(x, \bar{S})] \preceq [\underline{F}(x), \overline{F}(x)]$$

$$\text{for all } \bar{x} \in \bar{S} \text{ and } x \in S$$

$$\iff \quad \underline{F}(\bar{x}) + \alpha \operatorname{dist}(x, \bar{S}) \leq \underline{F}(x) \ \textit{and} \ \overline{F}(\bar{x}) + \alpha \operatorname{dist}(x, \bar{S}) \leq \overline{F}(x)$$

$$\textit{for all } \bar{x} \in \bar{S} \textit{ and } x \in S$$

$$\iff \quad \bar{S} \textit{ is a set of WSM of both } \underline{F} \textit{ and } \overline{F} \textit{ over } S \textit{ with modulus } \alpha > 0.$$

Example 2.2 *Let* $\boldsymbol{F} : \mathbb{R}^2 \to \overline{I(\mathbb{R})}$ *be an IVF defined by*

$$\boldsymbol{F}(x) = [5 - x_1x_2 - x_1, 10 - x_1^2x_2 - x_2^2x_1].$$

Let $S = [-a, 0] \times [-a, 0] \subseteq \mathbb{R}^2$ *and* $\bar{S} = \{0\} \times [-a, 0]$*, where* $a > 0$*. Thus,* $\bar{S} \subseteq S \subseteq \mathbb{R}^n$*. Clearly, the functions* $\underline{F}$ *and* $\overline{F}$ *are* $5 - x_1x_2 - x_1$ *and* $10 - x_1^2x_2 - x_2^2x_1$*, respectively. Note that for any* $\alpha > 0$*,*

$$\underline{F}(\bar{x}) + \alpha \operatorname{dist}(x, \bar{S}) \leq \underline{F}(x) \ \textit{and} \ \overline{F}(\bar{x}) + \alpha \operatorname{dist}(x, \bar{S}) \leq \overline{F}(x) \ \textit{for all } \bar{x} \in \bar{S} \textit{ and } x \in S.$$

Thus, $\bar{S} = \{0\} \times [-a, 0]$ *is a set of WSM of both* $\underline{F}$ *and* $\overline{F}$ *over* S *with modulus* α*, for any* $\alpha > 0$*. Therefore, by Remark 2.2,* $\bar{S}$ *is a set of WSM of* $\boldsymbol{F}$ *over* S *with modulus* $\alpha > 0$*.*

Let us consider an IOP

$$\min_{x \in S} \mathbf{F}(x), \tag{2.22}$$

where $\mathbf{F}$ and S are same as in Definition 2.3. Note that constrained IOP (2.22) can be converted into unconstrained IOP (2.23)

$$\min_{x \in \mathbb{R}^n} \mathbf{F}_o(x), \tag{2.23}$$

where $\mathbf{F}_o(x) = \begin{cases} \mathbf{F}(x), & \text{if } x \in S, \\ +\infty, & \text{otherwise.} \end{cases}$

Thus, we can solve the constrained and unconstrained IOP both using the concepts of

WSM. Regarding this, we give two characterizations primal and dual below.

Theorem 2.7 (Primal characterization). *Let* $\boldsymbol{F}$, *and* S *be as in IOP* (2.22). *Further, define an IVF* $\boldsymbol{F}_o : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ *as in IOP* (2.23). *Then, the set* $\bar{S}$ *is a set of WSM of* $\boldsymbol{F}$ *over the set* S *with modulus* $\alpha > 0$ *if and only if*

$$\alpha\,\mathrm{dist}(d, T_{\bar{S}}(x)) \preceq \boldsymbol{F}_{o\mathscr{D}}(x)(d) \text{ for all } x \in \bar{S} \text{ and } d \in \mathbb{R}^n. \tag{2.24}$$

Proof: Suppose $\bar{S}$ is a set of WSM of $\mathbf{F}$ over S with modulus $\alpha > 0$. Then, by Definition 2.3, for any $x \in \bar{S},\ d \in \mathbb{R}^n$, and $t > 0$, we have

$$\begin{aligned}
&\mathbf{F}_o(x) \oplus \alpha\,\mathrm{dist}(x+td, \bar{S}) \preceq \mathbf{F}_o(x+td)\\
\Longrightarrow\quad & \alpha\,\mathrm{dist}(x+td, \bar{S}) \preceq \mathbf{F}_o(x+td) \ominus_{gH} \mathbf{F}_o(x)\\
\Longrightarrow\quad & \frac{\alpha}{t}\left(\mathrm{dist}(x+td, \bar{S}) - \mathrm{dist}(x, \bar{S})\right) \preceq \frac{1}{t} \odot \left(\mathbf{F}_o(x+td) \ominus_{gH} \mathbf{F}_o(x)\right)\\
\Longrightarrow\quad & \lim_{t\to 0}\frac{\alpha}{t}\left(\mathrm{dist}(x+td, \bar{S}) - \mathrm{dist}(x, \bar{S})\right) \preceq \lim_{t\to 0}\frac{1}{t} \odot \left(\mathbf{F}_o(x+td) \ominus_{gH} \mathbf{F}_o(x)\right)\\
\Longrightarrow\quad & \alpha \lim_{t\to 0}\frac{1}{t}\left(\mathrm{dist}(x+td, \bar{S}) - \mathrm{dist}(x, \bar{S})\right) \preceq \mathbf{F}_{o\mathscr{D}}(x)(d) \text{ by Definition 1.10}\\
\Longrightarrow\quad & \alpha\,\mathrm{dist}(d, T_{\bar{S}}(x)) \preceq \mathbf{F}_{o\mathscr{D}}(x)(d) \text{ by part (ii) of Lemma 1.12.}
\end{aligned}$$

Thus,

$$\alpha\,\mathrm{dist}(d, T_{\bar{S}}(x)) \preceq \mathbf{F}_{o\mathscr{D}}(x)(d) \text{ for all } x \in \bar{S} \text{ and } d \in \mathbb{R}^n.$$

For the converse part, let $y \in S$ and $x \in \bar{S}$. Therefore, from Lemma 1.10, we get

$$\begin{aligned}
&\mathbf{F}_{o\mathscr{D}}(x)(y-x) \preceq \mathbf{F}_o(y) \ominus_{gH} \mathbf{F}_o(x)\\
\Longrightarrow\quad & \mathbf{F}_o(x) \oplus \mathbf{F}_{o\mathscr{D}}(x)(y-x) \preceq \mathbf{F}_o(y)\\
\Longrightarrow\quad & \mathbf{F}_o(x) \oplus \alpha\,\mathrm{dist}(y-x, T_{\bar{S}}(x)) \preceq \mathbf{F}_o(y) \text{ by (2.24)}\\
\Longrightarrow\quad & \mathbf{F}_o(x) \oplus \alpha\,\mathrm{dist}(y, x + T_{\bar{S}}(x)) \preceq \mathbf{F}_o(y).
\end{aligned}$$

Since $x \in \bar{S}$ is arbitrary, we have

$$
\begin{aligned}
& \mathbf{F}_o(x) \oplus \alpha \sup_{x \in \bar{S}} \operatorname{dist}(y, x + T_{\bar{S}}(x)) \preceq \mathbf{F}_o(y) \\
\Longrightarrow \quad & \mathbf{F}_o(x) \oplus \alpha \operatorname{dist}(y, \bar{S}) \preceq \mathbf{F}_o(y) \text{ for all } x \in \bar{S} \text{ and } \alpha > 0 \text{ by (i) of Lemma 1.12.}
\end{aligned}
$$

Hence, $\bar{S}$ is the set of WSM of $\mathbf{F}$ over S with modulus $\alpha > 0$, and the proof is complete. □

Theorem 2.8 (Dual characterizations). *Let $\boldsymbol{F}$, and S be as in IOP (2.22). Further, define an IVF $\boldsymbol{F}_o : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ as in IOP (2.23). Let $\bar{S}$ be the set of WSM of $\boldsymbol{F}$. Then, for any $\alpha > 0$, the following statements are equivalent.*

(a) *The set $\bar{S}$ is a set of WSM of $\boldsymbol{F}$ over the set S with modulus α.*

(b) *The normal cone inclusion holds. That is,*

$$\alpha\mathbb{B} \cap N_{\bar{S}}(x) \subseteq \boldsymbol{\partial F}_o(x) \text{ for all } x \in \bar{S}.$$

(c) *For all $x \in \bar{S}$ and $d \in T_S(x)$,*

$$\alpha \operatorname{dist}(d, T_{\bar{S}}(x)) \preceq \boldsymbol{F}_{\mathscr{D}}(x)(d).$$

(d) *The following inclusion holds,*

$$\alpha\mathbb{B} \bigcap \left(\bigcup_{x \in \bar{S}} N_{\bar{S}}(x) \right) \subseteq \bigcup_{x \in \bar{S}} \boldsymbol{\partial F}_o(x).$$

(e) *For all $x \in \bar{S}$ and $d \in T_S(x) \cap N_{\bar{S}}(x)$,*

$$\alpha \|d\| \preceq \boldsymbol{F}_{\mathscr{D}}(x)(d).$$

(f) *For all $y \in S$,*

$$\alpha\,\mathrm{dist}(y, \bar{S}) \preceq \boldsymbol{F}_{\mathscr{D}}(p)(y - p),$$

where $p \in P(y \mid \bar{S})$.

Proof: (a) $\iff$ (b). Let $x \in \bar{S}$. By hypothesis, $\bar{S}$ is a set of WSM of $\mathbf{F}$ over S. Therefore, by Theorem 2.7, we get

$$\alpha\,\mathrm{dist}(d, T_{\bar{S}}(x)) \preceq \mathbf{F}_{o\mathscr{D}}(x)(d) \text{ for all } d \in \mathbb{R}^n,$$

which along with Theorem 2.4 imply

$$\alpha\,\mathrm{dist}(d, T_{\bar{S}}(x)) \preceq \boldsymbol{\psi}^*_{\boldsymbol{\partial}\mathbf{F}_o(x)}(d) \text{ for all } d \in \mathbb{R}^n. \tag{2.25}$$

Notice that for all $x \in \bar{S}$ and $d \in \mathbb{R}^n$, we have

$$\begin{aligned}
\alpha\,\mathrm{dist}(d, T_{\bar{S}}(x)) &= \alpha\psi^*_{\mathbb{B}\cap N_{\bar{S}}(x)}(d) \text{ by (ii) of Lemma 1.12} \\
&= \alpha \sup\langle z, d\rangle, \text{ where } z \in \mathbb{B} \cap N_{\bar{S}}(x) \\
&= \sup\langle \alpha z, d\rangle, \text{ where } z \in \mathbb{B} \cap N_{\bar{S}}(x) \text{ and } \alpha > 0 \\
&= \sup\langle z, d\rangle, \text{ where } z \in \alpha\mathbb{B} \cap N_{\bar{S}}(x) \\
&= \psi^*_{\alpha\mathbb{B}\cap N_{\bar{S}}(x)}(d) \text{ for all } d \in \mathbb{R}^n.
\end{aligned}$$

That is,

$$\alpha\,\mathrm{dist}(d, T_{\bar{S}}(x)) = \psi^*_{\alpha\mathbb{B}\cap N_{\bar{S}}(x)}(d) \text{ for all } d \in \mathbb{R}^n. \tag{2.26}$$

Thus, by (2.25) and (2.26), we get

$$\psi^*_{\alpha\mathbb{B}\cap N_{\bar{S}}(x)}(d) \preceq \boldsymbol{\psi}^*_{\boldsymbol{\partial}\mathbf{F}_o(x)}(d). \tag{2.27}$$

Next, with the help of Lemma 2.2, we get the desired result

$$\alpha\mathbb{B} \cap N_{\bar{S}}(x) \subseteq \boldsymbol{\partial}\mathbf{F}_o(x) \text{ for all } d \in \mathbb{R}^n. \tag{2.28}$$

Conversely, we have

$$\begin{aligned} & \alpha\mathbb{B} \cap N_{\bar{S}}(x) \subseteq \boldsymbol{\partial}\mathbf{F}_o(x) \text{ for all } x \in \bar{S} & (2.29)\\ \implies \quad & \psi^*_{\alpha\mathbb{B}\cap N_{\bar{S}}(x)}(d) \preceq \boldsymbol{\psi}^*_{\boldsymbol{\partial}\mathbf{F}_o(x)}(d) \text{ for all } d \in \mathbb{R}^n \text{ by Lemma 2.1} & \\ \implies \quad & \alpha \operatorname{dist}(d, T_{\bar{S}}(x)) \preceq \boldsymbol{\psi}^*_{\boldsymbol{\partial}\mathbf{F}_o(x)}(d) \text{ for all } d \in \mathbb{R}^n \text{ by (2.26)}. & (2.30) \end{aligned}$$

Also, by Theorem 2.4, we have

$$\boldsymbol{\psi}^*_{\boldsymbol{\partial}\mathbf{F}_o(x)}(d) = \mathbf{F}_{o\mathscr{D}}(x)(d) \text{ for all } d \in \mathbb{R}^n.$$

Thus, by (2.30), we get

$$\alpha \operatorname{dist}(d, T_{\bar{S}}(x)) \preceq \mathbf{F}_{o\mathscr{D}}(x)(d) \text{ for all } d \in \mathbb{R}^n.$$

Therefore, by Theorem 2.7, $\bar{S}$ is a set of WSM of $\mathbf{F}$ over S with modulus α.

(a) $\iff$ (c). Let the statement (a) holds. Let $x \in \bar{S}$. Therefore, by Theorem 2.7, we have

$$\alpha \operatorname{dist}(d, T_{\bar{S}}(x)) \preceq \mathbf{F}_{o\mathscr{D}}(x)(d) \text{ for all } d \in T_S(x).$$

Note that for $x \in \bar{S}$, $\mathbf{F}_o(x) = \mathbf{F}(x)$. Thus,

$$\mathbf{F}_{o\mathscr{D}}(x)(d) = \mathbf{F}_{\mathscr{D}}(x)(d) \text{ for } x \in \bar{S} \text{ and } d \in T_S(x). \tag{2.31}$$

By (2.31) and Theorem 2.7, we get

$$\alpha\, \mathrm{dist}(d, T_{\bar{S}}(x)) \preceq \mathbf{F}_{\mathscr{D}}(x)(d) \text{ for all } d \in T_S(x) \text{ and } x \in \bar{S}.$$

Conversely, we are given that

$$\begin{aligned} & \alpha\, \mathrm{dist}(d, T_{\bar{S}}(x)) \preceq \mathbf{F}_{\mathscr{D}}(x)(d) \text{ for all } x \in \bar{S} \text{ and } d \in T_S(x) \\ \implies \quad & \alpha\, \mathrm{dist}(d, T_{\bar{S}}(x)) \preceq \boldsymbol{\psi}^*_{\boldsymbol{\partial}\mathbf{F}(x)}(d) \text{ for all } d \in T_S(x) \text{ by Theorem 2.4.} \end{aligned} \tag{2.32}$$

Note that for $x \in \bar{S}$, we have

$$\boldsymbol{\psi}^*_{\boldsymbol{\partial}\mathbf{F}(x)}(d) = \boldsymbol{\psi}^*_{\boldsymbol{\partial}\mathbf{F}_o(x)}(d) \text{ for all } d \in \mathbb{R}^n. \tag{2.33}$$

In view of (2.32) and (2.33), we have

$$\begin{aligned} & \alpha\, \mathrm{dist}(d, T_{\bar{S}}(x)) \preceq \boldsymbol{\psi}^*_{\boldsymbol{\partial}\mathbf{F}_o(x)}(d) \text{ for all } d \in T_S(x) \\ \implies \quad & \alpha\, \mathrm{dist}(d, T_{\bar{S}}(x)) \preceq \mathbf{F}_{o\mathscr{D}}(x)(d) \text{ for all } d \in T_S(x) \text{ by Theorem 2.4.} \end{aligned}$$

Hence, by Theorem 2.7, $\bar{S}$ is the set of WSM of $\mathbf{F}$ over S with modulus $\alpha > 0$.

(b) $\iff$ (d). If the statement (b) holds, then obviously the statement (d) also holds. Conversely, let the statement (d) holds. Let $x \in \bar{S}$ and $\widehat{G} \in \alpha\mathbb{B} \cap N_{\bar{S}}(x)$. Therefore, there exists a $\bar{y} \in \bar{S}$ such that $\widehat{G} \in \boldsymbol{\partial}\mathbf{F}_o(\bar{y})$. Thus, by Definition 2.2, we get

$$(z - \bar{y})^\top \odot \widehat{G} \preceq \mathbf{F}_o(z) \ominus_{gH} \mathbf{F}_o(\bar{y}) \text{ for all } z \in \mathbb{R}^n. \tag{2.34}$$

In particular, for any $z \in \bar{S}$, $\mathbf{F}_o(z) = \mathbf{F}_o(\bar{y})$. Thus, (2.34) reduces to

$$(z - \bar{y})^\top \odot \widehat{G} \preceq \mathbf{0} \text{ for all } z \in \bar{S}.$$

Since $\widehat{G} \in \mathbb{R}^n$, by using Remark 1.1, $(z - \bar{y})^\top \odot \widehat{G} = \left\langle \widehat{G}, z - \bar{y} \right\rangle \leq 0$ for all $z \in \bar{S}$. Therefore,

$$\begin{aligned} & \left\langle \widehat{G}, z \right\rangle \leq \left\langle \widehat{G}, \bar{y} \right\rangle \text{ for all } z \in \bar{S} \\ \Longrightarrow & \sup_{z \in \bar{S}} \left\langle \widehat{G}, z \right\rangle \leq \left\langle \widehat{G}, \bar{y} \right\rangle \\ \Longrightarrow & \psi^*_{\bar{S}}(\widehat{G}) = \left\langle \widehat{G}, \bar{y} \right\rangle \text{ because } \bar{y} \in \bar{S}. \end{aligned} \tag{2.35}$$

Since $\widehat{G} \in N_{\bar{S}}(x)$, by Definition 1.25, we have

$$\begin{aligned} & \left\langle \widehat{G}, z - x \right\rangle \leq 0 \text{ for all } z \in \bar{S} \\ \Longrightarrow & \psi^*_{\bar{S}}(\widehat{G}) = \left\langle \widehat{G}, x \right\rangle. \end{aligned} \tag{2.36}$$

Combining (2.35) and (2.36), we get

$$\left\langle \widehat{G}, x \right\rangle = \left\langle \widehat{G}, \bar{y} \right\rangle. \tag{2.37}$$

Note that

$$\begin{aligned} (z - x)^\top \odot \widehat{G} &= \left\langle \widehat{G}, z - x \right\rangle \text{ for all } z \in \mathbb{R}^n \\ &= \left\langle \widehat{G}, z - \bar{y} \right\rangle \text{ for all } z \in \mathbb{R}^n \text{ by (2.37)} \\ &= (z - \bar{y})^\top \odot \widehat{G} \text{ for all } z \in \mathbb{R}^n \text{ by Remark 1.1} \\ &\preceq \mathbf{F}_o(z) \ominus_{gH} \mathbf{F}_o(\bar{y}) \text{ for all } z \in \mathbb{R}^n \text{ by (2.34)} \\ &= \mathbf{F}_o(z) \ominus_{gH} \mathbf{F}_o(x) \text{ for all } z \in \mathbb{R}^n \text{ as } \mathbf{F}_o(x) = \mathbf{F}_o(\bar{y}). \end{aligned}$$

Hence, $\widehat{G} \in \boldsymbol{\partial}\mathbf{F}_o(x)$. Since $x \in \bar{S}$ is arbitrary, the statement (b) holds.

(c) $\implies$ (e). From the statement (c), we have

$$\begin{aligned} & \alpha \operatorname{dist}(d, T_{\bar{S}}(x)) \preceq \mathbf{F}_{\mathscr{D}}(x)(d) \text{ for all } d \in T_S(x) \text{ and } x \in \bar{S} \\ \implies & \alpha \|d\| \preceq \mathbf{F}_{\mathscr{D}}(x)(d) \text{ for all } d \in T_S(x) \cap N_{\bar{S}}(x) \text{ by (ii) of Lemma 1.12.} \end{aligned}$$

Hence, the statement (e) holds.

(e) $\implies$ (a). Let $y \in S$. Set $x = P(y \mid \bar{S})$, then $(y - x) \in T_S(x) \cap N_{\bar{S}}(x)$. Therefore, according to the hypothesis, we obtain

$$\begin{aligned} & \alpha \|y - x\| \preceq \mathbf{F}_{\mathscr{D}}(x)(y - x) \\ \implies & \alpha \operatorname{dist}(y, \bar{S}) \preceq \mathbf{F}_{\mathscr{D}}(x)(y - x) \text{ by Definition 1.22} \\ \implies & \alpha \operatorname{dist}(y, \bar{S}) \preceq \mathbf{F}(y) \ominus_{gH} \mathbf{F}(x) \text{ by Lemma 1.10} \\ \implies & \mathbf{F}(x) \oplus \alpha \operatorname{dist}(y, \bar{S}) \preceq \mathbf{F}(y) \text{ by (i) of Lemma 1.2,} \end{aligned}$$

which shows that $\bar{S}$ is a set of WSM of $\mathbf{F}$ over S.

(a) $\iff$ (f). Let the statement (a) holds. Let $y \in S$ and $p = P(y \mid \bar{S})$. Thus, the statement (a) gives

$$\mathbf{F}(p) \oplus \alpha \operatorname{dist}(y, \bar{S}) \preceq \mathbf{F}(y), \text{ i.e., } \mathbf{F}(p) \oplus \alpha \|y - p\| \preceq \mathbf{F}(y). \tag{2.38}$$

Define $z_\lambda = \lambda y + (1 - \lambda)p$ for $\lambda \in [0, 1]$. Then, $p = P(z_\lambda \mid \bar{S})$ for all $\lambda \in [0, 1]$. From (2.38), we have

$$\begin{aligned} & \mathbf{F}(p) \oplus \alpha \|z_\lambda - p\| \preceq \mathbf{F}(z_\lambda) \\ \implies & \mathbf{F}(p) \oplus \alpha\lambda \|y - p\| \preceq \mathbf{F}(z_\lambda) \\ \implies & \alpha \|x - p\| \preceq \tfrac{1}{\lambda} \odot \Big(\mathbf{F}(p + \lambda(y - p)) \ominus_{gH} \mathbf{F}(p)\Big). \end{aligned} \tag{2.39}$$

By taking limit as $\lambda \downarrow 0$ in (2.39), we get

$$\alpha \operatorname{dist}(y, \bar{S}) \preceq \mathbf{F}_{\mathscr{D}}(p)(y-p), \text{ where } p \in P(y \mid \bar{S}).$$

Conversely, let $y \in S$ and set $x = P(y \mid \bar{S})$. Then, from the statement (f), we get

$$\begin{aligned} & \alpha \operatorname{dist}(y, \bar{S}) \preceq \mathbf{F}_{\mathscr{D}}(x)(y-x) \\ \Longrightarrow \quad & \alpha \operatorname{dist}(y, \bar{S}) \preceq \mathbf{F}(y) \ominus_{gH} \mathbf{F}(x) \text{ by Lemma 1.10} \\ \Longrightarrow \quad & \mathbf{F}(x) \oplus \alpha \operatorname{dist}(y, \bar{S}) \preceq \mathbf{F}(y) \text{ by (i) of Lemma 1.2,} \end{aligned}$$

which is the required result. □

2.7 Applications of WSM

In this section, we present two applications of the proposed study.

2.7.1 Application 1

As a first application, we find the set of WSM of a minimum risk portfolio interval optimization problem (MRPIOP), where the returns and the components of the risk covariance matrix of returns are intervals.

The conventional minimum risk portfolio optimization problem of two assets is given by (see [18])

$$\begin{aligned} \min \quad & y^\top Q y \\ \text{s.t.} \quad & y_1 + y_2 = 1 \\ & 0 \leq y_i, i = 1, 2, \end{aligned}$$

where $y = (y_1, y_2)^\top$, y_i is the proportion of investment corresponding to the i-th asset, and Q is the risk covariance matrix of returns. Conventionally, the entries in Q are real numbers. However, in practice, for a portfolio optimization problems, realistic data involves uncertainty. We, thus, aim to formulate and solve an MRPIOP with interval-valued data.

We define an MRPIOP as

$$\min_{x\in[0,1]} \mathbf{F}(x), \tag{2.40}$$

where $\mathbf{F}(x) = x^2 \odot \mathbf{Q}_{11} \oplus (1-x)^2 \odot \mathbf{Q}_{21} \oplus x^2 \odot \mathbf{Q}_{12} \oplus (1-x)^2 \odot \mathbf{Q}_{22}$, and $\mathbf{Q}_{ik}$ for $i, k = 1, 2$ are (interval-valued) entries of the risk covariance matrix $\mathbf{Q}$ of interval-valued returns.

For instance, we consider two assets namely 'AAL' and 'DAL' of the companies American Airlines Group and Delta Airlines Inc., respectively, for the period January 2021 to December 2021. We calculate the interval-valued returns $\mathbf{R}_{ij}$ of every month using the lowest and highest indices of the i-th asset in the j-th month, for $i = 1, 2$ and $j = 1, 2, \ldots, 12$. These interval-valued returns are displayed in Table 2.1.

Periods	Returns of AAL ($\mathbf{R}_{1j}$)	Returns of DAL ($\mathbf{R}_{2j}$)
Jan-21	[1.1400,5.9200]	[2.4300,2.4700]
Feb-21	[0.9200,5.1900]	[0.9300,11.8500]
Mar-21	[2.8100,4.3400]	[2.8100,6.4600]
Apr-21	[1.2500,2.9500]	[1.0100,4.2500]
May-21	[1.3600,3.7400]	[0.4100,5.5500]
Jun-21	[0.6900,3.1900]	[1.0400,5.9000]
Jul-21	[1.0000,1.8500]	[1.7600,2.7300]
Aug-21	[1.3700,2.1800]	[1.7700,4.4900]
Sep-21	[1.0800,2.1500]	[2.0500,5.1400]
Oct-21	[2.1900,3.2600]	[4.5500,6.3700]
Nov-21	[2.1000,3.4700]	[3.1900,4.2400]
Dec-21	[0.9300,1.7900]	[3.0200,3.4700]

Table 2.1: Sample interval-valued returns of the given assets

We calculate the interval-valued mean return $\mathbf{R}_{i_M}$ for the i-th asset by the formula

$$\mathbf{R}_{i_M} = \tfrac{1}{12} \odot (\mathbf{R}_{i1} \oplus \mathbf{R}_{i2} \oplus \cdots \oplus \mathbf{R}_{i12}),$$

for $i = 1, 2$. The interval-valued mean returns are

$$\mathbf{R}_{1_M} = [1.4033, 3.3358] \text{ and } \mathbf{R}_{2_M} = [2.1183, 5.2433].$$

The interval components $\mathbf{Q}_{ik}$ of risk covariance matrix $\mathbf{Q}$ are computed by the formula

$$\mathbf{Q}_{ik} = \tfrac{1}{144} \odot \big[\big\{(\mathbf{R}_{i1} \ominus_{gH} \mathbf{R}_{i_M}) \otimes (\mathbf{R}_{k1} \ominus_{gH} \mathbf{R}_{k_M})\big\} \oplus \big\{(\mathbf{R}_{i2} \ominus_{gH} \mathbf{R}_{i_M}) \otimes (\mathbf{R}_{k2} \ominus_{gH} \mathbf{R}_{k_M})\big\} \oplus \cdots \oplus \big\{(\mathbf{R}_{i12} \ominus_{gH} \mathbf{R}_{i_M}) \otimes (\mathbf{R}_{k12} \ominus_{gH} \mathbf{R}_{k_M})\big\}\big],$$

for $i, k = 1, 2$, and are shown in Table 2.2.

Assets	AAL	DAL
AAL	[0.0091,0.1486]	[0.0092,0.1821]
DAL	[0.0092,0.1821]	[0.0475,0.4738]

Table 2.2: The risk covariance interval-valued matrix $\mathbf{Q}$ of the returns of the given assets

With the help of $\mathbf{Q}$, MRPIOP (2.40) can be written as

$$\begin{aligned}
&\min_{x\in[0,1]} \{x^2 \odot [0.0091, 0.1486] \oplus (1-x)^2 \odot [0.0092, 0.1821] \oplus x^2 \odot [0.0092, 0.1821] \\
&\qquad \oplus (1-x)^2 \odot [0.0475, 0.4738]\} \\
=\ &\min_{x\in[0,1]} \{x^2 \odot [0.0183, 0.3307] \oplus (1-x)^2 \odot [0.0567, 0.6559]\} \\
=\ &\min_{x\in[0,1]} [0.0750x^2 - 0.1134x + 0.0567, 0.9876x^2 - 1.3118x + 0.6559].
\end{aligned}$$

Thus,

$$\mathbf{F}(x) = [0.0750x^2 - 0.1134x + 0.0567, 0.9876x^2 - 1.3118x + 0.6559],\ x \in [0, 1].$$

Consider $\alpha = 0.075$, $\bar{S}_1 = [0, 0.6641]$ and $\bar{S}_2 = [0.7560, 1]$.

Then, with respect to the usual Euclidean distance, $\text{dist}(x, \bar{S}_1) = (x - 0.6641)^2$, and $\text{dist}(x, \bar{S}_2) = (x - 0.7560)^2$, where $x \in [0, 1]$. Thus, it is easy to notice that for any $x \in [0, 1]$,

$$\mathbf{F}(\bar{x}) \oplus \alpha\, \text{dist}(x, \bar{S}_1) \preceq \mathbf{F}(x) \text{ for all } \bar{x} \in \bar{S}_1$$
$$\text{and } \mathbf{F}(\bar{x}) \oplus \alpha\, \text{dist}(x, \bar{S}_2) \preceq \mathbf{F}(x) \text{ for all } \bar{x} \in \bar{S}_2.$$

Hence, by Definition 2.3 of WSM, $\bar{S}_1 = [0, 0.6641]$ and $\bar{S}_2 = [0.7560, 1]$ are the sets of WSM for (2.40) with the data in Table 2.2. Thus, the points belonging to $\bar{S}_1$ and $\bar{S}_2$ are preferable points for the investment.

2.7.2 Application 2

As a second application, we use the concept of WSM to find the weak efficient solutions of the following linear programming problem with interval-valued objective function:

$$\min_{x \in S'} \mathbf{F}(x), \tag{2.41}$$

where $\mathbf{F}$ is a linear IVF on $\mathbb{R}^n$ and S' is a polyhedral subset of $\mathbb{R}^n$. The above problem is extensively studied by several authors, for instance, see [108, 111, 114], and the references therein. The real-world applications of (2.41) are shown by several authors. For instance, Steuer [181] used (2.41) to study feedmix and blending problems. Wu et al. [199] proposed a method for the planning of a waste management system for the region of Hamilton, Ontario, Canada, with the help of the tools of (2.41).

Inspired by all these real-life applications of (2.41), we find out the weak efficient solutions of (2.41) with the help of the studied concept of WSM. Since the interval linear programming problem (ILPP) (2.41) has been solved by many researchers, we provide a theory to find the set of weak sharp minima of ILPP.

Theorem 2.9 (See [36]). *Suppose we have a linear programming problem (LPP)*

$$\min_{x \in S'} F(x), \tag{2.42}$$

where F is a linear real-valued function on $\mathbb{R}^n$ and S' is polyhedral subset of $\mathbb{R}^n$. Then, the solution set of LPP (2.42) is equal to the set of WSM of F over S'.

Theorem 2.10 *Let $\boldsymbol{F} = [\underline{F}, \overline{F}]$ be a linear IVF on $\mathbb{R}^n$. Then, the set of weak efficient solutions of (2.41), where S' is a polyhedral set in $\mathbb{R}^n$, is identical to the set of WSM of $\boldsymbol{F}$ over S'.*

Proof: Let $\bar{S}$ be the set of weak efficient solutions of (2.41). Then, every $\bar{x} \in \bar{S}$ minimizes $\mathbf{F}(x)$ over S'. Therefore, every $\bar{x} \in \bar{S}$ minimizes $\underline{F}(x)$ and $\overline{F}(x)$ as well. Hence, $\bar{S}$ is the solution set of both the linear programming problems

$$\min_{x \in S'} \underline{F} \text{ and } \min_{x \in S'} \overline{F}. \tag{2.43}$$

Therefore, by Theorem 2.9, we have

$$\begin{aligned} & \bar{S} \text{ is a set of WSM of both } \underline{F} \text{ and } \overline{F} \text{ over } S' \\ \implies \quad & \bar{S} \text{ is a set of WSM of } \mathbf{F} \text{ over } S' \text{ by Remark 2.2.} \end{aligned}$$

Next, if $\bar{x}$ belongs to the set of WSM of $\mathbf{F}$, then by Definition 2.3 of WSM, $\bar{x}$ is also a weak efficient solution of (2.43), which completes the proof.

□

In the next theorem (Theorem 2.11), we use the concept of WSM to solve the following IOP, need not to be an ILPP:

$$\min_{x \in S} \mathbf{F}_1(x), \tag{2.44}$$

where S is a closed convex subset of $\mathbb{R}^n$ and $\mathbf{F}_1$ is proper, extended, convex and gH-lsc IVF on $\mathbb{R}^n$. Note that it is not always an easy task to solve IOP (2.44) by finding WSM of $\mathbf{F}_1$ or by some other methods. In this case, we use perturbation to investigate the weak efficient solution of IOP (2.44). In the perturbation, we consider a different IOP (2.45), whose weak efficient solution is known or easy to find:

$$\min_{x \in S} \mathbf{F}_2(x), \tag{2.45}$$

where $\mathbf{F}_2$ is a proper extended IVF on $\mathbb{R}^n$. Now we consider a perturbed IOP:

$$\min_{x \in S} \{\mathbf{F}_1(x) \oplus \epsilon \odot \mathbf{F}_2(x)\}, \tag{2.46}$$

where ϵ is nonnegative real number.

Theorem 2.11 *Let $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and S be as in (2.46). Let $\bar{S}(\epsilon) \subseteq \bar{S}$ be the set of weak efficient solutions of perturbed IOP (2.46) and $\bar{S}$ be the set of WSM of $\boldsymbol{F}_1$ over S. Then, $\bar{S}(\epsilon) \subseteq \bar{S}_{\boldsymbol{F}_2}$, where $\bar{S}_{\boldsymbol{F}_2}$ is the set of weak efficient solutions of IOP (2.45). Moreover, if $\boldsymbol{F}_2$ is gH-locally Lipschitz continuous on $\mathbb{R}^n$, then $\bar{S}_{\boldsymbol{F}_2} = \bar{S}(\epsilon)$.*

Proof: Suppose $\bar{x} \in \bar{S}(\epsilon)$. Then, for any $y \in S$, we have

$$\mathbf{F}_1(\bar{x}) \oplus \epsilon \odot \mathbf{F}_2(\bar{x}) \preceq \mathbf{F}_1(y) \oplus \epsilon \odot \mathbf{F}_2(y). \tag{2.47}$$

Since $\bar{S} \subseteq S$, then (2.47) holds for any $\bar{y} \in \bar{S}$. Thus,

$$\begin{aligned}
& \mathbf{F}_1(\bar{x}) \oplus \epsilon \odot \mathbf{F}_2(\bar{x}) \preceq \mathbf{F}_1(\bar{y}) \oplus \epsilon \odot \mathbf{F}_2(\bar{y}) \\
\Longrightarrow \quad & \epsilon \odot \mathbf{F}_2(\bar{x}) \preceq \epsilon \odot \mathbf{F}_2(\bar{y}) \text{ because } \bar{S} \text{ is a set of WSM of } \mathbf{F}_1 \\
\Longrightarrow \quad & \mathbf{F}_2(\bar{x}) \preceq \mathbf{F}_2(\bar{y}) \text{ because } \epsilon > 0.
\end{aligned}$$

Thus, $\bar{x} \in \bar{S}_{\mathbf{F}_2}$. Since $\bar{x} \in \bar{S}(\epsilon)$ is arbitrarily chosen, we get the result.

Conversely, let $\bar{x} \in \bar{S}_{\mathbf{F}_2}$. In order to show $\bar{x} \in \bar{S}(\epsilon)$, we prove that

$$\mathbf{F}_1(\bar{x}) \oplus \epsilon \odot \mathbf{F}_2(\bar{x}) \preceq \mathbf{F}_1(\bar{y}) \oplus \epsilon \odot \mathbf{F}_2(\bar{y}) \text{ for } \bar{y} \in \bar{S} \tag{2.48}$$

and

$$\mathbf{F}_1(\bar{x}) \oplus \epsilon \odot \mathbf{F}_2(\bar{x}) \prec \mathbf{F}_1(y) \oplus \epsilon \odot \mathbf{F}_2(y) \text{ for } y \in S \setminus \bar{S}. \tag{2.49}$$

Note that $\bar{S}$ is a set of WSM of $\mathbf{F}_1$ over S and $\bar{x} \in \bar{S}_{\mathbf{F}_2}$, therefore (2.48) holds. To establish (2.49), let $x \in S \setminus \bar{S}$, thus $x \neq \bar{x}$. Then, we have

$$\begin{aligned}
& \epsilon \odot \mathbf{F}_2(\bar{x}) \preceq \epsilon \odot \mathbf{F}_2(\bar{x}) \\
\implies\ & \epsilon \odot \mathbf{F}_2(\bar{x}) \preceq \epsilon \odot \mathbf{F}_2(\bar{x}) \ominus_{gH} \epsilon \odot \mathbf{F}_2(x) \oplus \epsilon \odot \mathbf{F}_2(x) \\
\implies\ & \epsilon \odot \mathbf{F}_2(\bar{x}) \preceq \epsilon M \|\bar{x} - x\| \oplus \epsilon \odot \mathbf{F}_2(x) \text{ as } \mathbf{F} \text{ is } gH\text{-local Lipschitz continuous} \\
\implies\ & \epsilon \odot \mathbf{F}_2(\bar{x}) \prec \alpha \|\bar{x} - x\| \oplus \epsilon \odot \mathbf{F}_2(x), \text{ where } \epsilon M < \alpha \\
\implies\ & \mathbf{F}_1(\bar{x}) \oplus \epsilon \odot \mathbf{F}_2(\bar{x}) \prec \alpha \|\bar{x} - x\| \oplus \mathbf{F}_1(\bar{x}) \oplus \epsilon \odot \mathbf{F}(x) \\
\implies\ & \mathbf{F}_1(\bar{x}) \oplus \epsilon \odot \mathbf{F}_2(\bar{x}) \prec \mathbf{F}_1(x) \oplus \epsilon \odot \mathbf{F}(x) \text{ because } \bar{S} \text{ is a set of WSM of } \mathbf{F}_1.
\end{aligned}$$

Therefore, $\bar{x} \in \bar{S}(\epsilon)$. Since $\bar{x}$ is arbitrarily chosen, so $\bar{S}_{\mathbf{F}_2} \subseteq \bar{S}(\epsilon)$, and hence we get the desired result. □

2.8 Conclusion

In this chapter, the conventional concepts of support function and subdifferentiability have been extended for IVFs (Definitions 2.1 and Definition 2.2). Also, some important characteristics of the gH-subdifferential set like nonemptiness (Lemma 2.6), boundedness (Theorem 2.5), convexity and closedness (Theorem 2.2) have been presented. Subsequently, we have provided a few necessary results (Lemma 2.1, Theorem 2.1 and Lemma 2.2) based on the support function of a subset of $I(\mathbb{R})^n$. It has been reported that the gH-subdifferential set of a gH-differentiable convex IVF is a singleton set containing the gH-gradient (Theorem 2.3). The relationship between gH-directional derivative and the support function of gH-subdifferential set of convex IVF has also been established (Theorem 2.4). Further, we have introduced the notion of WSM for convex IVFs (Definition 2.3). With the help of the proposed concepts of gH-subdifferentiability and support function, a primal characterization (Theorem 2.7) and a few dual characterizations (Theorem 2.8) of WSM have been presented. Two applications of the proposed study have been given. In the first application, the sets of WSM of MRPIOP (2.40) have been given. In the second application, we provide a relationship between the WSM and weak efficient solutions of linear and nonlinear IOPs (Theorems 2.10 and 2.11).

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