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Numerical Solution of Three-Dimensional Time-Space Fractional Order Reaction–Advection–Diffusion Equation by Shifted Legendre–Gauss–Lobatto Collocation Method

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ABSTRACT

In the present article, a shifted Legendre–Gauss–Lobatto collocation method (SLGLCM) is introduced to derive an approximate solution for a three-dimensional time-space fractional order reaction–advection–diffusion equation (3D-TSFRADE) with initial and boundary conditions. The proposed method involves reducing the 3D-TSFRADE into a system of algebraic equations, which has been solved by applying Newton method. Two examples are provided to illustrate the accuracy and efficiency of the proposed method through error analysis between the numerical and exact solutions.

1 | Introduction

Groundwater serves as a crucial freshwater reservoir for people worldwide, catering to various needs including domestic, agricultural, and industrial purposes. Roughly one-third of the global population depends on groundwater as their primary source of drinking water. This resource holds particular significance in arid and semi-arid areas where surface water and rainfall are scarce, making it a lifeline for communities in such regions. Big problems occur nowadays for drinking water as it is getting polluted by many sources from humans and nature. When contaminants, like pesticides or gasoline, find their way into groundwater, they contaminate it and make it unsafe or unfit for human consumption. Groundwater contamination can occur from any activity that emits pollutants and chemicals into the environment, and human activity is nearly always the culprit. The most susceptible places for groundwater contamination are those with dense populations and intense land use. Numerous professionals, including environmentalists, mathematicians, soil and agricultural scientists, hydrologists, and chemical engineers, are deeply engaged in studying groundwater contamination issues. They employ various analytical tools, including mathematical models such as convection models and advection–reaction diffusion models, to comprehensively understand the physical dynamics of groundwater pollution. These models facilitate the analysis of solute transport within the aquifer, providing valuable insights into the mechanisms and behaviors associated with groundwater pollution. Groundwater solute transport mathematical modeling is a crucial field of study. Many engineers and scientists have predicted the movement of solute in the groundwater system through experiments and theoretical studies [1–3].

Fractional calculus is the generalization of integer one. The derivatives, ordering ordinary calculus, restrict the order of derivatives to natural numbers in fractional calculus; however, any real integer can be used. The development of fractional calculus has become a very hot topic of research. The motivation of the present scientific contribution is mainly to investigate the 3D-TSFRADE under suitable boundary conditions by using an effective numerical scheme shifted Legendre–Gauss–Lobatto collocation method (SLGLCM), which is first of its kind. Actually, the growing interests in FRADE in one- and two-dimensional cases to the researchers are due to its useful applications in the areas electro-magnetic, robotics and controls, acoustics, and viscoelastic damping. FRADE describes accurately the transporation of solute in complex media like porous aquifer [4–6]. Fractional integrals and fractional derivatives are nonlocal, so they are useful tools to describe real-life phenomena. The analyses of several physical, chemical, and biological phenomena have been modeled using fractional derivatives, such as the Caputo derivative, Riemann–Liouville derivative, and Riesz derivative [7–13]. Due to the nonlocal property of the fractional order derivative, the FRADE has greater flexibility, faster convergency, and more memory effects as compared to the integer order systems. FRADE describes anomalous diffusion which often found in many physical and engineering problems including solute transport, random, and disordered media, neural networks, and signal processing. Fractional calculus is highly effective in modeling the viscoelastic behavior of polymer solids without cross-linking. These materials exhibit time-dependent deformation behavior, such as creep and relaxation, which cannot be fully captured by classical (integer-order) differential models [14]. Speech signal modeling benefits from fractional calculus, particularly when linear predictive coding (LPC), a widely popular method, is used for encoding speech. Traditional LPC uses integer-order differential models to predict future speech signal values based on past samples [15].

Several numerical schemes have been proposed in the literature to obtain a solution to the fractional reaction advection diffusion equation (RADE) like, spectral collocation technique [16], finite difference method [17–20], operational matrix approach [21–23], and finite volume scheme [24]. The authors of [25] showed that a hybrid approach can solve fractional RADE with piece-wise fractional derivative. Lin et al. [26] have successfully solved fractional order reaction–advection–diffusion equations by using the effective Laguerre collocation method. Using a local meshless method based on the radial basis function, the approximate solutions of the fractional order diffusion equation were investigated in [27]. The Jacobi spectral approach is used to solve the fractional reaction–diffusion problem [28].

Several authors are working on different kinds of 3D-diffusion equations. Gupta and Zhang [29] have suggested solving a 3D-convection–diffusion equation by explicit fourth-order compact finite difference scheme. Ma et al. [30] introduced a high-order finite difference scheme incorporating Richardson–extrapolation for solving the 3D convection–diffusion equation. Galerkin and least squares methods are used to solve a variable coefficients 3D reaction-diffusion equation with convection [31]. Chen et al. [32] have suggested to solve the fractional 3D subdiffusion equation by using an alternating direction implicit scheme. Tian et al. [33] have used a novel radial basis function method for solving variable coefficients 3D diffusion equations linear and nonlinear with advection–reaction term. Liao [34] has solved the 3D reaction-diffusion equation by using a high-order alternating direction implicit finite difference scheme. In [35], an improved element-free Galerkin approach is used for handling variable coefficient 3D convection–diffusion–reaction problem.

SLGLCM is an efficient tool for solving different kinds of PDEs arising in various fields of science and technology. This approach has an advantage that it is highly precise and easy to implement. The said numerical method, consisting of Legendre polynomials, is motivated by the work of [36]. Therefore, keeping in view of the advantages, namely, efficiency, effectiveness, high precision rate, and easy to implement of the SL-GLC scheme in solving two-dimensional space-time fractional model, the authors have made a drive to extend the said scheme in an appropriate way to solve a complicated realistic mathematical model having practical importance, namely, three-dimensional space-time reactional–advection–diffusion equation, which is first of its kind.

The extension from two-dimensional to three dimensional space for solving time-space fractional order reaction–advection–diffusion equation is nontrivial and requires substantial modifications to the SL-GLC scheme. This involves addressing additional complexities in higher-dimensional spaces, including computational complexity, which is the novelty of the present scientific contribution.

Here an attempt is being made to employ the SLGLCM based on Legendre polynomials to explore the following 3D-TSFRADE given by

$$\begin{aligned}\frac{\partial^{\alpha}u(x,y,z,t)}{\partial t^{\alpha}} =& \frac{\partial^{1+\beta_1}u(x,y,z,t)}{\partial x^{1+\beta_1}} + \frac{\partial^{1+\beta_2}u(x,y,z,t)}{\partial y^{1+\beta_2}} \\ &+ \frac{\partial^{1+\beta_3}u(x,y,z,t)}{\partial z^{1+\beta_3}} + \nu_1\frac{\partial^{\beta_1}u(x,y,z,t)}{\partial x^{\beta_1}} \\ &+ \nu_2\frac{\partial^{\beta_2}u(x,y,z,t)}{\partial y^{\beta_2}} + \nu_3\frac{\partial^{\beta_3}u(x,y,z,t)}{\partial z^{\beta_3}} \\ &+ \lambda u(x,y,z,t) + f(x,y,z,t),\end{aligned} \tag{1}$$

under the initial and boundary conditions given as

$$\left.\begin{aligned}&u(x,y,z,0)=h_1(x,y,z), \quad u(0,y,z,t)=h_2(y,z,t),\\ &u(\mathcal{X},y,z,t)=h_3(y,z,t), \quad u(x,0,z,t)=h_4(x,z,t),\\ &u(x,\mathcal{Y},z,t)=h_5(x,z,t), \quad u(x,y,0,t)=h_6(x,y,t),\\ &u(x,y,Z,t)=h_7(x,y,t),\end{aligned}\right\} \tag{2}$$

where $0 \le x \le \mathcal{X}$, $0 \le y \le \mathcal{Y}$, $0 \le z \le Z$, $0 \le t \le T$, $0 < \alpha \le 1$, $0 < \beta_1 \le 1$, $0 < \beta_2 \le 1$, $0 < \beta_3 \le 1$. The function $u(x,y,z,t)$ represents the concentration of a solute in a three-dimensional medium at the position (x,y,z) and time t. ν_1, ν_2, and ν_3 are advection coefficients. The parameter λ is the reaction coefficient. $\lambda < 0$ represents sink term, while $\lambda > 0$ indicates source term. $\lambda = 0$ denotes conservative medium otherwise nonconservative medium. $f(x,y,z,t)$ is the force term; $h_1(x,y,z)$, $h_2(y,z,t)$, $h_3(y,z,t)$, $h_4(x,z,t)$, $h_5(x,z,t)$, $h_6(x,y,t)$, and $h_7(x,y,t)$ are sufficiently smooth functions. The considered model (1) has applications in transport of contamination in groundwater.

This article is prepared as follows. Some basic definitions and properties have been illustrated in Section 2. The implementation of the proposed method is discussed in Section 3. The error analyses are furnished in Section 4. Two examples are provided in Section 5 to confirm the accuracy and efficiency of the proposed method. Finally, Section 6 provides a brief conclusion.

2 | Preliminaries

2.1 | Caputo Fractional Derivative

Caputo fractional derivative of a function $h \in C_\mu^n$, $n = \lceil \alpha \rceil$ and $\mu \geq -1$ is defined as [37, 38]

$$
{}_0^cD_\xi^\alpha h(\xi) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^\xi \frac{1}{(\xi-\tau)^{\alpha-n+1}} \frac{\partial^n h(\tau)}{\partial \tau^n} d\tau, & t > 0, 0 \leq n-1 \leq \alpha < n \in \mathbb{N}, \\ \frac{\partial^n h(\xi)}{\partial \xi^n}, & \alpha = n \in \mathbb{N}. \end{cases} \tag{3}
$$

The power function property is

$$
{}_0^cD_\xi^\alpha \xi^{\mathrm{q}} = \begin{cases} 0, & \mathrm{q} = 0, \\ \frac{\Gamma(\mathrm{q}+1)}{\Gamma(\mathrm{q}+1-\alpha)} \xi^{\mathrm{q}-\alpha}, & \mathrm{q} \in \mathbb{N}. \end{cases}
$$

2.2 | Legendre Polynomial

Let $P_i(\varkappa)$ represents the Legendre polynomials (in the interval [−1,1]) of degree i. $P_i(\varkappa) = P_i\left(\frac{2\varkappa}{\mathcal{X}} - 1\right)$ be the shifted Legendre polynomials of degree i on the interval $[0, \mathcal{X}]$. The shifted Legendre polynomials $P_{\mathcal{X},i}(\varkappa)$ are defined by the recurrence relation as [39]

$$
\begin{aligned} P_{\mathcal{X},i}(\varkappa) &= \frac{2i+1}{i+1}\left(\frac{2\varkappa - \mathcal{X}}{\mathcal{X}}\right) P_{\mathcal{X},i}(\varkappa) \\ &\quad - \frac{i}{i+1} P_{\mathcal{X},i-1}(\varkappa), \quad i = 1,2,3,\ldots; \end{aligned}
$$

$$
P_{\mathcal{X},0}(\varkappa) = 1, \qquad P_{\mathcal{X},1}(\varkappa) = \frac{2\varkappa}{\mathcal{X}} - 1,
$$

which satisfies the orthogonality property

$$
\int_0^{\mathcal{X}} P_{\mathcal{X},i}(\varkappa) P_{\mathcal{X},j}(\varkappa) d\varkappa = \frac{\mathcal{X}}{2i+1}\delta_{ij},
$$

where δ_{ij} is the Kronecker function defined as

$$
\delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}
$$

The analytical form of the shifted Legendre polynomial of degree i is given by

$$
P_{\mathcal{X},i}(\varkappa) = \Sigma_{s=0}^{i} \frac{(i+s)!\varkappa^s}{(i-s)!(s!)^2\mathcal{X}^s}. \quad i = 0,1,2,\ldots. \tag{4}
$$

The Caputo fractional derivative of shifted Legendre polynomial of the order α is

$$
\begin{aligned} {}_0^cD_\varkappa^\alpha P_{\mathcal{X},i}(\varkappa) &= \Sigma_{s=1}^{i} (-1)^{(i+s)} \frac{(i+s)!\varkappa^{s-\alpha}}{(i-s)!(s!)\Gamma(s+1-\alpha)\mathcal{X}^s}. \\ & i = 1,2,3,\ldots; \end{aligned} \tag{5}
$$

3 | SLGLCM

In this section, the detailed algorithm is discussed to show how to use the SLGLCM to solve the 3D-TSFRADE. The approximate solutions for 3D-TSFRAD equations are defined as [36]

$$
\mathrm{u}(\varkappa, y, z, t) \simeq \Sigma_{i=0}^{\mathcal{N}} \Sigma_{j=0}^{\mathcal{M}} \Sigma_{k=0}^{\mathcal{K}} \Sigma_{l=0}^{\mathcal{L}} \mathrm{a}_{i,j,k,l}\, P_{T,l}(t)\, P_{\mathcal{X},i}(\varkappa)\, P_{\mathcal{Y},j}(\mathrm{y})\, P_{Z,k}(z), \tag{6}
$$

where $\mathrm{a}_{i,j,k,l}$ are unknown coefficients; $P_{T,l}(t)$, $P_{\mathcal{X},i}(\varkappa)$, $P_{\mathcal{Y},j}(\mathrm{y})$, and $P_{Z,k}(z)$ are the shifted Legendre polynomials given in Section 2.2; $\mathcal{N}$, $\mathcal{M}$, $\mathcal{K}$, and $\mathcal{L}$ are positive integers; and $\varkappa_{\mathcal{N},\mathrm{p}}$, $\mathrm{y}_{\mathcal{M},\mathrm{q}}$, $z_{\mathcal{K},\mathrm{r}}$, and $t_{\mathcal{L},\mathrm{s}}$ are the roots of shifted Legendre polynomials.

Derivatives in Caputo sense of $\mathrm{u}(\varkappa, y, z, t)$ are approximated at the SLGL interpolation nodes given as [36]

$$
\begin{aligned} \left.\frac{\partial^{1+\beta_1}\mathrm{u}(\varkappa, \mathrm{y}_{\mathcal{M},\mathrm{q}}, z_{\mathcal{K},\mathrm{r}}, t_{\mathcal{L},\mathrm{s}})}{\partial \varkappa^{1+\beta_1}}\right|_{\varkappa=\varkappa_{\mathcal{N},\mathrm{p}}} &\simeq \Sigma_{i=0}^{\mathcal{N}} \Sigma_{j=0}^{\mathcal{M}} \Sigma_{k=0}^{\mathcal{K}} \Sigma_{l=0}^{\mathcal{L}} \mathrm{a}_{i,j,k,l} \\ & P_{T,l}(t_{\mathcal{L},\mathrm{s}})\, {}_0^cD_\varkappa^{1+\beta_1} P_{\mathcal{X},i}(\varkappa)\Big|_{\varkappa=\varkappa_{\mathcal{N},\mathrm{p}}} \\ & P_{\mathcal{Y},j}(\mathrm{y}_{\mathcal{M},\mathrm{q}}) \times P_{Z,k}(z_{\mathcal{K},\mathrm{r}}), \end{aligned} \tag{7}
$$

$$
\begin{aligned} \left.\frac{\partial^{1+\beta_2}\mathrm{u}(\varkappa_{\mathcal{N},\mathrm{p}}, y, z_{\mathcal{K},\mathrm{r}}, t_{\mathcal{L},\mathrm{s}})}{\partial \mathrm{y}^{1+\beta_2}}\right|_{\mathrm{y}=\mathrm{y}_{\mathcal{M},\mathrm{q}}} &\simeq \Sigma_{i=0}^{\mathcal{N}} \Sigma_{j=0}^{\mathcal{M}} \Sigma_{k=0}^{\mathcal{K}} \Sigma_{l=0}^{\mathcal{L}} \mathrm{a}_{i,j,k,l}\, P_{T,l}(t_{\mathcal{L},\mathrm{s}}) \\ & P_{\mathcal{X},i}(\varkappa_{\mathcal{N},\mathrm{p}})\, {}_0^cD_\mathrm{y}^{1+\beta_2} P_{\mathcal{Y},j}(\mathrm{y})\Big|_{\mathrm{y}=\mathrm{y}_{\mathcal{M},\mathrm{q}}} \\ & \times P_{Z,k}(z_{\mathcal{K},\mathrm{r}}), \end{aligned} \tag{8}
$$

$$
\begin{aligned} \left.\frac{\partial^{1+\beta_3}\mathrm{u}(\varkappa_{\mathcal{N},\mathrm{p}}, \mathrm{y}_{\mathcal{M},\mathrm{q}}, z, t_{\mathcal{L},\mathrm{s}})}{\partial z^{1+\beta_3}}\right|_{z=z_{\mathcal{K},\mathrm{r}}} &\simeq \Sigma_{i=0}^{\mathcal{N}} \Sigma_{j=0}^{\mathcal{M}} \Sigma_{k=0}^{\mathcal{K}} \Sigma_{l=0}^{\mathcal{L}} \mathrm{a}_{i,j,k,l} \\ & P_{T,l}(t_{\mathcal{L},\mathrm{s}})\, P_{\mathcal{X},i}(\varkappa_{\mathcal{N},\mathrm{p}})\, P_{\mathcal{Y},j}(\mathrm{y}_{\mathcal{M},\mathrm{q}}) \\ & \times {}_0^cD_z^{1+\beta_3} P_{Z,k}(z)\Big|_{z=z_{\mathcal{K},\mathrm{r}}}, \end{aligned} \tag{9}
$$

$$
\begin{aligned} \left.\frac{\partial^{\beta_1}\mathrm{u}(\varkappa, \mathrm{y}_{\mathcal{M},\mathrm{q}}, z_{\mathcal{K},\mathrm{r}}, t_{\mathcal{L},\mathrm{s}})}{\partial \varkappa^{\beta_1}}\right|_{\varkappa=\varkappa_{\mathcal{N},\mathrm{p}}} &\simeq \Sigma_{i=0}^{\mathcal{N}} \Sigma_{j=0}^{\mathcal{M}} \Sigma_{k=0}^{\mathcal{K}} \Sigma_{l=0}^{\mathcal{L}} \mathrm{a}_{i,j,k,l}\, P_{T,l}(t_{\mathcal{L},\mathrm{s}}) \\ & {}_0^cD_\varkappa^{\beta_1} P_{\mathcal{X},i}(\varkappa)\Big|_{\varkappa=\varkappa_{\mathcal{N},\mathrm{p}}} P_{\mathcal{Y},j}(\mathrm{y}_{\mathcal{M},\mathrm{q}}) \\ & \times P_{Z,k}(z_{\mathcal{K},\mathrm{r}}), \end{aligned} \tag{10}
$$

$$
\begin{aligned} \left.\frac{\partial^{\beta_2}\mathrm{u}(\varkappa_{\mathcal{N},\mathrm{p}}, y, z_{\mathcal{K},\mathrm{r}}, t_{\mathcal{L},\mathrm{s}})}{\partial \mathrm{y}^{\beta_2}}\right|_{\mathrm{y}=\mathrm{y}_{\mathcal{M},\mathrm{q}}} &\simeq \Sigma_{i=0}^{\mathcal{N}} \Sigma_{j=0}^{\mathcal{M}} \Sigma_{k=0}^{\mathcal{K}} \Sigma_{l=0}^{\mathcal{L}} \mathrm{a}_{i,j,k,l}\, P_{T,l}(t_{\mathcal{L},\mathrm{s}}) \\ & P_{\mathcal{X},i}(\varkappa_{\mathcal{N},\mathrm{p}})\, {}_0^cD_\mathrm{y}^{\beta_2} P_{\mathcal{Y},j}(\mathrm{y})\Big|_{\mathrm{y}=\mathrm{y}_{\mathcal{M},\mathrm{q}}} \\ & \times P_{Z,k}(z_{\mathcal{K},\mathrm{r}}), \end{aligned} \tag{11}
$$

$$
\begin{aligned} \left.\frac{\partial^{\beta_3}\mathrm{u}(\varkappa_{\mathcal{N},\mathrm{p}}, \mathrm{y}_{\mathcal{M},\mathrm{q}}, z, t_{\mathcal{L},\mathrm{s}})}{\partial z^{\beta_3}}\right|_{z=z_{\mathcal{K},\mathrm{r}}} &\simeq \Sigma_{i=0}^{\mathcal{N}} \Sigma_{j=0}^{\mathcal{M}} \Sigma_{k=0}^{\mathcal{K}} \Sigma_{l=0}^{\mathcal{L}} \mathrm{a}_{i,j,k,l}\, P_{T,l}(t_{\mathcal{L},\mathrm{s}}) \\ & P_{\mathcal{X},i}(\varkappa_{\mathcal{N},\mathrm{p}})\, P_{\mathcal{Y},j}(\mathrm{y}_{\mathcal{M},\mathrm{q}}) \\ & \times {}_0^cD_z^{\beta_3} P_{Z,k}(z)\Big|_{z=z_{\mathcal{K},\mathrm{r}}}, \end{aligned} \tag{12}
$$

$$\left.\frac{\partial^{\alpha}\mathrm{u}(\varkappa_{\mathcal{N},\mathrm{p}},\mathrm{y}_{\mathcal{M},\mathrm{q}},z_{\mathcal{K},\mathrm{r}},t)}{\partial t^{\alpha}}\right|_{t=t_{\mathcal{L},s}} \simeq \Sigma_{i=0}^{\mathcal{N}}\Sigma_{j=0}^{\mathcal{M}}\Sigma_{k=0}^{\mathcal{K}}\Sigma_{l=0}^{\mathcal{L}} \mathrm{a}_{i,j,k,l}\,{}_0^cD_t^{\alpha}P_{T,l}(t)\Big|_{t=t_{\mathcal{L},s}} P_{\mathcal{X},i}(\varkappa_{\mathcal{N},\mathrm{p}}) P_{\mathcal{Y},j}(\mathrm{y}_{\mathcal{M},\mathrm{q}})\,P_{Z,k}(z_{\mathcal{K},\mathrm{r}}). \tag{13}$$

The aim of this scheme is to estimate the residuals of Equation (1) which are set to zero at $(\mathcal{N}-1)(\mathcal{M}-1)(\mathcal{K}-1)\mathcal{L}$ collocation points. So, substituting all these Equations (6–13) in Equation (1), we get

$$\begin{aligned}&\Sigma_{i=0}^{\mathcal{N}}\Sigma_{j=0}^{\mathcal{M}}\Sigma_{\mathcal{K}=0}^{\mathcal{K}}\Sigma_{l=0}^{\mathcal{L}} \mathrm{a}_{i,j,k,l}\,\Phi_{i,j,k,l}\\ &-\lambda\Sigma_{i=0}^{\mathcal{N}}\Sigma_{j=0}^{\mathcal{M}}\Sigma_{\mathcal{K}=0}^{\mathcal{K}}\Sigma_{l=0}^{\mathcal{L}} \mathrm{a}_{i,j,k,l}\,P_{T,l}(t)\,P_{\mathcal{X},i}(\varkappa)\,P_{\mathcal{Y},j}(\mathrm{y})\,P_{Z,k}(z)\\ &-f(\varkappa_{\mathcal{N},\mathrm{p}},\mathrm{y}_{\mathcal{M},\mathrm{q}},z_{\mathcal{K},\mathrm{r}},t_{\mathcal{L},\mathrm{s}})=0,\end{aligned} \tag{14}$$

where

$$\begin{aligned}\Phi_{i,j,k,l} =\ & {}_0^cD_t^{\alpha}P_{T,l}(t)|_{t=t_{\mathcal{L},s}}\,P_{\mathcal{X},i}(\varkappa_{\mathcal{N},\mathrm{p}})\,P_{\mathcal{Y},j}(\mathrm{y}_{\mathcal{M},\mathrm{q}})\,P_{Z,k}(z_{\mathcal{K},\mathrm{r}})\\ &-P_{T,l}(t_{\mathcal{L},\mathrm{s}})\,{}_0^cD_{\varkappa}^{1+\beta_1}P_{\mathcal{X},i}(\varkappa)|_{\varkappa=\varkappa_{\mathcal{N},\mathrm{p}}}\\ &\times P_{\mathcal{Y},j}(\mathrm{y}_{\mathcal{M},\mathrm{q}})\,P_{Z,k}(z_{\mathcal{K},\mathrm{r}})-P_{T,l}(t_{\mathcal{L},\mathrm{s}})\\ &P_{\mathcal{X},i}(\varkappa_{\mathcal{N},\mathrm{p}})\,{}_0^cD_{\mathrm{y}}^{1+\beta_2}P_{\mathcal{Y},j}(\mathrm{y})|_{\mathrm{y}=\mathrm{y}_{\mathcal{M},\mathrm{q}}}\,P_{Z,k}(z_{\mathcal{K},\mathrm{r}})\\ &-P_{T,l}(t_{\mathcal{L},\mathrm{s}})\,P_{\mathcal{X},i}(\varkappa_{\mathcal{N},\mathrm{p}})\,P_{\mathcal{Y},j}(\mathrm{y}_{\mathcal{M},\mathrm{q}})\,{}_0^cD_z^{1+\beta_3}P_{Z,k}(z)|_{z=z_{\mathcal{K},\mathrm{r}}}\\ &-\nu_1\,P_{T,l}(t_{\mathcal{L},\mathrm{s}})\,{}_0^cD_{\varkappa}^{\beta_1}P_{\mathcal{X},i}(\varkappa)|_{\varkappa=\varkappa_{\mathcal{N},\mathrm{p}}}\\ &\times P_{\mathcal{Y},j}(\mathrm{y}_{\mathcal{M},\mathrm{q}})\,P_{Z,k}(z_{\mathcal{K},\mathrm{r}})-\nu_2\,P_{T,l}(t_{\mathcal{L},\mathrm{s}})\\ &P_{\mathcal{X},i}(\varkappa_{\mathcal{N},\mathrm{p}})\,{}_0^cD_{\mathrm{y}}^{\beta_2}P_{\mathcal{Y},j}(\mathrm{y}_{\mathcal{M},\mathrm{q}})|_{\mathrm{y}=\mathrm{y}_{\mathcal{M},\mathrm{q}}}\,P_{Z,k}(z_{\mathcal{K},\mathrm{r}})\\ &-\nu_3\,P_{T,l}(t_{\mathcal{L},\mathrm{s}})\,P_{\mathcal{X},i}(\varkappa_{\mathcal{N},\mathrm{p}})\,P_{\mathcal{Y},j}(\mathrm{y}_{\mathcal{M},\mathrm{q}})\,{}_0^cD_z^{\beta_3}P_{Z,k}(z)|_{z=z_{\mathcal{K},\mathrm{r}}}.\end{aligned}$$

with $\mathrm{p}=1,\ldots,\mathcal{N}-1$, $\mathrm{q}=1,\ldots,\mathcal{M}-1$, $\mathrm{r}=1,\ldots,\mathcal{K}-\infty$, $\mathrm{s}=1,\ldots,\mathcal{L}$.

The above system give $(\mathcal{N}-1)(\mathcal{M}-1)(\mathcal{K}-1)\mathcal{L}$ equations and $(\mathcal{N}+1)(\mathcal{M}+1)(\mathcal{K}+1)(\mathcal{L}+1)$ unknown coefficients. So to find the unique solution, we need $\mathcal{NMK}+2(\mathcal{MKL}+\mathcal{NKL}+\mathcal{NML})+\mathcal{N}+\mathcal{M}+\mathcal{K}+2\mathcal{L}+2(\mathcal{MK}+\mathcal{NK}+\mathcal{NM})+1$ additional equations. Additional equations are derived from the initial and boundary conditions as

$$\Sigma_{i=0}^{\mathcal{N}}\Sigma_{j=0}^{\mathcal{M}}\Sigma_{k=0}^{\mathcal{K}}\Sigma_{l=0}^{\mathcal{L}} \mathrm{a}_{i,j,k,l}\,P_{T,l}(0)\,P_{\mathcal{X},i}(\varkappa_{\mathcal{N},p})\,P_{\mathcal{Y},j}(\mathrm{y}_{\mathcal{M},\mathrm{q}})\,P_{Z,k}(z_{\mathcal{K},r})=\mathrm{h}_1(\varkappa_{\mathcal{N},p},\mathrm{y}_{\mathcal{M},q},z_{\mathcal{K},r}), \tag{15}$$

where $\mathrm{p}=1,2,3,\ldots\ldots,\mathcal{N}-1$, $\mathrm{q}=1,2,3,\ldots,\mathcal{M}-1$, $\mathrm{r}=1,2,3,\ldots,\mathcal{K}-1$.

$$\begin{aligned}&\Sigma_{i=0}^{\mathcal{N}}\Sigma_{j=0}^{\mathcal{M}}\Sigma_{k=0}^{\mathcal{K}}\Sigma_{l=0}^{\mathcal{L}} \mathrm{a}_{i,j,k,l}\,P_{T,l}(t_{\mathcal{L},s})\,P_{\mathcal{X},i}(0)\,P_{\mathcal{Y},j}(\mathrm{y}_{\mathcal{M},\mathrm{q}})\,P_{Z,k}(z_{\mathcal{K},r})\\ &=\mathrm{h}_2(\mathrm{y}_{\mathcal{M},\mathrm{q}},z_{\mathcal{K},\mathrm{r}},t_{\mathcal{L},\mathrm{s}}),\\ &\Sigma_{i=0}^{\mathcal{N}}\Sigma_{j=0}^{\mathcal{M}}\Sigma_{k=0}^{\mathcal{K}}\Sigma_{l=0}^{\mathcal{L}} \mathrm{a}_{i,j,k,l}\,P_{T,l}(t_{\mathcal{L},s})\,P_{\mathcal{X},i}(1)\,P_{\mathcal{Y},j}(\mathrm{y}_{\mathcal{M},\mathrm{q}})\,P_{Z,k}(z_{\mathcal{K},r})\\ &=\mathrm{h}_3(\mathrm{y}_{\mathcal{M},\mathrm{q}},z_{\mathcal{K},\mathrm{r}}t_{\mathcal{L},\mathrm{s}}),\end{aligned} \tag{16}$$

where $\mathrm{q}=0,1,2,\ldots\ldots,\mathcal{M}$, $\mathrm{r}=0,1,2,\ldots,\mathcal{K}$, $\mathrm{s}=0,1,2,\ldots,\mathcal{L}$.

$$\begin{aligned}&\Sigma_{i=0}^{\mathcal{N}}\Sigma_{j=0}^{\mathcal{M}}\Sigma_{k=0}^{\mathcal{K}}\Sigma_{l=0}^{\mathcal{L}} \mathrm{a}_{i,j,k,l}\,P_{T,l}(t_{\mathcal{L},s})\,P_{\mathcal{X},i}(\varkappa_{\mathcal{N},p})\,P_{\mathcal{Y},j}(0)\,P_{Z,k}(z_{\mathcal{K},r})\\ &=\mathrm{h}_4(\varkappa_{\mathcal{N},\mathrm{p}},z_{\mathcal{K},\mathrm{r}}t_{\mathcal{L},\mathrm{s}}),\\ &\Sigma_{i=0}^{\mathcal{N}}\Sigma_{j=0}^{\mathcal{M}}\Sigma_{k=0}^{\mathcal{K}}\Sigma_{l=0}^{\mathcal{L}} \mathrm{a}_{i,j,k,l}\,P_{T,l}(t_{\mathcal{L},s})\,P_{\mathcal{X},i}(\varkappa_{\mathcal{N},p})\,P_{\mathcal{Y},j}(1)\,P_{Z,k}(z_{\mathcal{K},r})\\ &=\mathrm{h}_5(\varkappa_{\mathcal{N},\mathrm{p}},z_{\mathcal{K},\mathrm{r}}t_{\mathcal{L},\mathrm{s}}),\end{aligned} \tag{17}$$

where $\mathrm{p}=1,2,3,\ldots\ldots,\mathcal{N}-1$, $\mathrm{r}=1,2,3,\ldots,\mathcal{K}$, $\mathrm{s}=0,1,2,\ldots,\mathcal{L}$.

$$\begin{aligned}&\Sigma_{i=0}^{\mathcal{N}}\Sigma_{j=0}^{\mathcal{M}}\Sigma_{k=0}^{\mathcal{K}}\Sigma_{l=0}^{\mathcal{L}} \mathrm{a}_{i,j,k,l}\,P_{T,l}(t_{\mathcal{L},s})\,P_{\mathcal{X},i}(\varkappa_{\mathcal{N},p})\,P_{\mathcal{Y},j}(\mathrm{y}_{\mathcal{M},\mathrm{q}})\,P_{Z,k}(0)\\ &=\mathrm{h}_6(\varkappa_{\mathcal{N},\mathrm{p}},\mathrm{y}_{\mathcal{M},\mathrm{q}}t_{\mathcal{L},s}),\\ &\Sigma_{i=0}^{\mathcal{N}}\Sigma_{j=0}^{\mathcal{M}}\Sigma_{k=0}^{\mathcal{K}}\Sigma_{l=0}^{\mathcal{L}} \mathrm{a}_{i,j,k,l}\,P_{T,l}(t_{\mathcal{L},s})\,P_{\mathcal{X},i}(\varkappa_{\mathcal{N},p})\,P_{\mathcal{Y},j}(\mathrm{y}_{\mathcal{M},\mathrm{q}})\,P_{Z,k}(1)\\ &=\mathrm{h}_7(\varkappa_{\mathcal{N},\mathrm{p}},\mathrm{y}_{\mathcal{M},\mathrm{q}}t_{\mathcal{L},s}),\end{aligned} \tag{18}$$

where $\mathrm{p}=1,2,3,\ldots\ldots,\mathcal{N}-1$, $\mathrm{q}=1,2,3,\ldots,\mathcal{M}$, $\mathrm{s}=0,1,2,\ldots,\mathcal{L}$.

Thus, a system of $(\mathcal{N}+1)(\mathcal{M}+1)(\mathcal{K}+1)(\mathcal{L}+1)$ algebraic equations is obtained with $(\mathcal{N}+1)(\mathcal{M}+1)(\mathcal{K}+1)(\mathcal{L}+1)$ unknown coefficients, which can be easily handle by Newton method. After finding the coefficients, one may determine the solution of $\mathrm{u}(\varkappa,y,z,t)$ from the expression (6).

4 | Error Analyses

In this section, the error bounds have been estimated for the approximate solution. Let us consider $\mathrm{u}(\varkappa,y,z,t)$ sufficiently smooth function on domain $(\varkappa,y,z)\in[0,1]\times[0,1]\times[0,1]$, $t\in[0,1]$.

Theorem 1. ([36]). Let us consider that $U_{\mathcal{N},\mathcal{M},\mathcal{K},\mathcal{L}}(\varkappa,y,z,t)$ is the best approximation of function $\mathrm{u}(\varkappa,y,z,t)$. Then, we have

$$\begin{aligned}||\bar{\mathrm{u}}||_{L_2}\le(\mathcal{X}\mathcal{Y}ZT)^{\frac{1}{2}}\Bigg(&\frac{1}{4}\left(\frac{\mathcal{X}}{\mathcal{N}}\right)^{\mathcal{N}+1}\mathcal{C}_1+\frac{1}{4}\left(\frac{\mathcal{Y}}{\mathcal{M}}\right)^{\mathcal{M}+1}\mathcal{C}_2+\frac{1}{4}\left(\frac{Z}{\mathcal{K}}\right)^{\mathcal{K}+1}\mathcal{C}_3+\frac{1}{4}\left(\frac{T}{\mathcal{L}}\right)^{\mathcal{L}+1}\mathcal{C}_4\\ &+\frac{1}{256}\left(\frac{\mathcal{X}}{\mathcal{N}}\right)^{\mathcal{N}+1}\left(\frac{\mathcal{Y}}{\mathcal{M}}\right)^{\mathcal{M}+1}\left(\frac{Z}{\mathcal{K}}\right)^{\mathcal{K}+1}\left(\frac{T}{\mathcal{L}}\right)^{\mathcal{L}+1}\mathcal{C}_5,\end{aligned} \tag{19}$$

where $\bar{\mathrm{u}}=\mathrm{u}(\varkappa,y,z,t)-U_{\mathcal{N},\mathcal{M},\mathcal{K},\mathcal{L}}(\varkappa,y,z,t)$, $\mathcal{C}_1=\max\left|\frac{\partial^{\mathcal{N}+1}\mathrm{u}(\varkappa,y,z,t)}{\partial\varkappa^{\mathcal{N}+1}}\right|$, $\mathcal{C}_2=\max\left|\frac{\partial^{\mathcal{M}+1}\mathrm{u}(\varkappa,y,z,t)}{\partial\mathrm{y}^{\mathcal{M}+1}}\right|$, $\mathcal{C}_3=\max\left|\frac{\partial^{\mathcal{K}+1}\mathrm{u}(\varkappa,y,z,t)}{\partial z^{\mathcal{K}+1}}\right|$, $\mathcal{C}_4=\max\left|\frac{\partial^{\mathcal{L}+1}\mathrm{u}(\varkappa,y,z,t)}{\partial t^{\mathcal{L}+1}}\right|$, $\mathcal{C}_5=\max\left|\frac{\partial^{\mathcal{N}+\mathcal{M}+\mathcal{K}+\mathcal{L}+4}\mathrm{u}(\varkappa,y,z,t)}{\partial\varkappa^{\mathcal{N}+1}\partial\mathrm{y}^{\mathcal{M}+1}\partial z^{\mathcal{K}+1}\partial t^{\mathcal{L}+1}}\right|$ at $(\varkappa,y,z,t)\in[0,\mathcal{X}]\times[0,\mathcal{Y}]\times[0,Z]\times[0,T]$.

Theorem 2. ([36]). Consider that $U_{\mathcal{N},\mathcal{M},\mathcal{K},\mathcal{L}}(\varkappa,y,z,t)=\Sigma_{i=0}^{\mathcal{N}}\Sigma_{j=0}^{\mathcal{M}}\Sigma_{k=0}^{\mathcal{K}}\Sigma_{l=0}^{\mathcal{L}}\bar{\mathrm{a}}_{i,j,k,l}\,P_{T,l}(t)\,P_{\mathcal{X},i}(\varkappa)\,P_{\mathcal{Y},j}(\mathrm{y})\,P_{Z,k}(z)$ is the best approximations of the exact solutions of $\mathrm{u}(\varkappa,y,z,t)$. Assume that $\mathrm{u}_{\mathcal{N},\mathcal{M},\mathcal{K},\mathcal{L}}(\varkappa,y,z,t)=\Sigma_{i=0}^{\mathcal{N}}\Sigma_{j=0}^{\mathcal{M}}\Sigma_{k=0}^{\mathcal{K}}\Sigma_{l=0}^{\mathcal{L}}\mathrm{a}_{i,j,k,l}\,P_{T,l}(t)\,P_{\mathcal{X},i}(\varkappa)\,P_{\mathcal{Y},j}(\mathrm{y})\,P_{Z,k}(z)$ be the numerical solutions obtained by the proposed scheme. Then, we obtain

$$||\bar{\bar{u}}||_{L_2} \leq ||\bar{A} - A||_{L_2} \left(\Sigma_{i=0}^{N} \Sigma_{j=0}^{\mathcal{M}} \Sigma_{K=0}^{K} \Sigma_{l=0}^{\mathcal{L}} \frac{\mathcal{X}\mathcal{Y}ZT}{(2i+1)(2j+1)(2k+1)(2l+1)} \right)^{\frac{1}{2}}, \tag{20}$$

where $\bar{\bar{u}} = U_{\mathcal{N},\mathcal{M},\mathcal{K},\mathcal{L}}(\varkappa, y, z, t) - u_{\mathcal{N},\mathcal{M},\mathcal{K},\mathcal{L}}(\varkappa, y, z, t)$, $||\bar{A} - A||_{L_2} = \left(\Sigma_{i=0}^{N} \Sigma_{j=0}^{\mathcal{M}} \Sigma_{k=0}^{K} \Sigma_{l=0}^{\mathcal{L}} (|\bar{a}_{i,j,k,l} - a_{i,j,k,l}|^2) \right)^{\frac{1}{2}}$.

Theorem 3. ([36]). Let us consider that $u_{\mathcal{N},\mathcal{M},\mathcal{K},\mathcal{L}}(\varkappa, y, z, t)$ is the approximations of $u(\varkappa, y, z, t)$ obtained by the proposed method. Then, we obtain

$$\begin{aligned} ||\tilde{u}||_{L_2} \leq (\mathcal{X}\mathcal{Y}ZT)^{\frac{1}{2}} \Bigg(&\frac{1}{4}\left(\frac{\mathcal{X}}{\mathcal{N}}\right)^{\mathcal{N}+1} \mathcal{C}_1 \\ &+ \frac{1}{4}\left(\frac{\mathcal{Y}}{\mathcal{M}}\right)^{\mathcal{M}+1} \mathcal{C}_2 + \frac{1}{4}\left(\frac{Z}{\mathcal{K}}\right)^{\mathcal{K}+1} \mathcal{C}_3 + \frac{1}{4}\left(\frac{T}{\mathcal{L}}\right)^{\mathcal{L}+1} \mathcal{C}_4 \\ &+ \frac{1}{256}\left(\frac{\mathcal{X}}{\mathcal{N}}\right)^{\mathcal{N}+1} \left(\frac{\mathcal{Y}}{\mathcal{M}}\right)^{\mathcal{M}+1} \\ &\times \left(\frac{Z}{\mathcal{K}}\right)^{\mathcal{K}+1} \left(\frac{T}{\mathcal{L}}\right)^{\mathcal{L}+1} \mathcal{C}_5 + ||\bar{A} - A||_{L_2} \\ &\left(\Sigma_{i=0}^{N} \Sigma_{j=0}^{\mathcal{M}} \Sigma_{K=0}^{K} \Sigma_{l=0}^{\mathcal{L}} \frac{\mathcal{X}\mathcal{Y}ZT}{(2i+1)(2j+1)(2k+1)(2l+1)} \right)^{\frac{1}{2}}. \end{aligned} \tag{21}$$

where $\tilde{u} = u(\varkappa, y, z, t) - u_{\mathcal{N},\mathcal{M},\mathcal{K},\mathcal{L}}(\varkappa, y, z, t)$.

5 | Numerical Analysis

Two numerical examples of 3D-TSFRADE are solved with the SLGLCM and illustrate the validity and applicability of the approach. At t_m, the maximum absolute error (MAE) is evaluated as

$$MAE = \max_{1\leq i\leq \mathcal{N}, 1\leq j\leq \mathcal{M}, 1\leq k\leq \mathcal{K}} \left| u_E(\varkappa_i, y_j, z_k, t_m) - u_{\mathcal{N},\mathcal{M},\mathcal{K}}(\varkappa_i, y_j, z_k, t_m) \right|,$$

where u_E and $u_{\mathcal{N},\mathcal{M},\mathcal{K}}$ are the exact and approximate solutions, respectively.

Next, our aim is to validate the proposed numerical scheme while applying on the following two examples which are particular cases of our considered mathematical model.

Example 1. Let us consider the 3D-TSFRADE (1) on the domain $(\varkappa, y, z) \in [0,1] \times [0,1] \times [0,1]$, $t \in [0,1]$, with initial, boundary conditions and suitable forced function $f(x, y, z, t)$ whose solution is given by

$$u(\varkappa, y, z, t) = \frac{1}{4\pi} e^{-t} \varkappa^4 y^4 z^4 (t-1)(\varkappa - 1)(y-1)(z-1)(3\varkappa + 2y).$$

The above problem is solved for different values $(\mathcal{N}, \mathcal{M}, \mathcal{K}, \mathcal{L})$ and advection coefficients $\nu_1 = \nu_2 = \nu_3 = 0.5$, reaction coefficient $\lambda = -1$, and for different sets of values of $(\alpha, \beta_1, \beta_2, \beta_3) = (0.7, 0.8, 0.9, 0.8)$, $(\alpha, \beta_1, \beta_2, \beta_3) = (0.6, 0.7, 0.8, 0.7)$. Tables 1 and 2 represent that as $\mathcal{N}, \mathcal{M}, \mathcal{K}$, and $\mathcal{L}$ increase the errors decrease. Thus, it is claimed that the proposed method is computationally effective.

Example 2. Let us consider model (1) on the domain $(\varkappa, y, z) \in [0,1] \times [0,1] \times [0,1]$, $t \in [0,1]$, with suitable initial and boundary conditions, and forced function $f(x, y, z, t)$, whose solution is obtained as

$$u(\varkappa, y, z, t) = \frac{\sqrt{2}}{8} \varkappa^5 y^4 z^4 (1-\varkappa)^4 (1-y)^4 (1-z)^4 (1+\varkappa) t.$$

TABLE 1 | MAE with $\alpha = 0.7$, $\beta_1 = 0.8$, $\beta_2 = 0.9$ $\beta_3 = 0.8$ for Example 1.

t	$(\mathcal{N}, \mathcal{M}, \mathcal{K}, \mathcal{L}) = (3, 3, 3, 3)$	$(\mathcal{N}, \mathcal{M}, \mathcal{K}, \mathcal{L}) = (4, 4, 4, 4)$
0	1.55311×10^{-4}	2.12653×10^{-5}
0.2	9.701×10^{-5}	7.95557×10^{-6}
0.4	1.33191×10^{-4}	8.50602×10^{-6}
0.6	1.27227×10^{-4}	4.18045×10^{-5}
0.8	1.36643×10^{-4}	5.23258×10^{-5}
1	1.88176×10^{-4}	7.46781×10^{-5}

TABLE 2 | MAE with $\alpha = 0.6$, $\beta_1 = 0.7$, $\beta_2 = 0.8$ $\beta_3 = 0.7$ for Example 1.

t	$(\mathcal{N}, \mathcal{M}, \mathcal{K}, \mathcal{L}) = (3, 3, 3, 3)$	$(\mathcal{N}, \mathcal{M}, \mathcal{K}, \mathcal{L}) = (4, 4, 4, 4)$
0	1.05409×10^{-4}	2.71996×10^{-5}
0.2	8.40548×10^{-5}	1.57855×10^{-6}
0.4	1.67477×10^{-4}	2.17972×10^{-5}
0.6	2.20968×10^{-4}	3.73139×10^{-5}
0.8	2.84622×10^{-4}	4.80966×10^{-5}
1	3.98533×10^{-4}	1.9738×10^{-5}

TABLE 3 | MAE with $\alpha = 0.7$, $\beta_1 = 0.8$, $\beta_2 = 0.9$ $\beta_3 = 0.8$ for Example 2.

t	$(\mathcal{N},\mathcal{M},\mathcal{K},\mathcal{L}) = (\mathbf{3,3,3,3})$	$(\mathcal{N},\mathcal{M},\mathcal{K},\mathcal{L}) = (\mathbf{4,4,4,4})$
0	7.91382×10^{-7}	8.84448×10^{-8}
0.2	1.82392×10^{-6}	5.0162×10^{-8}
0.4	1.81351×10^{-6}	1.2152×10^{-7}
0.6	1.35146×10^{-6}	1.64348×10^{-7}
0.8	1.02907×10^{-6}	2.63936×10^{-7}
1	1.43765×10^{-6}	4.26753×10^{-7}

TABLE 4 | MAE with $\alpha = 0.3$, $\beta_1 = 0.9$, $\beta_2 = 0.8$ $\beta_3 = 0.9$ for Example 2.

t	$(\mathcal{N},\mathcal{M},\mathcal{K},\mathcal{L}) = (\mathbf{3,3,3,3})$	$(\mathcal{N},\mathcal{M},\mathcal{K},\mathcal{L}) = (\mathbf{4,4,4,4})$
0	1.50162×10^{-7}	7.99382×10^{-8}
0.2	2.91421×10^{-7}	2.66628×10^{-8}
0.4	5.71217×10^{-7}	8.09524×10^{-8}
0.6	8.28802×10^{-7}	1.64182×10^{-8}
0.8	1.10277×10^{-6}	1.83289×10^{-8}
1	2.3524×10^{-6}	8.37918×10^{-8}

Example 2 is solved by applying the proposed method for different sets of values $(\mathcal{N},\mathcal{M},\mathcal{K},\mathcal{L})$ and advection coefficients $v_1 = v_2 = v_3 = 0$, reaction coefficient $\lambda = 0$, and for different sets of values of $(\alpha, \beta_1, \beta_2, \beta_3) = (0.7,0.8,0.9,0.8)$,$(\alpha, \beta_1, \beta_2, \beta_3) = (0.3,0.9,0.8,0.9)$. Tables 3 and 4 show that the errors decrease when $\mathcal{N}, \mathcal{M}, \mathcal{K}$, and $\mathcal{L}$ increase.

6 | Conclusion

In this article, a numerical method called SLGLCM is used for solving the 3D-TSFRADE with prescribed initial and boundary conditions. This method effectively reduces the 3D-TSFRADE to a system of algebraic equations, which can be easily solved. The most important feature is the error analysis between the results obtained by the proposed numerical approach and the existing results which exhibit the efficiency and effectiveness of the scheme. High precision and easy implementation are the most significant advantages of these techniques. Keeping these are in view, the authors claim that a wide range of PDEs having physical importances can be solved with the help of the proposed numerical method.

Author Contributions

Manish Chopra: investigation, formal analysis, validation, visualization. **Subir Das:** supervision, investigation, writing – review and editing, validation, conceptualization, formal analysis.

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Conflicts of Interest

The authors declare no conflicts of interest.

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