Chapter 2

Mathematical preliminaries

Throughout the thesis, the notation used adheres to standard conventions. The symbol $\mathbb{R}$ represents the set of real numbers, while $\mathbb{R}_+$ denotes the set of positive real numbers, i.e., $\mathbb{R}_+ = \{z \in \mathbb{R} \mid z > 0\}$. Similarly, $\mathbb{R}_{>k}$ and $\mathbb{R}_{\geq k}$ represent the sets of real numbers greater than k and greater than or equal to k, respectively, where k is a real constant. The notation $\mathbb{R}^n$ denotes the n-dimensional Euclidean field. For any $p \geq 1$, the l_p-norm of a vector $y \in \mathbb{R}^n$ is defined as $\|y\|_p = (\sum_{i=1}^{n} |y_i|^p)^{\frac{1}{p}}$. When p is not explicitly mentioned, the norm is assumed to be the $2-$norm, denoted as $|y|$ which represents $\|y\|_2$. For matrices, we represent the transpose of matrix A as $A^\top$. Moreover, for a given vector $a = [a_1, a_2, \ldots, a_n] \in \mathbb{R}^n$, $\mathrm{diag}(a)$ denotes the diagonal matrix with elements $a_i \in \mathbb{R}$, where $i = 1, 2, ..., n$. In the context of a given set S, $\min\{S\}$ represents the smallest element within the set, whereas $\sup\{S\}$ denotes the supremum, which is the least upper bound, over the elements of S. If S_1, S_2, and S_3 are any sets, and we have two functions: $g_1 : S_1 \to S_2$ and $g_2 : S_2 \to S_3$, then their composition, denoted as $g_2 \circ g_1 : S_1 \to S_3$, is defined as $(g_2 \circ g_1)(\cdot) = g_2(g_1(\cdot))$. An n-dimensional open ball B_δ with radius δ is defined as the set of points in n-dimensional Euclidean space that are at a distance less than δ from a fixed point z_0. Mathematically, it can be represented as $B_\delta = \{z \in \mathbb{R}^n : \|z - z_0\| < \delta\}$, where z_0 is the fixed point.

To comprehend the findings presented in this study, it is essential for the reader to possess a fundamental comprehension of the language and terminology commonly employed in the analysis of stability in continuous-time dynamical systems. The definitions and concepts outlined in this chapter play a crucial role throughout the rest of this dissertation. The majority of the background information in this section has been adapted and rephrased from Hassan Khalil's well-known textbook, "Nonlinear Systems" [105], with appropriate citations provided to highlight the source of this material.

2.1 State space models for continuous-time dynamical systems

State space models are a mathematical representation of continuous-time dynamical systems. They describe the behavior of these systems in terms of a set of state variables and their derivatives. Indeed, many continuous-time dynamical systems can be modeled by a finite number of coupled first-order differential equations. This modeling approach decomposes the system into multiple variables, where each variable represents a state of the system. The general form of the state equation model for such systems can be expressed as

$$\begin{aligned}
\dot{z}_1(t) &= f_1(t, z_1(t), z_2(t), \ldots, z_n(t), u(t)) \\
\dot{z}_2(t) &= f_2(t, z_1(t), z_2(t), \ldots, z_n(t), u(t)) \\
\vdots \quad & \quad \vdots \\
\dot{z}_n(t) &= f_n(t, z_1(t), z_2(t), \ldots, z_n(t), u(t))
\end{aligned}$$

where $\dot{z}_i(t)$ represents the derivative of the i-th state variable $z_i(t)$ with respect to time t. $f_i(t, z_1(t), z_2(t), \ldots, z_n(t), u(t))$ represents the dynamics of the system, which depend on the current states $z_1(t), z_2(t), \ldots, z_n(t)$, the input $u(t)$ applied to the system at time t, and time t itself. By formulating the system as a set of coupled first-order differential equations, we can analyze and simulate its behavior using numerical integration techniques. Additionally, control strategies can be designed based on the state equation model to regulate and optimize the system's performance. This approach is widely used in various fields, including control systems, robotics, physics, and engineering, as it provides a flexible and modular framework for modeling and analyzing the dynamics of continuous-time systems.

A general form of the above state equation model for a continuous-time dynamical system can be represented as

$$\dot{z}(t) = f(t, z(t), u(t)) \tag{2.1}$$

where $\dot{z}(t) \in \mathbb{R}^n$ represents the derivative of the state vector $z(t) \in \mathbb{R}^n$ with respect to time t. $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a vector-valued function that defines the dynamics of the system. When the input $u(t) \in \mathbb{R}^m$ is absent or when $u(t) = \gamma(z(t))$ with $\gamma : \mathbb{R}^n \to \mathbb{R}^m$, equation (2.1) can be rewritten in terms of the state variables. The modified equation can be expressed as

$$\dot{z}(t) = f(t, z(t)) \tag{2.2}$$

which is called the unforced state equation. Furthermore, if the function f does not explicitly depend on t, i.e.,

$$\dot{z}(t) = f(z(t)) \tag{2.3}$$

the system is referred to as *autonomous* or *time-invariant*. Conversely, if f explicitly depends on t, then the system is considered *nonautonomous* or *time-varying*.

2.2 Comparison functions

We recall some definitions related to comparison functions that will be helpful in providing transparent definitions of stability.

Definition 2.1 ***(Class $\mathcal{K}$ function)*** *[105] A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is class $\mathcal{K}$ function, if it is continuous, strictly increasing and $\alpha(0) = 0$. In addition, if $\lim_{s \to \infty} \alpha(s) = \infty$, then it is $\mathcal{K}_\infty$ function.*

Definition 2.2 ***(Class $\mathcal{L}$ function)*** *[105] A function $\iota : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be class $\mathcal{L}$ function, if it is continuous, strictly decreasing and $\lim_{s \to \infty} \iota(s) = 0$.*

Definition 2.3 ***(Class $\mathcal{KL}$ function)*** *[105] A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class $\mathcal{KL}$ function if it is a class $\mathcal{K}$ function in its first argument and a class $\mathcal{L}$ function in its second argument.*

Lemma 2.4 *[105] Let α_1 and α_2 be class $\mathcal{K}$ functions on $[0, a)$, α_3 and α_4 be a class $\mathcal{K}_\infty$ functions, and β be a class $\mathcal{KL}$ function. Denote the inverse of α_i by α_i^{-1}. Then,*

- *α_1^{-1} is defined on $[0,\ \alpha_1(a))$ and belongs to class $\mathcal{K}$.*
- *α_3^{-1} is defined on $[0,\ \infty)$ and belongs to class $\mathcal{K}_\infty$.*

- $\alpha_1 \circ \alpha_2$ *belongs to class* $\mathcal{K}$.
- $\alpha_3 \circ \alpha_4$ *belongs to class* $\mathcal{K}_\infty$.

2.3 Stability notions

The main contributions of this work are centered around the stability and stabilization of nonlinear systems. In the upcoming section, we provide a concise overview of various stability concepts that are relevant to this study.

2.3.1 Equilibrium points

Consider the nonautonomous system

$$\dot{z}(t) = f(t, z(t)), \quad z(t_0) = z_0 \tag{2.4}$$

where $z(t) \in \mathbb{R}^n$ is the system state. The function $f\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ is nonlinear and satisfies $f(t, 0) = 0$, indicating that the origin $z(t) = 0$ serves as an equilibrium point for (2.4). $t_0 \geq 0$ denotes the initial time, and z_0 represents the initial state. Let us recall the existence theorem from Carathéodory [106]: It is known that if for all $z(t) \in \mathbb{R}^n$ and $t \in [t_0, \infty)$ (i) $f(t, z(t))$ is continuous in $z(t)$ for all fixed t, (ii) measurable in t for all fixed $z(t)$, and (iii) $\|f(t, z(t))\| \leq \mu(t)$, where $\mu(t)$ is integrable over $|t - t_0| \leq \alpha$, then for some $\alpha_1 > 0$, there exists a solution $z(t; z_0, t_0)$ for $|t - t_0| \leq \alpha_1$.

In this work, our focus is on studying equilibria at the origin. This choice is motivated by the fact that any nonzero equilibrium of the system can be transformed to the origin through a change of coordinates [105].

2.3.2 Stability concepts for general nonlinear systems

The definitions provided below are extracted from Chapter 4 of Khalil's *Nonlinear Systems* [105] and are widely recognized as standard definitions in the field of studying the stability of continuous-time dynamical systems.

Definition 2.5 *The equilibrium point* $z(t) = 0$ *of* (2.4) *is*

- *stable if, for every* $\epsilon > 0$, *there exists* $\delta = \delta(\epsilon, t_0) > 0$ *such that*

$$|z(t_0)| < \delta \implies |z(t)| < \epsilon, \quad \forall\ t \geq t_0 \geq 0 \tag{2.5}$$

- *unstable if it is not stable.*
- *uniformly stable if, for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$, independent of t_0, such that* (2.5) *is satisfied.*
- *asymptotically stable if it is stable and there exists a positive constant $k = k(t_0)$ such that $z(t) \to 0$ as $t \to \infty$, for all $|z(t_0)| < k$.*
- *uniformly asymptotically stable if it is uniformly stable and there exists a positive constant k, independent of t_0, such that for all $|z(t_0)| < c$, $z(t) \to 0$ as $t \to \infty$, uniformly in t_0; that is, for every $\eta > 0$, there exists $T = T(\eta) > 0$ such that*

$$|z(t)| < \eta, \quad \forall\ t \geq t_0 + T(\eta), \quad |z(t_0)| < k \tag{2.6}$$

- *globally uniformly asymptotically stable if it is uniformly stable, $\delta(\epsilon)$ can be chosen to satisfy $\lim_{\epsilon\to\infty} \delta(\epsilon) = \infty$, and, for each pair of positive numbers η and k, there exists $T = T(\eta, k) > 0$ such that*

$$|z(t)| < \eta, \quad \forall\ t \geq t_0 + T(\eta, c), \quad |z(t_0)| < k \tag{2.7}$$

Stability based on comparison functions

By employing the comparison functions outlined previously, we introduce stability concepts that serve as the building blocks for the more sophisticated stability notions that will be explored later. These initial stability concepts provide a fundamental framework for comprehending and analyzing the stability properties of dynamical systems.

Definition 2.6 *The equilibrium point $z(t) = 0$ of* (2.4) *is*

- *uniformly stable if and only if there exists a class $\mathcal{K}$ function β and a positive constant k, independent of t_0, such that*

$$|z(t)| \leq \beta(|z(t_0)|), \quad \forall\ t \geq t_0 \geq 0, \quad \forall\ |z(t_0)| < k \tag{2.8}$$

- *uniformly asymptotically stable if and only if there exists a class $\mathcal{KL}$ function β and a positive constant k, independent of t_0, such that*

$$|z(t)| \leq \beta(|z(t_0)|, t - t_0), \quad \forall\ t \geq t_0 \geq 0, \quad \forall\ |z(t_0)| < k \tag{2.9}$$

- *globally uniformly asymptotically stable if and only if it is uniformly asymptotically stable for any initial condition* $z(t_0)$.

Definition 2.7 *The equilibrium point* $z(t) = 0$ *of* (2.4) *is exponentially stable if there exist positive constants* k, c, λ *such that*

$$|z(t)| \leq c|z(t_0)|e^{-\lambda(t-t_0)}, \quad \forall\ |z(t_0)| < k \tag{2.10}$$

and globally exponentially stable if it is exponentially stable for any initial state $z(t_0)$.

The preceding stability notions are applicable to unforced systems. However, when an input is present, the following stability notion becomes more useful.

Definition 2.8 *The system* (2.1) *where* f *is piecewise continuous in* t *and locally Lipschitz in* $z(t)$ *and* $u(t)$, *is said to be input-to-state stable (ISS) if there exist a class* $\mathcal{KL}$ *function* β *and a class* $\mathcal{K}$ *function* γ, *such that for any initial state* $z(t_0)$, *and for any bounded input* $u(t)$, *the solution* $z(t)$ *exists for all* $t \geq t_0$ *and satisfies*

$$|z(t)| \leq \beta(|z(t_0)|, t - t_0) + \gamma\left(\sup_{t_0 \leq \tau \leq t} |u(\tau)|\right), \quad \forall\ 0 \leq t_0 \leq t \tag{2.11}$$

Definition 2.9 ***(Global finite-time stability)*** *[10] The origin of the system* (2.4) *is said to be globally finite-time stable if it is globally asymptotically stable and any solution* $z(t, t_0, x_0)$ *of* (2.4) *converges to the origin at some finite time, i.e.,* $\forall\, t \geq t_0 + T(t_0, z_0)$, $z(t, t_0, z_0) = 0$, *where* $T\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, *is the settling time function.*

Definition 2.10 ***(Fixed-time stability)*** *[10] The origin of the system* (2.4) *is said to be fixed-time stable if it is globally finite-time stable and the settling time function is bounded, i.e.,* $\exists\, T_{max} > 0 : \forall\, z_0 \in \mathbb{R}^n$ *and* $\forall\, t_0 \in \mathbb{R}_{\geq 0}, \quad T(t_0, z_0) \leq T_{max}$.

Definition 2.11 ***(Prescribed-time stability)*** *[17] The origin of the system* (2.4) *is said to be prescribed-time stable in time* t_p *if it is globally uniformly asymptotically stable and there exists a function* $\psi : [0, t_p) \to \mathbb{R}_+$ *with* ψ *increasing to* ∞ *as* $t \to t_p$ *and a class* $\mathcal{KL}$ *function* β *such that*

$$\|z(t)\| \leq \beta(\|z(t_0)\|, \psi(t)), \quad \forall\ t \in [0, t_p) \tag{2.12}$$

where t_p *is a finite number that can be prescribed by the designer.*

Remark 2.12 *It is important to note that the selection of $\psi(t)$, as defined in Definition 2.11, offers a range of options available to the designer. By appropriately specifying $\psi(t)$, the proposed control method allows for achieving prescribed finite time control with various convergence behaviors. Throughout this dissertation, we have employed different prescribed-time adjustment functions for prescribed-time control [17, 81]. This flexibility in design is a notable advantage of the proposed prescribed finite-time control method. For instance, the possible choices for $\psi(t)$ are:*

$$\psi(t) = \frac{\gamma}{t_p + t_0 - t}, \quad \psi(t) = \frac{t_p}{t_p + t_0 - t}, \quad \psi(t) = e^{\frac{t_p}{t_p + t_0 - t} - 1}, \quad \psi(t) = \left(\frac{t_p}{t_p + t_0 - t}\right)^{q^*}$$

where $\gamma \in \mathbb{R}_{\geq 1}$ and $q^* \geq 2$ being an integer.

Remark 2.13 *Please note that while there are subtle differences among prescribed-time, predefined-time, and arbitrary-time stability, this thesis will utilize the prescribed-time notion for all three types of stability concepts [107].*