

## Chapter 4

### Finite $C3$ Modules

An  $R$ -module  $N$  is called  $C2$  module if every submodule of  $N$  that is isomorphic to a direct summand of  $N$  is itself a direct summand of  $N$ . In 2019, Maurya and Gupta [39] generalized the class of  $C2$  modules to the class of finite  $C2$  modules. In this chapter, we introduce the new notion of Finite  $C3$  modules which serves as a generalization of the concept of finite  $C2$  modules and  $C3$  modules. We provide an example of a finite  $C3$  module that is not a finite  $C2$  module. Also, we provide several new characterizations of hereditary rings,  $V$ -rings, von Neumann regular rings,  $SI$ -rings, and semi-simple Artinian rings in respect of finite  $C3$  modules. We also introduced the concept of finite  $C3$  envelopes and finite  $C3$  covers of a module.

## 4.1 Finite C3 Modules

**Definition 4.1.1.** *An  $R$ -module  $N$  is called a finite C3 module if  $A$  and  $B$  are finitely generated direct summands of  $N$  with zero intersection, then  $A \oplus B$  is again a summand of  $N$ .*

**Example 4.1.2.** (i)  $\mathbb{Z}$  as  $\mathbb{Z}$ -module is finite C3.

(ii) Finite C2 modules and C3 modules are finite C3.

(iii) A module that has no finitely generated submodule or no finitely generated summand is a finite C3 module.

(iv) A module whose finitely generated submodule is a summand is a finite C3 module.

**Proposition 4.1.3.** *Let  $N$  be a finitely generated  $R$ -module. Then  $N$  is finite C3 if and only if it is a C3 module.*

*Proof.* Let  $N$  be a finite C3 module and  $K_1, K_2 \leq^\oplus N$  such that  $K_1 \cap K_2 = 0$ . As  $N$  is finitely generated,  $K_1$  and  $K_2$  are finitely generated and therefore  $K_1 \oplus K_2$  is also finitely generated, which implies  $K_1 \oplus K_2 \leq^\oplus N$  since  $N$  is finite C3. Thus,  $N$  is C3 module.

The converse easily follows from the definition of a finite C3 module that every C3 module is a finite C3 module.  $\square$

**Remark 4.1.4.** *The converse part of the above proposition holds without  $N$  being finitely generated. As we know every C3 module is a finite C3 module.*

**Proposition 4.1.5.** *Let  $N = N_1 \oplus N_2$  be a finite C3 module, where  $N_1$  and  $N_2$  are submodules of  $N$  and  $\phi : N_1 \rightarrow N_2$  be a homomorphism with  $\text{Ker}(\phi) \leq^\oplus N_1$  and  $\text{Im}(\phi)$  is finitely generated in  $N_2$ . Then  $\text{Im}(\phi) \leq^\oplus N_2$ .*

*Proof.* It follows easily from [Proposition 2.3, [2]].  $\square$

**Corollary 4.1.6.** *Let  $N = N_1 \oplus N_2$  be a finite C3 module, where  $N_1$  and  $N_2$  are submodules of  $N$  and  $\phi : N_1 \rightarrow N_2$  be a monomorphism and  $\text{Im}(\phi)$  is finitely generated in  $N_2$ . Then  $\text{Im}(\phi) \leq^\oplus N_2$ .*

**Corollary 4.1.7.** *Let  $N$  be an  $R$ -module. If  $N \oplus E(N)$  is finite C3 module then  $N$  is injective.*

*Proof.* Let  $i : N \rightarrow E(N)$  be the inclusion map. By Corollary 4.1.6,  $i(N) = N \leq^\oplus E(N)$  it implies  $N = E(N)$ . Hence  $N$  is injective.  $\square$

In the next proposition, the following statements show an equivalent condition of Proposition 4.1.5 and Corollary 4.1.7 for the direct sum of  $R$ -modules.

**Proposition 4.1.8.** *For an  $R$ -module  $N = N_1 \oplus N_2$  where  $N_1, N_2$  be finitely generated submodules of  $N$ , the following statements are equivalent:*

1. *If  $N$  is a finite C3 module and a homomorphism  $f : N_1 \rightarrow N_2$  with  $\text{Ker}(f) \leq^\oplus N_1$ , then  $\text{Im}(f) \leq^\oplus N_2$ ;*
2. *If  $N$  is a finite C3 module and a monomorphism  $f : N_1 \rightarrow N_2$ , then  $\text{Im}(f) \leq^\oplus N_2$ .*

*Proof.* (1  $\Rightarrow$  2) It is clear.

(2  $\Rightarrow$  1) Let a homomorphism  $f$  from  $N_1$  to  $N_2$  with  $\text{Ker}(f) \leq^\oplus N_1$ . Then  $N = (\text{Ker}(f) \oplus K) \oplus N_2$  for some submodule  $K \leq N_1$ . Now,  $f|_K : K \rightarrow N_2$  is a monomorphism such that  $\text{Im}(f|_K) = \text{Im}(f)$ . Hence from (2),  $\text{Im}(f) \leq^\oplus N_2$ .  $\square$

**Proposition 4.1.9.** *If  $N$  is a finite C3 module with  $K \leq^\oplus N$ , then  $K$  is also a finite C3 module.*

*Proof.* Let  $N$  be a finite C3 module and  $K \leq^{\oplus} N$ . Let  $K_1, K_2$  be finitely generated direct summands of  $K$  such that  $K_1 \cap K_2 = 0$ . As  $K_1, K_2$  are finitely generated direct summands,  $K_1 \oplus K_2$  is finitely generated submodule in  $K$ . Since,  $K_1 \oplus K_2$  is finitely generated in  $K \leq N$ ,  $K_1 \oplus K_2$  is finitely generated in  $N$ . As  $N$  is finite C3 module, it follows that  $K_1 \oplus K_2$  is summand of  $N$  i.e.,  $N = (K_1 \oplus K_2) \oplus L$  for some submodule  $L$  of  $N$ . By modular law,  $N \cap K = [(K_1 \oplus K_2) \oplus L] \cap K \Rightarrow K = (K_1 \oplus K_2) \oplus [L \cap K]$  (as  $K_1 \oplus K_2 \leq K$ ). So,  $(K_1 \oplus K_2) \leq^{\oplus} K$ . Hence,  $K$  is a finite C3 module.  $\square$

**Theorem 4.1.10.** *Any finite C2 (finite direct injective) module  $N$  is finite C3.*

*Proof.* Let  $A$  and  $B$  be any finitely generated direct summands of  $N$  with  $A \cap B = 0$ . Consider  $N = A \oplus X$  for a submodule  $X \subseteq N$ , and let  $p$  be a canonical projection from  $A \oplus X$  to  $X$ . Then  $p(B) \cong B \leq^{\oplus} N$ ,  $p(B) \leq^{\oplus} N$  ( $N$  is finite C2 module). As,  $p(B) \leq^{\oplus} N$  and  $p(B) \leq X$ , it follows that  $X = p(B) \oplus L$ . So,  $p(B) \leq^{\oplus} X$ . Thus  $N = A \oplus X = A \oplus p(B) \oplus L = (A \oplus B) \oplus L$  for a submodule  $L \leq N$ . Hence,  $N$  is finite C3 module.  $\square$

It is natural to ask whether the converse of the above Proposition 4.1.10 is true. The answer to this question is no. We illustrate this by the following example.

**Example 4.1.11.** *Let  $\mathbb{Z}$  as  $\mathbb{Z}$ -module. We know that it is a C3 module, so it is a finite C3 module, but since  $2\mathbb{Z} \cong \mathbb{Z}$  which is finitely generated submodule of  $\mathbb{Z}$  but not a summand of  $\mathbb{Z}$ . So, it is not a finite direct injective module. Hence by the above example, we can say that the finite C3 module is a proper generalization of the finite direct injective module.*

It is interesting to notice that the direct sum of finite C3 modules need not be finite C3. The following example shows this:

**Example 4.1.12.** 1. Let  $N_1 = \mathbb{Z}_2$ ,  $N_2 = \mathbb{Z}_4$  be  $\mathbb{Z}$ -modules.  $N_1$  and  $N_2$  are finite C3 modules because both do not have finitely generated direct summands of  $N_1$  and  $N_2$  modules. However,  $N = N_1 \oplus N_2$  is not a finite C3 module. For instance, if we take  $f : N_1 \rightarrow N_2$  such that  $f(x) = 2x$ , then  $\text{Ker}(f) = \{0\} \leq^\oplus N_1$  but  $\text{Im}(f) = 2\mathbb{Z}_4 \not\leq^\oplus N_2$ . Hence, by (Proposition 4.1.5),  $N$  is not finite C3.

2. Let  $S$  be a simple module which is not injective. Then  $S$  and its injective hull  $E(S)$  are finite C3 modules whereas  $S \oplus E(S)$  is not finite C3 by Proposition 4.1.7.

Now, it's natural to ask under what conditions the direct sum of finite C3 modules is finite C3.

**Proposition 4.1.13.** *If  $N \oplus N$  is a C3 module then  $N$  is a finite C3 module.*

*Proof.* By Proposition 2.14, [39],  $N$  is a finite C2. So, by Proposition 4.1.10,  $N$  is a finite C3 module.  $\square$

**Proposition 4.1.14.** *Any direct sum of injective modules is a finite C3 module.*

*Proof.* Let  $N = \bigoplus_{i \in I} E_i$  be a direct sum of injective modules. Let  $A$  and  $B$  be any finitely generated direct summands of  $N$  with  $A \cap B = 0$ . Since  $A$  and  $B$  are finitely generated then  $A \leq^\oplus (\bigoplus_{i \in F} E_i)$  and  $B \leq^\oplus (\bigoplus_{i \in F} E_i)$  for some finite  $F \subset I$ . As we know, the finite direct sum of injective modules is injective, it implies  $A$  and  $B$  are injective. So,  $A \oplus B$  is injective then  $A \oplus B \leq^\oplus \bigoplus_{i \in I} E_i$ . Hence,  $A \oplus B \leq^\oplus N$ .  $\square$

**Proposition 4.1.15.** *Let  $R$  be a ring such that every finite C3 module is a C3 module then  $R$  must be a Noetherian ring.*

*Proof.* As we know, for a Noetherian ring  $R$ , any direct sum of injective  $R$ -modules is injective. Let  $N$  be a direct sum of injective  $R$ -modules. So, by Proposition 4.1.14,  $N \oplus E(N)$  is a finite  $C3$  module. Now, by the given hypothesis,  $N \oplus E(N)$  is  $C3$  module. Now, the inclusion map  $i : N \rightarrow E(N)$  splits i.e.  $N = E(N)$  implies  $N$  is injective. Thus,  $R$  is a Noetherian ring.  $\square$

## 4.2 Characterization of rings using finite $C3$ modules

This section presents several interesting characterizations of finite  $C3$  modules. In the following proposition, we obtain a new characterization of hereditary rings in terms of finite  $C3$  modules.

**Theorem 4.2.1.** *For a ring  $R$ , the following conditions are equivalent:*

1.  $R$  is hereditary;
2. Every factor module of an injective module is a finite  $C3$  module;
3. The sum of two injective submodules is a finite  $C3$  module.

*Proof.* (1)  $\Rightarrow$  (2) By the given hypothesis, every factor module of an injective module is injective, and hence every injective is finite  $C3$ .

(2)  $\Rightarrow$  (1). Let  $N$  be an injective module and  $K \leq N$ . Since  $N \oplus E(N/K)$  is injective, by the given hypothesis  $(N \oplus E(N/K))/(K \oplus 0) = N/K \oplus E(N/K)$  is finite  $C3$  module. Now, By Corollary 4.1.7,  $N/K$  is the injective module. Hence, (1) holds.

(2)  $\Rightarrow$  (3). Let  $N$  be an  $R$ -module and  $K_1, K_2$  be injective submodules of  $N$ . As we know  $K_1 + K_2 \cong (K_1 \oplus K_2)/K$  for some submodule  $K$  of  $K_1 \oplus K_2$  (By surjective morphism from  $K_1 \oplus K_2$  to  $K_1 + K_2$ ). As  $K_1 \oplus K_2$  is injective. By the given hypothesis,  $(K_1 \oplus K_2)/K$  is finite C3. So,  $K_1 + K_2$  is finite C3 module.

(3)  $\Rightarrow$  (2). Let  $N$  be an injective  $R$ -module and  $K \leq N$ . Let  $f : N \rightarrow N/K \oplus E(N)$  with  $f(n) = (n + K, n)$ . Here  $f$  is a monomorphism and  $f(N)$  is an injective  $R$ -module. As  $f(N) = N/K \oplus N$  and  $0 \oplus E(N)$  are injective, so  $N/K \oplus E(N)$  is a finite C3 module by the given hypothesis. Now, by Proposition 4.1.9,  $N/K$  is a finite C3 module.

□

**Proposition 4.2.2.** *For a ring  $R$ , the following conditions are equivalent.*

1.  $R$  is a  $V$ -ring;
2. Every cofinitely generated  $R$ -module is finite C3.

*Proof.* (1)  $\Rightarrow$  (2) Let  $R$  be a  $V$ -ring. Then every cofinitely generated  $R$ -module is injective (Theorem 23.1, [57]). Since the injective module implies finite C3, the cofinitely generated  $R$ -module is finite C3.

(2)  $\Rightarrow$  (1) Let  $N$  be a simple  $R$ -module. Then  $N \oplus E(N)$  is finitely cogenerated [Proposition 10.7,[3]]. By the given hypothesis,  $N \oplus E(N)$  is a finite C3 module. Hence, by Corollary 4.1.7,  $N$  is an injective module. Since every simple  $R$ -module is injective,  $R$  is a right  $V$ -ring.

□

**Theorem 4.2.3.** *For a ring  $R$ , the following conditions are equivalent:*

1.  $R$  is a von Neumann regular ring;

2. Every finitely generated projective  $R$ -module is finite C3;
3. Every finitely generated projective  $R$ -module is dual rickart;
4. Every finitely generated projective  $R$ -module has SSP property;
5. Every finitely generated submodule of a finitely generated projective  $R$ -module is a summand.

*Proof.* (1)  $\Rightarrow$  (2) Let  $R$  be a regular ring and  $N$  be a finitely generated projective  $R$ -module. Since,  $N$  is a finitely generated projective  $R$ -module over the regular ring, every finitely generated submodule of  $N$  is a summand. Hence  $N$  is a finite C3 module.

(2)  $\Rightarrow$  (3) Let  $N$  be a finitely generated projective  $R$ -module and  $S = \text{End}_R(N)$ . For an element  $s \in S$ ,  $N \oplus s(N)$  is finitely generated projective, so by assumption it is finite C3. Now, by Corollary 4.1.7,  $s(N) \leq^\oplus N$ . Hence  $N$  is dual Rickart.

(3)  $\Rightarrow$  (4) Every dual Rickart module satisfies the SSP property by [Proposition 2.11,[33]]. So (4) holds.

(4)  $\Rightarrow$  (5) Let  $K$  be a finitely generated projective  $R$ -module such that  $K \leq N$ ,  $K \oplus N$  has SSP property. So  $K \oplus N$  is finite C3 module by Corollary 4.1.6,  $K \leq^\oplus N$ .

(5)  $\Rightarrow$  (1) As  $R$  is a finitely generated projective  $R$ -module. By the given hypothesis, Every principal ideal of  $R$  is a summand of  $R$ . Hence,  $R$  is a regular ring.  $\square$

In the next proposition, we give a new characterization of  $SI$ -rings in respect of finite C3 modules.

**Proposition 4.2.4.** *For a ring  $R$ , the following conditions are equivalent :*

1.  $R$  is a  $SI$ -ring;

2. Every singular  $R$ -module is finite C3-module.

*Proof.* (1)  $\Rightarrow$  (2) By definition of  $SI$ -ring, every singular  $R$ -module is injective. Hence, finite C3.

(2)  $\Rightarrow$  (1) Let  $N$  be a singular  $R$ -module. So,  $N \oplus E(N)$  is a singular module. By the given hypothesis,  $N \oplus E(N)$  is a finite C3 module. Now, by Corollary 4.1.7,  $N$  is injective. Thus, Every singular  $R$ -module is injective. Hence,  $R$  is  $SI$ -ring.  $\square$

**Proposition 4.2.5.** *For a ring  $R$ , the following conditions are equivalent:*

1.  $R$  is a semi-simple Artinian;
2. Every  $R$ -module is finite C3.

*Proof.* (1)  $\Rightarrow$  (2) Let  $N$  be an  $R$ -module. So every submodule of  $N$  is a summand of  $N$  as  $R$  is a semi-simple Artinian ring. So, every finitely generated submodule is a summand. Hence  $N$  is a finite C3.

(2)  $\Rightarrow$  (1) Let  $K$  be a cyclic  $R$ -module. As every  $R$ -module is finite C3,  $K \oplus E(N)$  is a finite C3 module. Hence,  $K$  is injective by Corollary 4.1.7, which implies  $R$  is semi-simple Artinian (Osofsky Theorem, [50]).  $\square$

In the next proposition, we prove that a ring for which every finite C3 module is endoregular must be a semisimple Artinian ring.

**Theorem 4.2.6.** *For a ring  $R$ , the following statements are equivalent:*

1.  $R$  is a semi-simple Artinian;
2. Every finite C3  $R$ -module is endoregular;

3. Every C3  $R$ -module is endoregular;

4. Every injective  $R$ -module is endoregular.

*Proof.* (1)  $\Rightarrow$  (2) It follows from (Proposition 2.17 [35]).

(2)  $\Rightarrow$  (3)  $\Rightarrow$  (4) Clear

(4)  $\Rightarrow$  (1) It follows from (Proposition 2.18 [35]).  $\square$

Motivated by [2], and [4], the notion of a Finite C3 Envelope and Finite C3 Cover of a module are introduced here.

### 4.3 Finite C3 Envelopes and Finite C3 Covers of a Module

In this section, we define the finite C3 envelopes and finite C3 covers of a module and study their properties.

**Definition 4.3.1.** An  $R$ -homomorphism  $f : N \longrightarrow E$  is called a finite C3 envelope of an  $R$ -module  $N$ , if  $E$  is a finite C3 module, and  $g : N \longrightarrow F$  such that any diagram

$$\begin{array}{ccc} N & \xrightarrow{f} & E \\ g \downarrow & \nearrow h & \\ & & F \end{array}$$

with  $F$  as a finite C3 module, can be completed by a homomorphism  $h : E \longrightarrow F$ , and the diagram

$$\begin{array}{ccc} N & \xrightarrow{f} & E \\ g \downarrow & \nearrow k & \\ & & E \end{array}$$

can be completed by an automorphism  $k : E \rightarrow E$ .

**Remark 4.3.2.** *The injective  $R$ -module implies the finite C3 envelope of an  $R$ -module.*

**Proposition 4.3.3.** *The following statements are equivalent:*

1. *Every finite C3-module is injective;*
2. *Every  $R$ -module has a finite C3 envelope;*
3. *The direct sum of any two finite C3 modules is a finite C3 module.*

*Proof.* (1)  $\Rightarrow$  (2) As we know, every  $R$ -module has an injective envelope. By the given hypothesis, every finite C3 module is injective. Hence, every  $R$ -module has a finite C3 envelope.

(2)  $\Rightarrow$  (1) Let  $N$  be a finite C3 module and  $f : N \oplus E(N) \rightarrow F$  be a finite C3 envelope of  $N \oplus E(N)$ . Now,  $\pi_1 : N \oplus E(N) \rightarrow N$  and  $\pi_2 : N \oplus E(N) \rightarrow E(N)$  be canonical projections. Since  $N$  and  $E(N)$  be finite C3 modules so there exist homomorphisms  $\theta_1 : F \rightarrow N$  and  $\theta_2 : F \rightarrow E(N)$  such that the following diagrams commute.

$$\begin{array}{ccc} N \oplus E(N) & \xrightarrow{\mu} & F \\ \pi_1 \downarrow & \swarrow \theta_1 & \\ N & & \end{array}$$

$$\begin{array}{ccc} N \oplus E(N) & \xrightarrow{\mu} & F \\ \pi_2 \downarrow & \swarrow \theta_2 & \\ E(N) & & \end{array}$$

Now, from diagrams  $\theta_1\mu = \pi_1$  and  $\theta_2\mu = \pi_2$ . Also define  $\theta : F \rightarrow N \oplus E(N)$  by  $\theta(x) = \theta_1(x) + \theta_2(x)$  such that  $\theta\mu = id_F$ , it implies  $\mu$  is a monomorphism, and  $N \oplus E(N) \cong Im(\mu) \leq^\oplus F$ . By Proposition 4.1.9,  $N \oplus E(N)$  is finite C3 module. Thus,  $N$  is an injective module by Corollary 4.1.7.

(1)  $\Rightarrow$  (3) Let  $N_1$  and  $N_2$  be two finite C3 modules. By the given hypothesis,  $N_1$  and  $N_2$  are injective then  $N_1 \oplus N_2$  is an injective module. Hence,  $N_1 \oplus N_2$  is finite C3 module.

(3)  $\Rightarrow$  (1) Let  $N$  be a finite C3 module, by the given hypothesis  $N \oplus E(N)$  is finite C3. By Corollary 4.1.7,  $N$  is injective.

□

**Theorem 4.3.4.** *The following statements are equivalent for a module  $N$  over a ring  $R$ :-*

1.  $R$  is a semisimple Artinian ring;
2. Every 2-generated  $R$  module has a finite C3 envelope.

*Proof.* (1)  $\Rightarrow$  (2) Let  $N$  be a 2-generated  $R$ -module over a semi-simple Artinian ring. Then  $N$  is a semi-simple module. Hence  $N$  is a finite C3 module and its envelope is trivial.

(2)  $\Rightarrow$  (1) We will prove that every simple module over ring  $R$  is injective. Let  $N$  be a simple right  $R$ -module and  $T = N \oplus E(N)$ . Then  $T$  is 2-generated, it has a finite C3 envelope, so there exist a homomorphism  $g : T \rightarrow Y$  a finite C3 envelope such that the following diagrams:

$$\begin{array}{ccc} T & \xrightarrow{g} & Y \\ \phi \downarrow & \swarrow h & \\ E(N) & & \end{array}$$

$$\begin{array}{ccc}
 T & \xrightarrow{g} & Y \\
 \phi' \downarrow & \nearrow h' & \\
 N & & 
 \end{array}$$

commutes where  $\phi$  and  $\phi'$  are the canonical projections. Since both  $N$  and  $E(N)$  are finite C3 modules and  $g : N \oplus E(N) \rightarrow Y$  is a finite C3 envelope, there exist  $R$ -homomorphisms  $h : Y \rightarrow E(N)$  and  $h' : Y \rightarrow N$  with  $h \circ g = \phi$  and  $h' \circ g = \phi'$ . Define  $f : Y \rightarrow T$  by  $f(x) = h(x) + h'(x)$ ,  $x \in Y$ . Clearly,  $h$  is a well-defined  $R$ -homomorphism,  $f \circ g = i_T$  and  $f : Y \rightarrow T$  is a split  $R$ -epimorphism. Thus  $T \leq^{\oplus} Y$ . Since  $Y$  is finite C3, we infer that  $T = N \oplus E(N)$  is also finite C3. By Proposition 4.1.5, the inclusion map  $i_N : N \rightarrow E(N)$  splits; consequently,  $N$  is injective.  $\square$

**Definition 4.3.5.** An  $R$ -homomorphism  $f : C \rightarrow N$  is called a finite C3 cover of a module  $N$  if  $C$  is a finite C3 module such that any diagram.

$$\begin{array}{ccc}
 & & C \\
 & \nearrow g & \downarrow f \\
 C' & \xrightarrow{h} & N
 \end{array}$$

with  $C'$  is a finite C3 module can be completed

$$\begin{array}{ccc}
 & & C \\
 & \nearrow g & \downarrow f \\
 C & \xrightarrow{h} & N
 \end{array}$$

and above diagram commutes where  $g : C \rightarrow C'$  automorphism.

**Proposition 4.3.6.** Let  $N$  be a finitely generated C3 module, then it has a finite C3 cover.

---

*Proof.* Every finitely generated  $C3$  module is finite  $C3$  module. As every finite  $C3$  module has a finite  $C3$  cover. Hence,  $N$  has a finite  $C3$  cover.  $\square$

**Proposition 4.3.7.** *If  $R$  is a  $V$ -ring, then every cofinitely generated  $R$ -module has a finite  $C3$  cover.*

*Proof.* From Proposition 4.2.2, Every cofinitely generated  $R$ -module is finite  $C3$ . As every finite  $C3$  module has a finite  $C3$  cover. Hence, every module has a finite  $C3$  cover.  $\square$