Chapter 2

An asymptotic expansion for a Lambert series associated to Siegel cusp forms of degree 2

In this chapter, we derive an exact formula for certain Lambert series involving Fourier-Jacobi coefficients of Siegel cusp forms of degree 2 . We also give its asymptotic expansion.

2.1 Introduction

Let $\zeta(s)$ and $\mu(n)$ denote the Riemann zeta function and the Möbius function, respectively. It is well-known that the reciprocal of the Riemann zeta function is the Dirichlet series associated to $\mu(n)$. Ramanujan, in his second notebook (see [61, p. 312] and [10, p. 470, eq. 37.3]) mentioned the following identity associated to $\mu(n)$:

For a positive real number x,

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \exp\left(-\frac{x}{n^2}\right) = \sqrt{\frac{\pi}{x}} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \exp\left(-\frac{\pi^2}{n^2 x}\right). \tag{2.1}$$

However, the above identity is not correct. The corrected version of the identity (2.1) was obtained by Hardy and Littlewood [27, p. 156]. By assuming that all the non-trivial zeros of $\zeta(s)$ are simple, they proved that the following identity holds:

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\exp\left(-\frac{1}{\pi}\left(\frac{\alpha}{n}\right)^2\right)-\sqrt{\beta}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\exp\left(-\frac{1}{\pi}\left(\frac{\beta}{n}\right)^2\right)$$
$$=-\frac{\sqrt{\pi}}{2\sqrt{\beta}}\sum_{\rho}\frac{\Gamma\left(\frac{1-\rho}{2}\right)}{\zeta'(\rho)}\left(\frac{\beta}{\sqrt{\pi}}\right)^{\rho}, \tag{2.2}$$

where α and β are positive real numbers with $\alpha\beta = 1$, and the sum runs over all the non-trivial zeros ρ of $\zeta(s)$. In the same paper, Hardy and Littlewood mentioned that they were unable to prove the convergence of this series even after assuming the Riemann hypothesis. However, in an indirect way, they showed that the series in (2.2) is convergent under the assumption of the following bracketing condition on the non-trivial zeros of $\zeta(s)$, namely, the non-trivial zeros ρ_1 and ρ_2 of $\zeta(s)$ are included in the same bracket if they satisfy

$$|\Im(\rho_1)-\Im(\rho_2)| < \exp\left(-\frac{A_0\Im(\rho_1)}{\log(\Im(\rho_1))}\right)+\exp\left(-\frac{A_0\Im(\rho_2)}{\log(\Im(\rho_2))}\right), \tag{2.3}$$

where A_0 is some positive constant. The identity (2.2) also inspired Hardy and Littlewood to obtain the following equivalent criterion for the Riemann hypothesis (RH):

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\exp\left(-\frac{x}{n^2}\right)=O\left(x^{-\frac{1}{4}+\epsilon}\right), \quad \text{as } x\to\infty. \tag{2.4}$$

Over the time, the above equivalent criterion for RH has inspired mathematicians to find different generalizations of (2.2) and (2.4) for L-functions associated to various objects. For more details on this topic, we refer to [1, 9, 19, 20, 21, 24, 25]. We would like to mention here that we encounter similar infinite series in the main result of this chapter (and that of chapters 3 and 4) analogous to the series appearing in (2.2).

The famous Ramanujan tau function $\tau(n)$ was introduced and extensively studied by Ramanujan in his seminal paper [60]. It is the n-th Fourier coefficient of the Ramanujan delta function $\Delta(\tau)$ which is the unique cusp form of weight 12 for the full modular group $\mathrm{SL}_2(\mathbb{Z})$ with Fourier series expansion given in (1.9). Based on the numerical values of $\tau(n)$, Ramanujan observed several properties (congruences, recursive relations and an estimate) for $\tau(n)$, which were later proved by mathematicians over the time. In particular, Ramanujan conjectured that $\tau(n)$ satisfies the estimate $\tau(n) = O\left(n^{11/2+\epsilon}\right)$ and this was proved by Deligne in 1974.

Rankin (1939) and Selberg (1940), independently developed a method to prove the analytic continuation of automorphic L-functions (having Euler product representation) which decay rapidly, widely known as the Rankin-Selberg method. More precisely, they proved that the Mellin transform of the constant term of the Fourier series of an automorphic function of rapid decay, can be expressed as scalar product of the automorphic function in consideration with certain non-holomorphic Eisenstein series. Note that cusp forms are functions of rapid decay but modular forms that are not cusp forms need not be rapidly decaying. Zagier [71] extended the Rankin-Selberg method to the automorphic functions which are not of rapid decay. In the same paper, Zagier conjectured the asymptotic behaviour of the Lambert series associated to Ramanujan tau function.

2.2 Zagier's conjecture

In this section, we explain the conjecture of Zagier. For more details we refer to [71].

Let $\mathcal{F}$ be a continuous complex-valued function defined on $\mathbb{H}$ which is invariant under $\Gamma = SL_2(\mathbb{Z})$ and has the following Fourier series expansion:

$$\mathcal{F}(\tau) = \sum_{n=-\infty}^{\infty} a_n(v)e(nu), \quad \tau = u + iv. \tag{2.5}$$

The usual non-holomorphic Eisenstein series for the group $SL_2(\mathbb{Z})$ is defined by

$$E(\tau, s) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \Im(\gamma\tau)^s, \quad \tau \in \mathbb{H}, \; s \in \mathbb{C} \text{ with } \Re(s) > 1. \tag{2.6}$$

Using the standard unfolding trick, one obtains the following:

$$\int_{\Gamma\backslash\mathbb{H}} \mathcal{F}(\tau)E(\tau, s)d\mu = \int_0^\infty a_0(v)v^{s-2}dv, \tag{2.7}$$

where $d\mu = v^{-2}dudv$ is an invariant volume element. If $\mathcal{F}(\tau) \to 0$ as $\tau \to \infty$, then the integral (2.7) is absolutely convergent for $\Re(s) > 1$. Next, for $\Re(s) > 1$, define the functions $R(\mathcal{F}, s)$ and $R^*(\mathcal{F}, s)$ as follows:

$$R(\mathcal{F}, s) := \int_0^\infty a_0(v)v^{s-2}dv, \tag{2.8}$$

$$R^*(\mathcal{F}, s) := \Lambda(2s)R(\mathcal{F}, s). \tag{2.9}$$

The function $R(\mathcal{F}, s)$ can be holomorphically continued to the whole complex plane except for simple poles at $s = 1$ and $s = \frac{\rho}{2}$ for all the non-trivial zeros ρ of $\zeta(s)$. Now, (2.7) together with (2.8) yields

$$\operatorname*{Res}_{s=1} R(\mathcal{F}, s) = \frac{3}{\pi} \int_{\Gamma\backslash\mathbb{H}} \mathcal{F}(\tau)d\mu. \tag{2.10}$$

In particular, if we take $\mathcal{F}(\tau) = v^{12}|\Delta(\tau)|^2$, where $\Delta(\tau)$ is the Ramanujan delta cusp form, then the constant term of $\mathcal{F}(\tau)$ is given by the following Lambert series:

$$a_0(v) = v^{12} \sum_{n=1}^{\infty} \tau^2(n) \exp(-4\pi n v). \tag{2.11}$$

This gives

$$\frac{(4\pi)^s}{\Gamma(s)} R(\mathcal{F}, s-11) = \sum_{n=1}^{\infty} \frac{\tau^2(n)}{n^s}. \tag{2.12}$$

Conjecture (Zagier's Conjecture): Zagier [[71]: p. 417, [72]: p. 271] speculated that the constant term of the automorphic function $\mathcal{F}(\tau) := v^{12}|\Delta(u+iv)|^2$, i.e., the Lambert series given in (2.11) associated to $\tau(n)$ has its asymptotic expansion in terms of the non-trivial zeros of $\zeta(s)$ and satisfies the following asymptotic expansion:

$$a_0(v) \sim \mathcal{A} + \sum_{\rho} v^{1-\frac{\rho}{2}} A_\rho, \tag{2.13}$$

where the sum runs over the non-trivial zeros ρ of $\zeta(s)$, the constant A_ρ depends on ρ and $\mathcal{A} = \frac{3}{\pi}\langle \Delta, \Delta \rangle$.

Further, under the assumption of the Riemann hypothesis, he claimed that the constant term $a_0(v)$ has an oscillatory behaviour when $v \to 0^+$. The following oscillatory behaviour of $a_0(v)$ was also predicted by Zagier:

$$a_0(v) \sim C + v^{\frac{3}{4}} \sum_{m=1}^{\infty} a_m \cos\left(\frac{1}{2}\gamma_m \log(v) + \phi_m\right) \qquad \text{as } v \to 0^+, \tag{2.14}$$

where $\gamma_m = \mathrm{Im}(\rho)$, a_m and ϕ_m are real constants.

The above conjecture of Zagier as well as the oscillatory behaviour was proved by Hafner and Stopple [26] by studying a certain heat kernel associated with Ramanujan tau function. More precisely, Hafner and Stopple proved the following result

Theorem 2.1. *[26, p. 127, Corollary 2.3] Assume that all the non-trivial zeros of $\zeta(s)$ are simple. Then for $v \to 0^+$,*

$$v^{12}\sum_{n=1}^{\infty}\tau^2(n)\exp(-vn) = 12\Gamma(11) + v^{3/4}\sum_{\rho} v^{1/4-\rho/2}\Gamma\left(\frac{\rho}{2}+11\right)\frac{\zeta(\rho/2)}{\zeta'(\rho)}D_{\Delta}\left(\frac{\rho}{2}\right) + O(v^{3/2}), \tag{2.15}$$

where $D_{\Delta}(s)$ is the symmetric square L-function associated to the Ramanujan delta function and ρ runs through the non-trivial zeros of $\zeta(s)$.

Note that under the assumption of Riemann hypothesis Hafner and Stopple stated that the term $y^{1/4-\rho/2}$ is purely imaginary which shows oscillatory behaviour of (2.15), proving numerical calculation of Zagier given in (2.14).

Recently, Chakraborty, Kanemitsu and Maji [14] extended the result of Hafner and Stopple [26] for normalized Hecke eigenforms, where the authors proved an analogous result for a Lambert series associated to a normalized Hecke eigenform and obtained their asymptotic expansion. More precisely, they studied the following Lambert series:

$$v^k\sum_{n=1}^{\infty}|a_f^2(n)|\exp(-4\pi nv), \tag{2.16}$$

when $v \to 0^+$, and proved that Lambert series (2.16) can also be expressed in terms of the non-trivial zeros of $\zeta(s)$, where $a_f(n)$ denotes the n-th Fourier coefficient of a normalized Hecke eigenform f over $SL_2(\mathbb{Z})$. More precisely, they proved the following result.

Theorem 2.2. *[14, p. 295, Corollary 1] Let $f \in S_k(SL_2(\mathbb{Z}))$ be a normalized Hecke eigenform with the Fourier series expansion given by $f(\tau) = \sum\limits_{n=1}^{\infty} a_f(n)e(n\tau)$. Assume* *that the non-trivial zeros of* $\zeta(s)$ *are simple and satisfy the bracketing condition* (2.3). *Then for* $v \to 0^+$,

$$\begin{aligned}\sum_{n=1}^{\infty} |a_f^2(n)| \exp(-4\pi v n) &= \frac{\Gamma(k) D_f(k)}{(4\pi v)^k \zeta(2)} + \sum_{\rho} \frac{\Gamma(\frac{\rho}{2}+k-1)\zeta(\frac{\rho}{2}) D_f(\frac{\rho}{2}+k-1)}{\zeta'(\rho)(4\pi v)^{\frac{\rho}{2}+k-1}} \\ &+ O\left(|v|^{\frac{3}{2}-k}\right), \qquad (2.17)\end{aligned}$$

where ρ *runs over the non-trivial simple zeros of* $\zeta(s)$.

Since then this problem has been studied for various automorphic forms. For more details about similar problems inspired from the above conjecture of Zagier, we refer to [2, 8, 14, 15, 34, 35, 53, 54]. The main aim of this chapter is to study the above mentioned problem for Siegel cusp forms of degree 2.

2.3 Statement of results

As already mentioned in the introduction, in this chapter we are interested in studying certain Lambert series associated to Siegel cusp forms of degree 2. More precisely, let $F, G \in S_k(\Gamma_2)$ with their Fourier Jacobi coefficients ϕ_m and ψ_m respectively, as given in (2.19). For $\alpha > 0$, we study the following Lambert series:

$$\sum_{m=1}^{\infty} \langle \phi_m, \psi_m \rangle \exp(-4\pi m \alpha). \qquad (2.18)$$

Before stating the main result of this chapter, we define the Rankin-Selberg type Dirichlet series associated to Siegel cusp forms of degree 2. Let $F, G \in S_k(\Gamma_2)$ with their Fourier-Jacobi expansions given by

$$F(Z) = \sum_{m=1}^{\infty} \phi_m(\tau, z) e(m\tau'), \qquad G(Z) = \sum_{m=1}^{\infty} \psi_m(\tau, z) e(m\tau'). \tag{2.19}$$

The Dirichlet series of Rankin-Selberg type associated to F and G is defined as

$$D_{F,G}(s) := \zeta(2s - 2k + 4) \sum_{m=1}^{\infty} \frac{\langle \phi_m, \psi_m \rangle}{m^s}. \tag{2.20}$$

We complete the Dirichlet series $D_{F,G}(s)$ as follows:

$$D^*_{F,G}(s) := (2\pi)^{-2s} \Gamma(s) \Gamma(s - k + 2) D_{F,G}(s).$$

The analytic properties of the Dirichlet series $D_{F,G}(s)$ have been studied by Kohnen and Skoruppa [40] and they proved the following result:

Theorem 2.3. *[40, p. 544, Theorem 1] The Dirichlet series $D_{F,G}(s)$ can be analytically continued to an entire function if $\langle F, G \rangle = 0$. However, if $\langle F, G \rangle \neq 0$, then $D_{F,G}(s)$ has a simple pole at $s = k$ with residue $\frac{4^k \pi^{k+2}}{(k-1)!} \langle F, G \rangle$. Further, it satisfies the following functional equation:*

$$D^*_{F,G}(s) = D^*_{F,G}(2k - s - 2). \tag{2.21}$$

We next define an arithmetical function $a_{F,G}(m)$ that satisfies the following generating function:

$$\sum_{m=1}^{\infty} \frac{a_{F,G}(m)}{m^s} = \frac{D_{F,G}(s)}{\zeta(2s + 1 - 2k)}. \tag{2.22}$$

In view of the Theorem 1.3 and (1.22), we see that the above series is absolutely convergent for $\Re(s) > k$. We now state the main result of this chapter.

Theorem 2.4. *[6, Theorem 3.1] Let F and G be two Siegel cusp forms of weight k and degree 2 with their Fourier-Jacobi expansions as given in (2.19) and satisfying (1.22). Let α and β be positive real numbers with $\alpha\beta = 1$. Under the assumption of the simplicity hypothesis of the non-trivial zeros of $\zeta(s)$, we have*

$$\sum_{m=1}^{\infty} \langle \phi_m, \psi_m \rangle \exp(-4\pi m\alpha) = \frac{\beta^{2k-2}}{\pi^{3/2}} \sum_{m=1}^{\infty} \frac{a_{F,G}(m)}{(4\pi m\beta)^{\frac{k-1}{2}}} \exp(-2\pi m\beta) W_{\frac{k}{2},\frac{k}{2}-1}(4\pi m\beta) + R_k + \sum_{\rho} \frac{\Gamma\left(\frac{\rho}{2}+k-2\right) D_{F,G}\left(\frac{\rho}{2}+k-2\right)(4\pi\alpha)^{2-k-\frac{\rho}{2}}}{\zeta'(\rho)},$$

where $W_{a,b}$ is the Whittaker function, the sum runs over non-trivial zeros ρ of $\zeta(s)$ and satisfy the bracketing condition (2.3), and the term R_k is given by

$$R_k = \begin{cases} 0, & \text{if } \langle F, G \rangle = 0, \\ \frac{90\langle F,G\rangle}{\pi^2 \alpha^k}, & \text{if } \langle F, G \rangle \neq 0. \end{cases} \tag{2.23}$$

Remark 2.1. In the case of Siegel modular forms of degree one, the Fourier coefficients coincide with the Fourier-Jacobi coefficients. In both the theorems, i.e., Theorem 2.2 and Theorem 2.4, the asymptotic expansions of Lambert series in consideration are expressed in terms of the non-trivial zeros of Riemann zeta function as mentioned in the Zagier's conjecture. Thus if we take $F = G$, Theorem 2.4, can be seen as generalization of Theorem 2.2 to degree two Siegel modular forms.

As an application of the above theorem, we have the following asymptotic expansion of the Lambert series defined in (2.18).

Corollary 2.1. *[6, Corollary 3.2] Let F and G be as in Theorem 2.4. If $\langle F, G \rangle \neq 0$, then for $\alpha \to 0^+$ we have*

$$\alpha^k \sum_{m=1}^{\infty} \langle \phi_m, \psi_m \rangle \exp(-4\pi m\alpha) \sim \frac{90\langle F, G \rangle}{\pi^2}. \tag{2.24}$$

2.4 Proof of results

In this section, we present a proof of Theorem 2.4. We first give a brief outline of the proof. We first express the Lambert series $\sum_{m=1}^{\infty}\langle \phi_m, \psi_m\rangle \exp(-4\pi m\alpha)$ in terms of the Dirichlet series $D_{F,G}(s)$ utilizing Inverse Mellin's transform. Next we use the analytic properties of the Dirichlet series $D_{F,G}(s)$ and method of contour integration to calculate the residual terms. Making appropriate change of variable and reducing the existing integral to the Meijer G-function yields the required result.

Proof of Theorem 2.4. Using Mellin transform of $\Gamma(s)$ and the definition of $D_{F,G}(s)$ we see that for $\Re(s) > k$,

$$\Gamma(s)D_{F,G}(s) = \int_0^{\infty} \zeta(2s-2k+4)\sum_{m=1}^{\infty}\langle \phi_m, \psi_m\rangle \exp(-mx)x^{s-1}dx. \tag{2.25}$$

Now, applying inverse Mellin transform of the Gamma function $\Gamma(s)$, we see that the inverse Mellin transform of the function $\frac{\Gamma(s)D_{F,G}(s)(4\pi)^{-s}}{\zeta(2s-2k+4)}$ is the infinite series $\sum\limits_{m=1}^{\infty}\langle \phi_m, \psi_m\rangle \exp(-4\pi m\alpha)$. Therefore, for some large positive constant $c > k$, we have

$$\sum_{m=1}^{\infty}\langle \phi_m, \psi_m\rangle \exp(-4\pi m\alpha) = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s)D_{F,G}(s)(4\pi\alpha)^{-s}}{\zeta(2s-2k+4)}ds. \tag{2.26}$$

Throughout the proof the symbol (c) denotes the line integral $c - i\infty$ to $c + i\infty$. Now we recall from Theorem 2.3 that the Dirichlet series $D_{F,G}(s)$ can be extended to an entire function if $\langle F, G\rangle = 0$. However, if $\langle F, G\rangle \neq 0$, then $D_{F,G}(s)$ has a simple pole at $s = k$ with residue $\frac{4^k\pi^{k+2}}{(k-1)!}\langle F, G\rangle$. Moreover, from the functional equation of $D_{F,G}(s)$ it is easy to deduce that the poles of $\Gamma(s)$ are neutralized by $D_{F,G}(s)$. Nevertheless, the non-trivial zeros of $\zeta(2s-2k+4)$ will give us infinitely many poles of the integrand function in the strip $k-2 < \Re(s) < k-\frac{3}{2}$, and the trivial zeros of $\zeta(2s-2k+4)$ are at $k-n$ for $n \geq 3$. Thus, we construct a closed rectangular contour $\mathcal{C}$ consisting of the end points $c-iT, c+iT, c_1+iT, c_1-iT$, where T is some large positive constant. Here, we take $c_1 \in (k-3, k-2)$, so that the non-trivial zeros of $\zeta(2s-2k+4)$ lie inside the contour $\mathcal{C}$ and the trivial zeros of $\zeta(2s-2k+4)$ lie outside the contour $\mathcal{C}$. Now, applying the Cauchy's residue theorem, we obtain

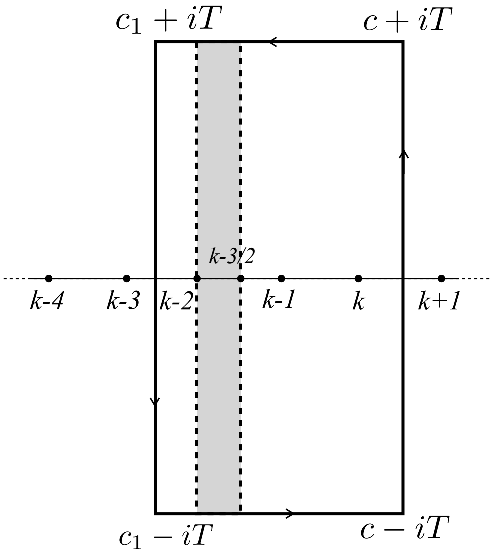

Figure 2.1: $\mathcal{C}$

$$\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\Gamma(s)D_{F,G}(s)(4\pi\alpha)^{-s}}{\zeta(2s-2k+4)}ds = R_k + \sum_{|\Im(\rho)|<T} R_\rho, \tag{2.27}$$

where R_k denotes the residual term at $s=k$ and R_ρ denotes the residual term at the non-trivial zero ρ of $\zeta(s)$. The residual term R_k is given by

$$R_k = \frac{\Gamma(k)(4\pi\alpha)^{-k}}{\zeta(4)} \operatorname*{Res}_{s\to k} D_{F,G}(s) = \frac{\Gamma(k)(4\pi\alpha)^{-k}}{\zeta(4)} \frac{4^k\pi^{k+2}}{(k-1)!}\langle F, G\rangle.$$

Thus,

$$R_k = \begin{cases} 0, & \text{if } \langle F, G\rangle = 0, \\ \frac{90\langle F,G\rangle}{\pi^2\alpha^k}, & \text{if } \langle F, G\rangle \neq 0. \end{cases} \tag{2.28}$$

Here, we have used the fact that $\zeta(4) = \pi^4/90$. Now we assume the simplicity hypothesis, which states that the non-trivial zeros of the Riemann zeta function $\zeta(s)$ are simple. This gives us the following evaluation of R_ρ:

$$R_\rho = \frac{\Gamma\left(\frac{\rho}{2}+k-2\right) D_{F,G}\left(\frac{\rho}{2}+k-2\right)(4\pi\alpha)^{2-k-\frac{\rho}{2}}}{\zeta'(\rho)}. \tag{2.29}$$

Next, we prove that the horizontal integrals H_1 and H_2 vanish as $T \to \infty$, where

$$H_1 := \frac{1}{2\pi i}\int_{c+iT}^{c_1+iT} \frac{\Gamma(s)D_{F,G}(s)(4\pi\alpha)^{-s}}{\zeta(2s-2k+4)}ds \tag{2.30}$$

and

$$H_2 := \frac{1}{2\pi i}\int_{c_1-iT}^{c-iT} \frac{\Gamma(s)D_{F,G}(s)(4\pi\alpha)^{-s}}{\zeta(2s-2k+4)}ds. \tag{2.31}$$

Writing $s = \sigma + iT$, (2.30) becomes

$$H_1 = \frac{1}{2\pi i}\int_{c+iT}^{c_1+iT} \frac{\Gamma(\sigma+iT)D_{F,G}(\sigma+iT)(4\pi\alpha)^{-(\sigma+iT)}}{\zeta(2(\sigma+iT)-2k+4)}d\sigma. \tag{2.32}$$

Thus,

$$|H_1| \ll \int_{c}^{c_1} \frac{|\Gamma(\sigma+iT)||D_{F,G}(\sigma+iT)|(4\pi\alpha)^{-\sigma}}{|\zeta(2\sigma-2k+4+2iT))|}d\sigma. \tag{2.33}$$

Employing Lemma 1.2 and using Stirling's bound (1.5) for $\Gamma(s)$ in (2.33), we obtain $H_1 \to 0$ as $T \to \infty$. Similarly, one can show $H_2 \to 0$ as $T \to \infty$. This shows that both the horizontal integrals will vanish. Finally, using (2.26), we obtain

$$\sum_{m=1}^{\infty}\langle\phi_m,\psi_m\rangle\exp(-4\pi m\alpha)=\frac{1}{2\pi i}\int_{(c_1)}\frac{\Gamma(s)D_{F,G}(s)(4\pi\alpha)^{-s}}{\zeta(2s-2k+4)}ds+R_k+\sum_{\rho}R_{\rho},\tag{2.34}$$

where the sum over ρ runs through the non-trivial zeros of $\zeta(s)$. It is known (from Hardy's theorem [66, p. 257]) that there exist infinitely many zeros of $\zeta(s)$. Therefore, the sum over ρ is an infinite sum. However, the convergence of this series is quite delicate as we do not know much about the lower bound for $\zeta'(s)$. Thus, under the assumption of convergence of the sum in (2.34) we proceed further. Now our main aim is to simplify the following left vertical integral:

$$V_k(\alpha):=\frac{1}{2\pi i}\int_{(c_1)}\frac{\Gamma(s)D_{F,G}(s)(4\pi\alpha)^{-s}}{\zeta(2s-2k+4)}ds.\tag{2.35}$$

From the functional equation of $D_{F,G}(s)$, i.e., Theorem 2.3, we obtain

$$\Gamma(s)D_{F,G}(s)=\frac{(2\pi)^{4s-4k+4}\Gamma(2k-s-2)\Gamma(k-s)D_{F,G}(2k-2-s)}{\Gamma(s-k+2)}.\tag{2.36}$$

Also, the functional equation of the Riemann zeta function (given in Theorem 1.2) implies that

$$\zeta(2s-2k+4)=\frac{\pi^{2s-2k+4}\Gamma\left(k-s-\frac{3}{2}\right)\zeta(2k-2s-3)}{\sqrt{\pi}\Gamma(s-k+2)}.\tag{2.37}$$

Now substituting (2.36) and (2.37) in (2.35), we obtain

$$V_k(\alpha)=\frac{\sqrt{\pi}}{16^{k-1}\pi^{2k}}\frac{1}{2\pi i}\int_{(c_1)}\frac{\Gamma(2k-2-s)\Gamma(k-s)D_{F,G}(2k-2-s)}{\Gamma\left(k-s-\frac{3}{2}\right)\zeta(2k-2s-3)}(4\pi\beta)^{s}ds,\tag{2.38}$$

where $\alpha\beta = 1$. To simplify further, we make a change of variable $w = 2k - 2 - s$ and upon simplification, we obtain

$$V_k(\alpha) = \frac{\beta^{2k-2}}{\pi^{3/2}} \frac{1}{2\pi i} \int_{(d_1)} \frac{\Gamma(w)\Gamma(w-k+2)D_{F,G}(w)(4\pi\beta)^{-w}}{\Gamma\left(w-k+\frac{1}{2}\right)\zeta(2w+1-2k)} dw. \tag{2.39}$$

It is easy to verify that $k < \Re(w) = d_1 < k+1$, as we have assumed $k-3 < \Re(s) = c_1 < k-2$. This allows us to write the infinite series expansions for $D_{F,G}(w)$ and $1/\zeta(2w+1-2k)$, as both the series are absolutely and uniformly convergent in the region $\Re(w) > k$. Let us write

$$\frac{D_{F,G}(w)}{\zeta(2w+1-2k)} = \sum_{m=1}^{\infty} \frac{a_{F,G}(m)}{m^w}, \quad \Re(w) > k. \tag{2.40}$$

Now plugging the above series expansion (2.40) in (2.39) and interchanging the summation and integration yields

$$V_k(\alpha) = \frac{\beta^{2k-2}}{\pi^{3/2}} \sum_{m=1}^{\infty} a_{F,G}(m) I_k(m,\beta), \tag{2.41}$$

where

$$I_k(m,\beta) := \frac{1}{2\pi i} \int_{(d_1)} \frac{\Gamma(w)\Gamma(w-k+2)(4\pi m\beta)^{-w}}{\Gamma\left(w-k+\frac{1}{2}\right)} dw. \tag{2.42}$$

Now invoking the Definition 1.6 with $m = 2, n = 0, p = 1, q = 2$ and $a_1 = \frac{1}{2} - k, b_1 = 0, b_2 = 2 - k$, we see that the above integral is the following Meijer G-function:

$$I_k(m,\beta) = G_{1,2}^{2,0}\left(\begin{matrix} \frac{1}{2} - k \\ 0, 2-k \end{matrix} \middle| 4\pi m\beta \right). \tag{2.43}$$

One can easily verify that the above Meijer G-function does satisfy all the convergence criteria. Now employing Lemma 1.5, we obtain

$$I_k(m,\beta) = (4\pi m\beta)^{\frac{1-k}{2}} W_{k/2,k/2-1}(4\pi m\beta)\exp(-2\pi m\beta). \tag{2.44}$$

Substituting (2.44) in (2.41), the final expression of the left vertical integral becomes

$$V_k(\alpha) = \frac{\beta^{2k-2}}{\pi^{3/2}} \sum_{m=1}^{\infty} a_{F,G}(m)(4\pi m\beta)^{\frac{1-k}{2}} W_{\frac{k}{2},\frac{k}{2}-1}(4\pi m\beta)\exp(-2\pi m\beta). \tag{2.45}$$

Now applying the asymptotic expansion of the Whittaker function (Lemma 1.4), one can easily verify the convergence of the above infinite series since the arithmetical function $a_{F,G}(m)$ has polynomial growth and the Whittaker function decays exponentially. Finally, considering the expression (2.45) of the left vertical integral $V_k(\alpha)$ in (2.34), together with residual terms (2.28) and (2.29), we complete the proof of Theorem 2.4. $\square$

Proof of Corollary 2.1. In Theorem 2.4, we have started with the assumption on α and β such that $\alpha\beta = 1$. Therefore, $\alpha \to 0^+$ implies that $\beta \to \infty$. Making use of Lemma 1.4 and multiplying by α^k, we obtain

$$\begin{aligned} &\frac{\alpha^k \beta^{2k-2}}{\pi^{3/2}} \sum_{m=1}^{\infty} \frac{a_{F,G}(m)}{(4\pi m\beta)^{\frac{k-1}{2}}} \exp(-2\pi m\beta) W_{\frac{k}{2},\frac{k}{2}-1}(4\pi m\beta) \\ &\ll \beta^{k-2} \sum_{m=1}^{\infty} \frac{a_{F,G}(m)}{(4\pi m\beta)^{\frac{k-1}{2}}} \exp(-4\pi m\beta)(4\pi m\beta)^{\frac{k}{2}} \ll \frac{1}{\beta^M} \end{aligned} \tag{2.46}$$

for some large positive number M. This shows that the above infinite series vanishes as $\beta \to \infty$. Again, multiplying by α^k in the infinite residual term in Theorem 2.4, which involves non-trivial zeros of $\zeta(s)$, we arrive at

$$\sum_{\rho} \frac{\Gamma\left(\frac{\rho}{2}+k-2\right) D_{F,G}\left(\frac{\rho}{2}+k-2\right)(4\pi)^{2-k-\frac{\rho}{2}}\alpha^{2-\frac{\rho}{2}}}{\zeta'(\rho)}. \tag{2.47}$$

It is well-known that the non-trivial zeros of $\zeta(s)$ lie in the critical strip $0 < \Re(s) < 1$. Therefore, we have $3/2 < \Re\left(2-\frac{\rho}{2}\right) < 2$, and this indicates that the above infinite sum (2.47) vanishes as $\alpha \to 0^+$. Thus, applying above two observations in Theorem 2.4 and considering the residual term (2.23), we complete the proof. □

2.5 Conclusion

In this chapter, we have studied the asymptotic behaviour of a Lambert series associated to Fourier Jacobi coefficients of Siegel cusp forms of weight k and degree 2; utilising Cauchy's residue theorem, analytic properties of associated Dirichlet series and bound of the defined Dirichlet series, bound for Riemann zeta function and Stirling's bound for Gamma function. Further, we have established the connection of defined Lambert series with non-trivial zeros of Riemann zeta function. To conclude the main result we have used particular value of Meijer G-function and asymptotic result of Meijer G-function in terms of Whittaker function.

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