Chapter 6

The analytical solutions of the Riemann problem to 1-D non-ideal flow of dusty gas with external force *

"Mathematics is the most beautiful
and most powerful creation of the human spirit."

–Stefan Banach.

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6.1 Introduction

The present chapter concerns with the analytical solution to the Riemann problem (RP) for a 1-D non-ideal flow of a dusty gas with friction term. There are numerous applications of the flow of dusty gas in physical sciences and modern engineering [167, 11, 113]. Generally, this is a mixture of gas and tinny solid particles. Also the particles are assumed to be identical in size and contribute less than 5% to the total volume of the mixture. Recent research studies have examined how dust particles present in gases affect the solution of gas dynamic equations [100, 168, 169]. Using ideal dusty gas, mixture of ideal polytropic gas and small solid particles, Pai et al. [112] studied similarity solution of the equations governing the shock wave motion. More information about shock wave motion in ideal dusty gas can be found in [113, 170, 171]. Vishwakarma and Nath [119, 172] have also analyzed the self-similar solution for a shock wave motion in a non-ideal dusty gas containing small solid particles. Further analysis was made by Nath [173, 174] on similarity solution of the equations governing the propagation of a spherical shock wave in non-ideal flow of dusty gas with gravitational field. In the applications of the Riemann Problem, mathematicians are continuously studying its solution for the various systems of PDEs and analyzing the behavior of the solution obtained for the system. In various flows, such as Granular flow, shallow water flow, traffic flow, and many other models, the Riemann Problem is continuously studied by ongoing research. Several analytical and numerical methods have been developed in the past to explore the solution of the RP for the system of PDEs [14, 36, 127]. Solving the Riemann problem for a dusty gas involves solving the system of PDEs with appropriate initial conditions. The solution to the Riemann problem provides valuable insight into the behavior of the dusty gas, such as shock wave formation, expansion waves, and rarefaction waves [142]. This inspires us to explore the RP for the flow of a dusty gas.

The conservative system for the non-ideal flow of dusty gas with external force is given as [175, 152]

$$\begin{cases} (\rho)_t + (\rho\nu)_x = 0, \\ (\rho\nu)_t + (\rho\nu^2 + p)_x = h\rho, \\ \left(\frac{\rho\nu^2}{2} + \rho e\right)_t + \left(\frac{\rho\nu^3}{2} + \rho\nu e + p\nu\right)_x = h\rho\nu, \end{cases} \tag{6.1}$$

where ν, ρ, and p denote the velocity, density, and pressure of non-ideal flow of dusty gas, respectively. t is time, x is spatial coordinate, and $h = h(t)$ represents the external force ($\neq 0$ continuous function of time). There have been many studies of the Riemann problem with different pressure laws in the system (6.1). As $p \equiv 0$, [176] presented a special solution to the Riemann problem for (6.1) that has become known as the delta shock wave solution, whereas Cheng [41] has completely shown the solution to this problem. In [18] Courant and Friedrichs studied the RP and shown that the ideal polytropic gas regulates the interaction of the elementary (fundamental) waves in equation (6.1). The formation of shock waves is one of the most interesting problem of non-linear wave theory in gasdynamics. Various researchers have conducted detailed investigations into the wave interaction phenomenon, including ([177, 178, 179, 180]). The process of wave interaction in various type of fields, such as shallow water, magnetogasdynamics, elastic solids, non-ideal gas, relaxing gas, etc., has been studied by the authors Radha et al. [181], Ruggeri [182], Virgopia and Ferraioli [183], Pandey and Sharma ([184, 185]). The RP for an isentropic case of the system (6.1) in an ideal dusty gas has also been explored by ([141, 142]). Then a quasilinear hyperbolic system of equations governing the 1-D unsteady flow of an ideal polytropic gas containing dust particle was solved by Nath et al. [170]. However, as demonstrated in [186], the assumption that the ideal gas is deceptive in low-density regions or high-temperature ranges, which encourages us to explore the non-ideal flow of dusty gas. The constant external force term (Coulomb-like friction term) appearing in the momentum equation of the model was used for the first time in [187]. The advantage of the source term appearing in the model (6.1) is that the nonhomogeneous model (6.1) can be reformulated into the homogeneous conservation form, which enables us to determine the solution of the RP for the model that causes to bent all the waves including shock wave, contact discontinuity, rarefaction wave and delta shock wave into the parabolic shape, and the solution of the RP for the model (6.1) is not self-similar solution.

The internal energy of the non-ideal flow of dusty gas is expressed as [119]

$$e = \frac{p(1-Z)}{\rho(\Gamma-1)(1+\bar{b}\rho)}. \tag{6.2}$$

Here, volume fraction Z and mass fraction k_p are connected as $Z = V_{sp}/V = \theta\rho$, where V is the total volume of mixture and V_{sp} is total volume of the small solid particles, and $\theta = k_p/\rho_{sp}$ is constant, with ρ_{sp} denoting the species density of small solid particles. $\bar{b} = (1-k_p)b$ with $k_p = m_{sp}/m$ (constant), where b is internal volume of molecules in gas, and m total mass of the mixture and m_{sp} total mass of the small solid particles. Γ is the Grüneisen coefficient, defined as $\Gamma = \gamma(1+\lambda\beta)/(1+\lambda\beta\gamma)$, with $\beta = C_{sp}/C_p, \lambda = k_p/(1-k_p)$, and $\gamma = C_p/C_v > 1$, is adiabatic index, C_{sp} is specific heat of solid particles, C_p is the specific heat at constant pressure and C_v is the specific heat at constant volume.

Using (6.2) in (6.1), the RP for non-ideal flow of dusty gas is obtained as

$$\begin{cases} (\rho)_t + (\rho\nu)_x = 0, \\ (\rho\nu)_t + (\rho\nu^2 + p)_x = h\rho, \\ \left(\frac{\rho\nu^2}{2} + \frac{p(1-Z)}{(\Gamma-1)(1+\bar{b}\rho)}\right)_t + \left(\frac{\rho\nu^3}{2} + \frac{p(1-Z)}{(\Gamma-1)(1+\bar{b}\rho)}\nu + p\nu\right)_x = h\rho\nu, \end{cases} \tag{6.3}$$

with the initial condition

$$(\rho, \nu, p)(0, x) = \begin{cases} (\rho_-, \nu_-, p_-), & x < 0, \\ (\rho_+, \nu_+, p_+), & x > 0, \end{cases} \tag{6.4}$$

where $\rho_\pm$, $\nu_\pm$ and $p_\pm$ are used as corresponding constant values. There are many discusions available in the literature, for homogeneous system ($h(t) = 0$) in (6.3) using the initial condition (6.4). In ([41, 50, 128]), the RP for the zero pressure homogeneous Euler equation is solved with the presence of vacuum and the delta shock. It has also been shown that there are weak global solutions by Brenier and Grenier [188] and Weinan et al. [158]. Using the viscous vanishing approach, Sheng and Zhang [130] determined the vacuum and delta shock waves in the Riemann solutions. See ([189, 190, 42, 191]) for more information about delta shock waves. With $h = -1$, Edwards et al. [192] solved the RP and described the mass tube by a delta shock wave. Additionally, Shen [139] got the Riemann solutions as $h = \beta$ (β = constant), and showed how the constant affects the Riemann solutions. Ding and Huang [193] used an Oleinik-type entropy criterion to prove the weak solution's uniqueness when $h = x$. With Coulomb-type friction, Chaturvedi and Singh [155] have derived the classical wave solutions for the logrithmic model and analyzed the solution's behavior. Recently, Shobhit et al. [152] studied the RP for a non-homogeneous 1-D compressible dusty gas flow system.

Now we consider modifying the state variable u, which is specified as [194]

$$u = \nu - H(t) \triangleq \nu - \int_0^t h(y)dy. \tag{6.5}$$

Equ. (6.3) can be reduced to the following conservative form by using (6.5):

$$\begin{cases} (\rho)_t + (\rho(u + H(t)))_x = 0, \\ (\rho u)_t + (\rho u(u + H(t)) + p)_x = 0, \\ \left(\frac{\rho\nu^2}{2} + \frac{p(1-Z)}{(\Gamma-1)(1+\bar{b}\rho)}\right)_t + \left((u + H(t))\left(\frac{\rho u^2}{2} + \frac{p(1-Z)}{(\Gamma-1)(1+\bar{b}\rho)}\right) + pu\right)_x = 0, \end{cases} \tag{6.6}$$

and the corresponding piecewise initial data is

$$(\rho, u, p)(0, x) = \begin{cases} (\rho_-, \nu_-, p_-), & x < 0, \\ (\rho_+, \nu_+, p_+), & x > 0. \end{cases} \tag{6.7}$$

This study investigates how non-ideal dusty gas flow affects the Riemann Problem solution in a non-homogeneous system. The presented chapter uses a modification to simplify the given original system and then determines the solution to the simplified system. Furthermore, it has also been observed that the curves of rarefaction, contact-discontinuity, and shock waves take a curved shape under the time-dependent external force. Therefore, the solutions are not self-similar. This chapter will discuss the following sections: In sect.6.2, the elementary wave curves corresponding to the Riemann invariants of the eigenvectors were analyzed and calculated. In sect.6.3, we evaluated the modified homogeneous system of equations' RP solution under specific conditions. The sect.6.4 deals with the given non-homogeneous system solution using the given initial condition, and we examine the solutions' structures. The final sect.6.5 of this chapter summarizes the complete examination of this work.

6.2 Behaviour of the Riemann solutions

The quasi-linear form of the system (6.6) is

$$W_t + QW_x = 0, \tag{6.8}$$

where

$$W = \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}, \text{ and } Q = \begin{bmatrix} u + H(t) & \rho & 0 \\ 0 & u + H(t) & 1/\rho \\ 0 & \frac{p(\Gamma(1+\bar{b}\rho)^2 - \bar{b}\rho^2(\theta+\bar{b}))}{(1-z)(1+\bar{b}\rho)} & u + H(t) \end{bmatrix}.$$

The eigenvalues of the above system can be obtained as

$$\begin{cases} \lambda_1 = u + H(t) - \sqrt{\frac{p(\Gamma(1+\bar{b}\rho)^2 - \bar{b}\rho^2(\theta+\bar{b}))}{\rho(1-z)(1+\bar{b}\rho)}}, \\ \lambda_2 = u + H(t) + \sqrt{\frac{p(\Gamma(1+\bar{b}\rho)^2 - \bar{b}\rho^2(\theta+\bar{b}))}{\rho(1-z)(1+\bar{b}\rho)}}, \\ \lambda_3 = u + H(t). \end{cases} \tag{6.9}$$

With respect to these eigenvalues, the right eigenvectors are

$$\begin{cases} r_1 &= \left(\rho, -\sqrt{\frac{p(\Gamma(1+\bar{b}\rho)^2 - \bar{b}\rho^2(\theta+\bar{b}))}{\rho(1-z)(1+\bar{b}\rho)}}, \frac{p(\Gamma(1+\bar{b}\rho)^2 - \bar{b}\rho^2(\theta+\bar{b}))}{(1-\theta\rho)(1+\bar{b}\rho)}\right)^T, \\ r_2 &= \left(\rho, \sqrt{\frac{p(\Gamma(1+\bar{b}\rho)^2 - \bar{b}\rho^2(\theta+\bar{b}))}{\rho(1-z)(1+\bar{b}\rho)}}, \frac{p(\Gamma(1+\bar{b}\rho)^2 - \bar{b}\rho^2(\theta+\bar{b}))}{(1-\theta\rho)(1+\bar{b}\rho)}\right)^T, \\ r_3 &= (1, 0, 0)^T. \end{cases} \tag{6.10}$$

We see that, $\nabla\lambda_i.r_i \neq 0$, $i = 1, 2$. Here, $\nabla = (\frac{\partial}{\partial\rho}, \frac{\partial}{\partial u}, \frac{\partial}{\partial p})$, which shows that the characteristics fields λ_1 and λ_2 are genuinely non-linear. In this instance, the waves are therefore either rarefaction or shock waves, which are symbolised by R and S, respectively. While the third characteristic field λ_3 is linearly degenerate. The

Riemann invariants (RI) for these characteristic fields are defined as

$$RI:\begin{cases}1:\left(u+\int_{\rho_-}^{\rho}\frac{1}{\rho}\sqrt{\frac{p(\Gamma(1+\bar{b}\rho)^2-\bar{b}\rho^2(\theta+\bar{b}))}{\rho(1-z)(1+\bar{b}\rho)}}d\rho,\ \frac{p}{1+\bar{b}\rho}(\frac{1}{\rho}-\theta)^{\Gamma}(1-\theta\rho)^{(\Gamma-1)\bar{b}/\theta}\right),\\ 2:\left(u-\int_{\rho_+}^{\rho}\frac{1}{\rho}\sqrt{\frac{p(\Gamma(1+\bar{b}\rho)^2-\bar{b}\rho^2(\theta+\bar{b}))}{\rho(1-z)(1+\bar{b}\rho)}}d\rho,\ \frac{p}{1+\bar{b}\rho}(\frac{1}{\rho}-\theta)^{\Gamma}(1-\theta\rho)^{(\Gamma-1)\bar{b}/\theta}\right),\\ 3:(u,p).\end{cases} \tag{6.11}$$

6.2.1 Continuous solutions

We consider self-similar solutions $U(t,x)=U(\chi)$ with $\chi=x/t$. Under the given transformation, both (6.6) and (6.7) are invariant. Hence, the systems (6.6)-(6.7) are converted to the equations

$$\begin{cases}-\chi(\rho)_\chi+(\rho(u+H(t)))_\chi=0,\\ -\chi(\rho u)_\chi+(\rho u(u+H(t))+p)_\chi=0,\\ -\chi\left(\frac{\rho u^2}{2}+\rho e\right)_\chi+\left(\left(\frac{\rho u^2}{2}+\rho e\right)(u+H(t))+pu\right)_\chi=0,\end{cases} \tag{6.12}$$

with

$$(\rho,u,p)(\pm\infty)=(\rho_\pm,u_\pm,p_\pm).$$

The equations (6.12) is determined in the following form to obtain a smooth solution

$$\begin{bmatrix}-\chi+u+H(t) & \rho & 0\\ 0 & \rho(-\chi+u+H(t)) & 1\\ 0 & \frac{p(\Gamma(1+\bar{b}\rho)^2-\bar{b}\rho^2(\theta+\bar{b}))}{(1-z)(1+\bar{b}\rho)} & -\chi+u+H(t)\end{bmatrix}\begin{bmatrix}\rho_\chi\\ u_\chi\\ p_\chi\end{bmatrix}=0. \tag{6.13}$$

On solving, it gives, besides constant state $(\rho, \nu)(\chi)$ = constant, $(\rho > 0)$, and a vacuum state

$$(\rho, u, p) = (0, u(\chi), 0), \tag{6.14}$$

Since $u(\chi)$ is an arbitrarily smooth function. The smooth solutions satisfy the expressions

$$\begin{cases} \chi = u + H(t) - \sqrt{\frac{p(\Gamma(1+\bar{b}\rho)^2 - \bar{b}\rho^2(\theta+\bar{b}))}{\rho(1-z)(1+\bar{b}\rho)}}, \\ \frac{du}{d\rho} = -\frac{1}{\rho}\sqrt{\frac{p(\Gamma(1+\bar{b}\rho)^2 - \bar{b}\rho^2(\theta+\bar{b}))}{\rho(1-z)(1+\bar{b}\rho)}}, \\ \frac{dp}{d\rho} = \frac{p(\Gamma(1+\bar{b}\rho)^2 - \bar{b}\rho^2(\theta+\bar{b}))}{\rho(1-z)(1+\bar{b}\rho)}, \end{cases} \tag{6.15}$$

or

$$\begin{cases} \chi = u + H(t) + \sqrt{\frac{p(\Gamma(1+\bar{b}\rho)^2 - \bar{b}\rho^2(\theta+\bar{b}))}{\rho(1-z)(1+\bar{b}\rho)}}, \\ \frac{du}{d\rho} = \frac{1}{\rho}\sqrt{\frac{p(\Gamma(1+\bar{b}\rho)^2 - \bar{b}\rho^2(\theta+\bar{b}))}{\rho(1-z)(1+\bar{b}\rho)}}, \\ \frac{dp}{d\rho} = \frac{p(\Gamma(1+\bar{b}\rho)^2 - \bar{b}\rho^2(\theta+\bar{b}))}{\rho(1-z)(1+\bar{b}\rho)}. \end{cases} \tag{6.16}$$

On integrating (6.15), using the left state $U_- = (\rho_-, u_-, p_-)$ and a given condition $\lambda_1(U_-) < \lambda_1(U)$, we have 1-rarefaction wave curves

$$R_1(p; U_-): \begin{cases} \chi = \lambda_1 = u + H(t) - \sqrt{\frac{p(\Gamma(1+\bar{b}\rho)^2 - \bar{b}\rho^2(\theta+\bar{b}))}{\rho(1-z)(1+\bar{b}\rho)}}, \\ u = \nu_- - \int_{\rho_-}^{\rho} \frac{1}{\rho}\sqrt{\frac{p(\Gamma(1+\bar{b}\rho)^2 - \bar{b}\rho^2(\theta+\bar{b}))}{\rho(1-z)(1+\bar{b}\rho)}} d\rho, \\ p = \frac{p_-}{1+\bar{b}\rho_-}(1+\bar{b}\rho)\left(\frac{\rho(1-\theta\rho_-)}{\rho_-(1-\theta\rho)}\right)^{\Gamma}\left(\frac{(1-\theta\rho_-)}{(1-\theta\rho)}\right)^{(\Gamma-1)\bar{b}/\theta}, \quad 0 \le p \le p_-. \end{cases} \tag{6.17}$$

Now, we will determine the value of ρ_{r_1}, u_{r_1} and p_{r_1} inside the 1-Rarefaction wave. Calculating the following initial value problem

$$\frac{dx}{dt} = \lambda_1(\rho_{r_1}, u_{r_1}, p_{r_1}), x(0) = 0,$$

we get,

$$\left(u_{r_1}+\sqrt{\frac{p(\Gamma(1+\bar{b}\rho)^2-\bar{b}\rho^2(\theta+\bar{b}))}{\rho(1-z)(1+\bar{b}\rho)}}\right)t = x-\int_0^t H(y)dy. \tag{6.18}$$

In the 1-rarefaction wave, suppose U_{r_1} represents the values of U on (t,x) point. So U_{r_1} is obtained as

$$\begin{cases} \rho_{r_1}=\rho_{r_1}(t,x), \;\; p_{r_1}=p(\rho_{r_1}), \\ u_{r_1}=\nu_- - \int_{\rho_-}^{\rho_{r_1}} \frac{1}{\rho}\sqrt{\frac{p_-(\Gamma(1+\bar{b}\rho)^2-\bar{b}\rho^2(\theta+\bar{b}))}{\rho(1-z)(1+\bar{b}\rho)}\left(\frac{\rho(1-\theta\rho_-)}{\rho_-(1-\theta\rho)}\right)^{\Gamma}\left(\frac{(1-\theta\rho_-)}{(1-\theta\rho)}\right)^{(\Gamma-1)\bar{b}/\theta}}d\rho, \end{cases} \tag{6.19}$$

where $\rho_{r_1}=\rho_{r_1}(t,x)$, is uniquely determined by

$$\begin{aligned} &\frac{x}{t}+\sqrt{\frac{p_-(\Gamma(1+\bar{b}\rho_{r_1})^2-\bar{b}\rho_{r_1}^2(\theta+\bar{b}))}{\rho_{r_1}(1-z)(1+\bar{b}\rho_{r_1})}\left(\frac{\rho_{r_1}(1-\theta\rho_-)}{\rho_-(1-\theta\rho_{r_1})}\right)^{\Gamma}\left(\frac{(1-\theta\rho_-)}{(1-\theta\rho_{r_1})}\right)^{(\Gamma-1)\bar{b}/\theta}}-\nu_- \\ &+\int_{\rho_-}^{\rho_{r_1}}\frac{1}{\rho}\sqrt{\frac{p_-(\Gamma(1+\bar{b}\rho)^2-\bar{b}\rho^2(\theta+\bar{b}))}{\rho(1-z)(1+\bar{b}\rho)}\left(\frac{\rho(1-\theta\rho_-)}{\rho_-(1-\theta\rho)}\right)^{\Gamma}\left(\frac{(1-\theta\rho_-)}{(1-\theta\rho)}\right)^{(\Gamma-1)\bar{b}/\theta}}d\rho \\ &-\frac{1}{t}\int_0^t H(y)dy=0. \end{aligned} \tag{6.20}$$

Similarly, under the restriction $\lambda_2(U_+)>\lambda_2(U)$, as a result of integrating (6.16) and using the right state $U_+=(\rho_+,u_+,p_+)$, 2-rarefaction wave curves are:

$$R_2(p;U_-):\begin{cases} \chi=\lambda_2=u+H(t)+\sqrt{\frac{p(\Gamma(1+\bar{b}\rho)^2-\bar{b}\rho^2(\theta+\bar{b}))}{\rho(1-z)(1+\bar{b}\rho)}}, \\ u=\nu_+ + \int_{\rho_+}^{\rho}\frac{1}{\rho}\sqrt{\frac{p(\Gamma(1+\bar{b}\rho)^2-\bar{b}\rho^2(\theta+\bar{b}))}{\rho(1-z)(1+\bar{b}\rho)}}d\rho, \\ p=\frac{p_+}{1+\bar{b}\rho_+}(1+\bar{b}\rho)\left(\frac{\rho(1-\theta\rho_+)}{\rho_+(1-\theta\rho)}\right)^{\Gamma}\left(\frac{(1-\theta\rho_+)}{(1-\theta\rho)}\right)^{(\Gamma-1)\bar{b}/\theta}, \;\; 0\le p\le p_+. \end{cases} \tag{6.21}$$

In 2-rarefaction wave, suppose U_{r_2} represents the values of U on (t, x) point. By using the formulas (6.16) and (6.21), one can find the expression of U_{r_2} as

$$\begin{aligned}&\frac{x}{t} - \sqrt{\frac{p_+(\Gamma(1+\bar{b}\rho_{r_2})^2 - \bar{b}\rho_{r_2}^2(\theta+\bar{b}))}{\rho_{r_2}(1-z)(1+\bar{b}\rho_{r_2})}\left(\frac{\rho_{r_2}(1-\theta\rho_+)}{\rho_+(1-\theta\rho_{r_2})}\right)^{\Gamma}\left(\frac{(1-\theta\rho_+)}{(1-\theta\rho_{r_2})}\right)^{(\Gamma-1)\bar{b}/\theta}} - \nu_+\\&-\int_{\rho_+}^{\rho_{r_2}}\frac{1}{\rho}\sqrt{\frac{p_+(\Gamma(1+\bar{b}\rho)^2 - \bar{b}\rho^2(\theta+\bar{b}))}{\rho(1-z)(1+\bar{b}\rho)}\left(\frac{\rho(1-\theta\rho_+)}{\rho_+(1-\theta\rho)}\right)^{\Gamma}\left(\frac{(1-\theta\rho_+)}{(1-\theta\rho)}\right)^{(\Gamma-1)\bar{b}/\theta}}d\rho\\&-\frac{1}{t}\int_0^t H(y)dy = 0.\end{aligned} \tag{6.22}$$

The expression (6.22) admits uniquely a function $\rho = \rho_{r_2}(t, x)$. Accordingly, U_{r_2} is provided as:

$$\begin{cases}\rho_{r_2} = \rho_{r_2}(t,x), \quad p_{r_2} = p(\rho_{r_2}),\\ u_{r_2} = \nu_+ + \int_{\rho_+}^{\rho_{r_2}}\frac{1}{\rho}\sqrt{\frac{p_+(\Gamma(1+\bar{b}\rho)^2-\bar{b}\rho^2(\theta+\bar{b}))}{\rho(1-z)(1+\bar{b}\rho_+)}\left(\frac{\rho(1-\theta\rho_+)}{\rho_+(1-\theta\rho)}\right)^{\Gamma}\left(\frac{(1-\theta\rho_+)}{(1-\theta\rho)}\right)^{(\Gamma-1)\bar{b}/\theta}}d\rho.\end{cases} \tag{6.23}$$

Now we study how elementary wave curves behave.

Lemma 6.1. *Consider the curve $R_1(p; U_-)$, using the above values of Γ, θ, and $\bar{b}$, the ρ is an increasing function of p and goes to 0 as $p \to 0^+$, but the function u is a decreasing function of p and tends to $\bar{u}$ as $p \to 0^+$, where*

$$\bar{u} = \nu_- - \int_{\rho_-}^{0}\frac{1}{\rho}\sqrt{\frac{(\Gamma(1+\bar{b}\rho)^2 - \bar{b}\rho^2(\theta+\bar{b}))p_-}{\rho(1-z)(1+\bar{b}\rho_-)}\left(\frac{(1-\theta\rho_-)\rho}{\rho_-(1-\theta\rho)}\right)^{\Gamma}\left(\frac{(1-\theta\rho_-)}{(1-\theta\rho)}\right)^{(\Gamma-1)\bar{b}/\theta}}d\rho.$$

Therewith, on $R_2(p; U_+)$ using the given values of Γ, θ, and $\bar{b}$, the ρ is an increasing function of p and goes to 0 as $p \to 0^+$, but the function u is an increasing function of p and tends to $\bar{\bar{u}}$ as $p \to 0^+$, where

$$\bar{\bar{u}} = \nu_+ + \int_{\rho_+}^{0}\frac{1}{\rho}\sqrt{\frac{p_+(\Gamma(1+\bar{b}\rho)^2 - \bar{b}\rho^2(\theta+\bar{b}))}{\rho(1-z)(1+\bar{b}\rho_+)}\left(\frac{\rho(1-\theta\rho_+)}{\rho_+(1-\theta\rho)}\right)^{\Gamma}\left(\frac{(1-\theta\rho_+)}{(1-\theta\rho)}\right)^{(\Gamma-1)\bar{b}/\theta}}d\rho.$$

Proof. The values of Γ, θ, and $\bar{b}$ can be used to determine from (6.17) so that,

$$\frac{\partial \rho}{\partial p} = \frac{\left(\left(\frac{\rho(1-\theta\rho_-)}{\rho_-(1-\theta\rho)}\right)^{\Gamma}\left(\frac{(1-\theta\rho_-)}{(1-\theta\rho)}\right)^{(\Gamma-1)\bar{b}/\theta}\right)^{-1}}{\frac{p_-(\Gamma(1+\bar{b}\rho)^2-\bar{b}(\theta+\bar{b})\rho^2)}{\rho(1+\bar{b}\rho_-)(1-\theta\rho)}} > 0,$$

which implies that

$$\rho < \rho_- \text{ as } p < p_-. \tag{6.24}$$

Now, on $R_1(p;U_-)$, the specified value of Γ, θ, and $\bar{b}$ are used. Furthermore, we have

$$\frac{\partial u}{\partial p} = \frac{\partial u}{\partial \rho}\frac{\partial \rho}{\partial p} = -\frac{1}{\rho}\sqrt{\frac{p_-(\Gamma(1+\bar{b}\rho)^2-\bar{b}(\theta+\bar{b})\rho^2)}{\rho(1+\bar{b}\rho_-)(1-\theta\rho)}\left(\frac{\rho(1-\theta\rho_-)}{\rho_-(1-\theta\rho)}\right)^{\Gamma}\left(\frac{(1-\theta\rho_-)}{(1-\theta\rho)}\right)^{(\Gamma-1)\bar{b}/\theta}}\frac{\partial \rho}{\partial p} < 0.$$

In addition, from $(6.17)_3$ demonstrates that $\rho \to 0$ as $p \to 0^+$. According to the convergence of the integral

$$\int_{\rho_-}^{0} \frac{1}{\rho^a} d\rho, \quad a = \max\left\{\frac{1}{4}, \frac{3-\Gamma}{2}\right\}.$$

By analysing the integral we have

$$\int_{\rho_-}^{0} \frac{1}{\rho}\sqrt{\frac{p_-(\Gamma(1+\bar{b}\rho)^2-\bar{b}(\theta+\bar{b})\rho^2)}{\rho(1+\bar{b}\rho_-)(1-\theta\rho)}\left(\frac{\rho(1-\theta\rho_-)}{\rho_-(1-\theta\rho)}\right)^{\Gamma}\left(\frac{(1-\theta\rho_-)}{(1-\theta\rho)}\right)^{(\Gamma-1)\bar{b}/\theta}} d\rho,$$

is convergent. Hence $u \to \bar{u}$ as $p \to 0^+$. Similarly, we can also prove the properties of $R_2(p;U_+)$, using the specified value of Γ, θ, and $\bar{b}$.

6.2.2 Vacuum solution

It is also possible to have a vacuum, a continuous solution of a modified homogeneous system (6.6), as

$$(\rho, u, p) = (0, u(t,x), 0). \tag{6.25}$$

Here, an arbitrary smooth function is represented by $u(t,x)$.

6.2.3 Bounded discontinuous solutions

For a discontinuous solution of equation (6.6), the R-H conditions are obtained by

$$\begin{cases} -\sigma[\rho] + [\rho(u + H(t))] = 0, \\ -\sigma[\rho u] + [\rho u(u + H(t)) + p] = 0, \\ -\sigma\left[\frac{\rho u^2}{2} + \rho e\right] + \left[\left(\frac{\rho u^2}{2} + \rho e\right)(u + H(t)) + pu\right] = 0, \end{cases} \tag{6.26}$$

where $\sigma = x'(t)$, is the speed of the discontinuity, and $[G] = G - G_-$, is the jumps of the function G about the discontinuity $x = x(t)$, whereas $G_- = G(t, x(t) - 0)$ and $G = G(t, x(t) + 0)$. Solving (6.26), we obtain

$$\begin{cases} \sigma_1 = u_- + H(t) - \sqrt{\frac{(\Gamma-1)(1+\bar{b}\rho_+)(p_+ + p_-)}{2\rho_-(1-\theta\rho_+)} + \frac{p_+(1+\bar{b}\rho_-)(1-\theta\rho_+) + p_-\rho_+(\theta+\bar{b})}{\rho_-(1-\theta\rho_+)(1+\bar{b}\rho_-)}}, \\ \sqrt{\frac{(\Gamma-1)(1+\bar{b}\rho_+)(p_+ + p_-)}{2\rho_-(1-\theta\rho_+)} + \frac{p_+(1+\bar{b}\rho_-)(1-\theta\rho_+) + p_-\rho_+(\theta+\bar{b})}{\rho_-(1-\theta\rho_+)(1+\bar{b}\rho_-)}}[\rho] + \rho[u] = 0, \\ \rho_-\sqrt{\frac{(\Gamma-1)(1+\bar{b}\rho_+)(p_+ + p_-)}{2\rho_-(1-\theta\rho_+)} + \frac{p_+(1+\bar{b}\rho_-)(1-\theta\rho_+) + p_-\rho_+(\theta+\bar{b})}{\rho_-(1-\theta\rho_+)(1+\bar{b}\rho_-)}}[u] + [p] = 0, \end{cases} \tag{6.27}$$

$$\begin{cases} \sigma_2 = u_- + H(t) + \sqrt{\frac{(\Gamma-1)(1+\bar{b}\rho_+)(p_+ + p_-)}{2\rho_-(1-\theta\rho_+)} + \frac{p_+(1+\bar{b}\rho_-)(1-\theta\rho_+) + p_-\rho_+(\theta+\bar{b})}{\rho_-(1-\theta\rho_+)(1+\bar{b}\rho_-)}}, \\ -\sqrt{\frac{(\Gamma-1)(1+\bar{b}\rho_+)(p_+ + p_-)}{2\rho_-(1-\theta\rho_+)} + \frac{p_+(1+\bar{b}\rho_-)(1-\theta\rho_+) + p_-\rho_+(\theta+\bar{b})}{\rho_-(1-\theta\rho_+)(1+\bar{b}\rho_-)}}[\rho] + \rho[u] = 0, \\ -\rho_-\sqrt{\frac{(\Gamma-1)(1+\bar{b}\rho_+)(p_+ + p_-)}{2\rho_-(1-\theta\rho_+)} + \frac{p_+(1+\bar{b}\rho_-)(1-\theta\rho_+) + p_-\rho_+(\theta+\bar{b})}{\rho_-(1-\theta\rho_+)(1+\bar{b}\rho_-)}}[u] + [p] = 0, \end{cases} \tag{6.28}$$

$$\begin{cases} \sigma_3 = u_- + H(t), \\ [u] = 0, [p] = 0, \\ [\rho] \neq 0. \end{cases} \tag{6.29}$$

6.2.3.1 1-Shock wave curve

Using (6.27), under the stability condition $\lambda_1(U) < \sigma_1 < \lambda_1(U_-),\ \ \lambda_3(U) > \sigma_1$, with left state U_-, the 1-Shock wave curve is obtained by

$$S_1(p;U_-):\begin{cases}\sigma_1 = \nu_- + H(t) - \sqrt{\frac{(\Gamma-1)(1+\bar{b}\rho)(p+p_-)}{2\rho_-(1-\theta\rho)} + \frac{p(1+\bar{b}\rho_-)(1-\theta\rho)+p_-\rho(\theta+\bar{b})}{\rho_-(1-\theta\rho)(1+\bar{b}\rho_-)}},\\ u = \nu_- - \sqrt{p_-\left(-\frac{(\Gamma-1)(\rho-\rho_-)+2\rho(1-\theta\rho_-)/(1+\bar{b}\rho_-)}{(\Gamma-1)(\rho-\rho_-)-2\rho_-(1-\theta\rho)/(1+\bar{b}\rho)} - 1\right)\frac{\rho-\rho_-}{\rho\rho_-}},\\ p = -\frac{(\Gamma-1)(\rho-\rho_-)+2\rho(1-\theta\rho_-)/(1+\bar{b}\rho_-)}{(\Gamma-1)(\rho-\rho_-)-2\rho_-(1-\theta\rho)/(1+\bar{b}\rho)}p_-,\ \ p_- < p.\end{cases} \tag{6.30}$$

6.2.3.2 2-Shock wave curve

Under the stability condition $\lambda_2(U_+) < \sigma_2 < \lambda_2(U),\ \ \lambda_3(U) < \sigma_2$, with right state U_+, using (6.28), we derive 2-shock wave curve as

$$S_2(p;U_+):\begin{cases}\sigma_2 = \nu_+ + H(t) + \sqrt{\frac{(\Gamma-1)(1+\bar{b}\rho)(p+p_+)}{2\rho_+(1-\theta\rho)} + \frac{p(1+\bar{b}\rho_+)(1-\theta\rho)+p_+\rho(\theta+\bar{b})}{\rho_+(1-\theta\rho)(1+\bar{b}\rho_+)}},\\ u = \nu_+ + \sqrt{p_+\left(-\frac{(\Gamma-1)(\rho-\rho_+)+2\rho(1-\theta\rho_+)/(1+\bar{b}\rho_+)}{(\Gamma-1)(\rho-\rho_+)-2\rho_+(1-\theta\rho)/(1+\bar{b}\rho)} - 1\right)\frac{\rho-\rho_+}{\rho\rho_+}},\\ p = -\frac{(\Gamma-1)(\rho-\rho_+)+2\rho(1-\theta\rho_+)/(1+\bar{b}\rho_+)}{(\Gamma-1)(\rho-\rho_+)-2\rho_+(1-\theta\rho)/(1+\bar{b}\rho)}p_+,\ \ p_+ < p.\end{cases} \tag{6.31}$$

6.2.3.3 3-contact discontinuity

With the left state U_-, by using (6.29), we obtain a 3-contact discontinuity as

$$J_3(p;U_-):\begin{cases}\sigma_3 = \nu_- + H(t),\\ u = \nu_-, p = p_-,\\ \rho \neq \rho_-.\end{cases} \tag{6.32}$$

Lemma 6.2. *The behavior of ρ and u w.r.t. the values of p at the curve $S_1(p, U_-)$ corresponding to the values of Γ, θ, and $\bar{b}$, are given as: the values of u decreases as p increases and tends to $-\infty$ as $p \to +\infty$, whereas ρ increases as p increases and tends to $\bar{\rho}$ as $p \to +\infty$, therefore*

$$\bar{\rho} = \frac{\sqrt{((1-\bar{b}\rho_-)(\Gamma-1)+2\theta\rho_-)^2 + 4\bar{b}\rho_-(\Gamma^2-1)} - ((1-\bar{b}\rho_-)(\Gamma-1)+2\theta\rho_-)}{2(\Gamma-1)\bar{b}}.$$

Besides, the behavior of ρ and u w.r.t. the values of p at the curve $S_2(p, U_+)$ corresponding to the values of Γ, θ, and $\bar{b}$, are given as: the values of u increases as p increases and tends to $+\infty$ as $p \to +\infty$, whereas ρ increases as p increases and tends to $\bar{\bar{\rho}}$ as $p \to +\infty$, therefore

$$\bar{\bar{\rho}} = \frac{\sqrt{((1-\bar{b}\rho_+)(\Gamma-1)+2\theta\rho_+)^2 + 4\bar{b}\rho_+(\Gamma^2-1)} - ((1-\bar{b}\rho_+)(\Gamma-1)+2\theta\rho_+)}{2(\Gamma-1)\bar{b}}.$$

Proof. From (6.30), we can write

$$u = \nu_- - \sqrt{(p-p_-)\left(\frac{1}{\rho_-} - \frac{1}{\rho}\right)}, \tag{6.33}$$

which gives

$$\frac{\partial u}{\partial p} = \frac{1}{2\sqrt{(p-p_-)\left(\frac{1}{\rho_-} - \frac{1}{\rho}\right)}} \left(\left(\frac{1}{\rho_-} - \frac{1}{\rho}\right) + \frac{(p-p_-)}{\rho^2}\frac{\partial \rho}{\partial p}\right) > 0, \tag{6.34}$$

that is, $u \to -\infty$ as $p \to +\infty$.

Now, it follows from the second and third part of (6.27), with Γ, θ, and $\bar{b}$, are given so that

$$\frac{[\rho]}{[p]} = \frac{\rho}{\rho_-}\sqrt{\frac{(\Gamma-1)(1+\bar{b}\rho)(p+p_-)}{2\rho_-(1-\theta\rho)} + \frac{p(1+\bar{b}\rho_-)(1-\theta\rho)+p_-\rho(\theta+\bar{b})}{\rho_-(1+\bar{b}\rho_-)(1-\theta\rho)}} > 0, \tag{6.35}$$

which implies that

$$\frac{\partial\rho}{\partial p} \geq 0. \tag{6.36}$$

So, we have

$$\rho > \rho_- \text{ as } p > p_-, \tag{6.37}$$

along the curve $S_1(p;U_-)$, using the corrsponding values of Γ, θ, and $\bar{b}$, and the equality of (6.30)$_3$, we have

$$\rho \to \frac{\sqrt{((1-\bar{b}\rho_-)(\Gamma-1)+2\theta\rho_-)^2+4\bar{b}\rho_-(\Gamma^2-1)}-((1-\bar{b}\rho_-)(\Gamma-1)+2\theta\rho_-)}{2(\Gamma-1)\bar{b}}.$$
$$< 1/\theta \text{ as } p \to +\infty. \tag{6.38}$$

Similar proofs can be provided for the results along the curve $S_2(p;U_+)$ with the specified values for Γ, θ, and $\bar{b}$.

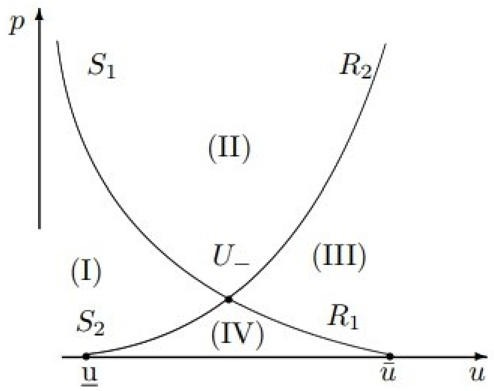

FIGURE 6.1: The (u, p) phase plane

6.3 Solution of transformed homogeneous systems (6.6)-(6.7) to the Riemann problem

The (u, p)-plane is divided into four distinct regions, labelled I, II, III, and IV, as shown in Fig.6.1 by the projection of the curves $S_1(p; U_-)$, $S_2(p; U_+)$, $R_1(p; U_-)$, and $R_2(p; U_+)$.

The expression $\underline{u} = \nu_- - \sqrt{\frac{2p_-(1-\theta\rho_-)}{\rho_-(\Gamma-1)(1+\bar{b}\rho_-)}}$ (See Fig.6.1). Along the space-(ρ, u, p), using $\rho > 0$ and $p > 0$, $R_2(p; U_-)$, $S_2(p; U_-)$ are expressed as

$$R_2(p;U_-):\begin{cases}\frac{dx}{dt} = u + H(t) + \sqrt{\frac{(\Gamma(1+\bar{b}\rho)^2 - \bar{b}\rho^2(\theta+\bar{b}))}{\rho(1-\theta\rho)(1+\bar{b}\rho)}p},\\ u = \nu_- + \int_{\rho_-}^{\rho}\frac{1}{\rho}\sqrt{\frac{(\Gamma(1+\bar{b}\rho)^2 - \bar{b}\rho^2(\theta+\bar{b}))}{\rho(1-\theta\rho)(1+\bar{b}\rho)}p}\,d\rho,\\ p = \frac{p_-}{1+\bar{b}\rho_-}(1+\bar{b}\rho)\left(\frac{\rho(1-\theta\rho_-)}{\rho_-(1-\theta\rho)}\right)^{\Gamma}\left(\frac{(1-\theta\rho_-)}{(1-\theta\rho)}\right)^{(\Gamma-1)\bar{b}/\theta}, \quad p_- \le p,\end{cases} \tag{6.39}$$

and

$$S_2(p;U_-):\begin{cases}\sigma_2 = \nu_- + H(t) + \sqrt{\frac{(\Gamma-1)(1+\bar{b}\rho)(p+p_-)}{2\rho_-(1-\theta\rho)} + \frac{p(1+\bar{b}\rho_-)(1-\theta\rho)+p_-\rho(\theta+\bar{b})}{\rho_-(1-\theta\rho)(1+\bar{b}\rho_-)}},\\ u = \nu_- + \sqrt{p_-\left(-\frac{(\Gamma-1)(\rho-\rho_-)+2\rho(1-\theta\rho_-)/(1+\bar{b}\rho_-)}{(\Gamma-1)(\rho-\rho_-)-2\rho_-(1-\theta\rho)/(1+\bar{b}\rho)} - 1\right)\frac{\rho-\rho_-}{\rho\rho_-}},\\ p = -\frac{(\Gamma-1)(\rho-\rho_-)+2\rho(1-\theta\rho_-)/(1+\bar{b}\rho_-)}{(\Gamma-1)(\rho-\rho_-)-2\rho_-(1-\theta\rho)/(1+\bar{b}\rho)}p_-, \quad p_- > p.\end{cases} \tag{6.40}$$

We study the solution to the Riemann problem for transformed systems (6.6)-(6.7) and followed all possible cases.

Case-1: When the state $(\rho_+, \nu_+, p_+) \in \mathrm{I}$, it is possible to express the solution as

$$(\rho, u, p)(t, x) = \begin{cases} (\rho_-, \nu_-, p_-), & x < x_1(t), \\ (\rho_{*_1}, u_*, p_*), & x_1(t) < x < x_2(t), \\ (\rho_{*_2}, u_*, p_*), & x_2(t) < x < x_3(t), \\ (\rho_+, \nu_+, p_+), & x > x_3(t), \end{cases} \tag{6.41}$$

and (u_*, p_*) can be obtained from the equation

$$\begin{cases} u_* = \nu_+ + \sqrt{(p_* - p_-)\left(\frac{1}{\rho_+} - \frac{1}{\rho_*}\right)}, \\ u_* = \nu_- - \sqrt{(p_* - p_-)\left(\frac{1}{\rho_-} - \frac{1}{\rho_*}\right)}, \end{cases} \tag{6.42}$$

where

$$\begin{cases} p_* = -\frac{(\Gamma-1)(\rho_{*_1}-\rho_+)+2\rho_{*_1}(1-\theta\rho_+)/(1+\bar{b}\rho_+)}{(\Gamma-1)(\rho_{*_1}-\rho_+)-2\rho_+(1-\theta\rho_{*_1})/(1+\bar{b}\rho_{*_1})} p_+, \\ p_* = -\frac{(\Gamma-1)(\rho_{*_2}-\rho_-)+2\rho_{*_2}(1-\theta\rho_-)/(1+\bar{b}\rho_-)}{(\Gamma-1)(\rho_{*_2}-\rho_-)-2\rho_-(1-\theta\rho_{*_2})/(1+\bar{b}\rho_{*_2})} p_-, \end{cases} \tag{6.43}$$

$$x_1(t) = \left(\nu_- - \sqrt{\frac{(\Gamma-1)(1+\bar{b}\rho_{*_1})(p_*+p_-)}{2\rho_-(1-\theta\rho_{*_1})} + \frac{p_*(1+\bar{b}\rho_-)(1-\theta\rho_{*_1})+p_-\rho_{*_1}(\theta+\bar{b})}{\rho_-(1-\theta\rho_{*_1})(1+\bar{b}\rho_-)}}\right) t + \int_0^t H(y)dy,$$

is 1-shock wave,

$$x_3(t) = \left(\nu_+ + \sqrt{\frac{(\Gamma-1)(1+\bar{b}\rho_{*_2})(p_*+p_+)}{2\rho_+(1-\theta\rho_{*_2})} + \frac{p_*(1+\bar{b}\rho_+)(1-\theta\rho_{*_2})+p_+\rho_{*_2}(\theta+\bar{b})}{\rho_+(1-\theta\rho_{*_2})(1+\bar{b}\rho_+)}}\right) t + \int_0^t H(y)dy,$$

is 2-shock wave, and

$$x_2(t) = u_* t + \int_0^t H(y)dy,$$

is 3-contact discontinuity.

Case-2: Here, the state $(\rho_+, \nu_+, p_+) \in \text{II}$. The possible solution is constructed by

$$(\rho, u, p)(t,x) = \begin{cases} (\rho_-, \nu_-, p_-), & x < x_1(t), \\ (\rho_{*_1}, u_*, p_*), & x_1(t) < x < x_2(t), \\ (\rho_{*_2}, u_*, p_*), & x_2(t) < x < x_3^-(t), \\ (\rho_{r_2}, u_{r_2}, p_{r_2}), & x_3^-(t) \le x \le x_3^+(t), \\ (\rho_+, \nu_+, p_+), & x > x_3^+(t), \end{cases} \tag{6.44}$$

where (u_*, p_*) can be determined from the equation

$$\begin{cases} u_* = \nu_- - \int_{\rho_-}^{\rho_{*_1}} \frac{1}{\rho} \sqrt{\frac{p_*(\Gamma(1+\bar{b}\rho)^2 - \bar{b}\rho^2(\theta+\bar{b}))}{\rho(1-\theta\rho)(1+\bar{b}\rho)}} d\rho, \\ u_* = \nu_- - \sqrt{(p_* - p_-)\left(\frac{1}{\rho_-} - \frac{1}{\rho_*}\right)}, \end{cases} \tag{6.45}$$

and

$$\begin{cases} p_* = \frac{p_-}{1+\bar{b}\rho_-}(1+\bar{b}\rho_{*_1})\left(\frac{\rho_{*_1}(1-\theta\rho_-)}{\rho_-(1-\theta\rho_{*_1})}\right)^{\Gamma}\left(\frac{(1-\theta\rho_-)}{(1-\theta\rho_{*_1})}\right)^{(\Gamma-1)\bar{b}/\theta}, \\ p_* = -\frac{(\Gamma-1)(\rho_{*_2}-\rho_-)+2\rho_{*_2}(1-\theta\rho_-)/(1+\bar{b}\rho_-)}{(\Gamma-1)(\rho_{*_2}-\rho_-)-2\rho_-(1-\theta\rho_{*_2})/(1+\bar{b}\rho_{*_2})} p_-, \\ x_3^-(t) = \left(u_* + \sqrt{\frac{p_*(\Gamma(1+\bar{b}\rho_{*_2})^2 - \bar{b}\rho_{*_2}^2(\theta+\bar{b}))}{\rho_{*_2}(1-\theta\rho_{*_2})(1+\bar{b}\rho_{*_2})}}\right) t + \int_0^t H(y)dy, \\ x_3^+(t) = \left(u_+ + \sqrt{\frac{p_+(\Gamma(1+\bar{b}\rho_+)^2 - \bar{b}\rho_+^2(\theta+\bar{b}))}{\rho_+(1-\theta\rho_+)(1+\bar{b}\rho_+)}}\right) t + \int_0^t H(y)dy, \end{cases} \tag{6.46}$$

$x_1(t)$ is 1-shock wave and $x_2(t) = u_* t + \int_0^t H(y)dy$, is 3-contact discontinuity.

Case-3: Consider the state $(\rho_+, \nu_+, p_+) \in \text{III}$. There arise three cases , which can be determined as follows

Case-3(a): $\bar{u} > \bar{\bar{u}}$, the curves $R_1(\rho_-, \nu_-, p_-)$ and $R_2(\rho_+, \nu_+, p_+)$ intersect in the plane (u,p) which is shown as (u_*, p_*), and is defined by

$$\begin{cases} u_* = \nu_- - \int_{\rho_-}^{\rho_{*1}} \frac{1}{\rho}\sqrt{\frac{p_*(\Gamma(1+\bar{b}\rho)^2-\bar{b}\rho^2(\theta+\bar{b}))}{\rho(1-\theta\rho)(1+\bar{b}\rho)}}d\rho, \\ p_* = \frac{p_-}{1+\bar{b}\rho_-}(1+\bar{b}\rho_{*1})\left(\frac{\rho_{*1}(1-\theta\rho_-)}{\rho_-(1-\theta\rho_{*1})}\right)^{\Gamma}\left(\frac{(1-\theta\rho_-)}{(1-\theta\rho_{*1})}\right)^{(\Gamma-1)\bar{b}/\theta}. \end{cases} \tag{6.47}$$

Therefore, the Riemann solution on the (t,x)-plane is expressed as

$$(\rho,u,p)(t,x) = \begin{cases} (\rho_-, \nu_-, p_-), & x < x_1^-(t), \\ (\rho_{r_1}, u_{r_1}, p_{r_1}), & x_1^-(t) \le x \le x_1^+(t), \\ (\rho_{*1}, u_*, p_*), & x_1^+(t) < x < x_2(t), \\ (\rho_{*2}, u_*, p_*), & x_2(t) < x < x_3^-(t), \\ (\rho_{r_2}, u_{r_2}, p_{r_2}), & x_3^-(t) \le x \le x_3^+(t), \\ (\rho_+, \nu_+, p_+), & x > x_3^+(t), \end{cases} \tag{6.48}$$

where

$$\begin{cases} x_1^-(t) = \left(\nu_- - \sqrt{\frac{p_-(\Gamma(1+\bar{b}\rho_-)^2-\bar{b}\rho_-^2(\theta+\bar{b}))}{\rho_-(1-\theta\rho_-)(1+\bar{b}\rho_-)}}\right)t + \int_0^t H(y)dy, \\ x_1^+(t) = \left(u_* - \sqrt{\frac{p_*(\Gamma(1+\bar{b}\rho_{*1})^2-\bar{b}\rho_{*1}^2(\theta+\bar{b}))}{\rho_{*1}(1-\theta\rho_{*1})(1+\bar{b}\rho_{*1})}}\right)t + \int_0^t H(y)dy, \\ x_3^-(t) = \left(u_* + \sqrt{\frac{p_*(\Gamma(1+\bar{b}\rho_{*2})^2-\bar{b}\rho_{*2}^2(\theta+\bar{b}))}{\rho_{*2}(1-\theta\rho_{*2})(1+\bar{b}\rho_{*2})}}\right)t + \int_0^t H(y)dy, \\ x_3^+(t) = \left(\nu_+ + \sqrt{\frac{p_+(\Gamma(1+\bar{b}\rho_+)^2-\bar{b}\rho_+^2(\theta+\bar{b}))}{\rho_+(1-\theta\rho_+)(1+\bar{b}\rho_+)}}\right)t + \int_0^t H(y)dy. \end{cases} \tag{6.49}$$

Here, $(\rho_{r_1}, u_{r_1}, p_{r_1})$, is 1-rarefaction wave, which is given by (6.19) and $(\rho_{r_2}, u_{r_2}, p_{r_2})$, is 2-rarefaction wave governed by (6.23), and $x_2(t) = u_*t + \int_0^t H(y)dy$, is 3-contact discontinuity.

Case-3(b): $\bar{u} < \bar{\bar{u}}$, here, the curves $R_1(\rho_-, \nu_-, p_-)$ and $R_2(\rho_+, \nu_+, p_+)$ meet the $u-$axis at the points $(u_{*1}, 0)$ and $(u_{*2}, 0)$ and they do not intersect both the curves, we have

$$\begin{cases} u_{*_1} = \nu_- - \int_{\rho_-}^{0} \frac{1}{\rho}\sqrt{\frac{p_-(\Gamma(1+\bar{b}\rho)^2-\bar{b}\rho^2(\theta+\bar{b}))}{\rho(1-z)(1+\bar{b}\rho_-)}\left(\frac{\rho(1-\theta\rho_-)}{\rho_-(1-\theta\rho)}\right)^{\Gamma}\left(\frac{(1-\theta\rho_-)}{(1-\theta\rho)}\right)^{(\Gamma-1)\bar{b}/\theta}}\,d\rho, \\ u_{*_2} = \nu_+ + \int_{\rho_+}^{0} \frac{1}{\rho}\sqrt{\frac{p_+(\Gamma(1+\bar{b}\rho)^2-\bar{b}\rho^2(\theta+\bar{b}))}{\rho(1-z)(1+\bar{b}\rho_+)}\left(\frac{\rho(1-\theta\rho_+)}{\rho_+(1-\theta\rho)}\right)^{\Gamma}\left(\frac{(1-\theta\rho_+)}{(1-\theta\rho)}\right)^{(\Gamma-1)\bar{b}/\theta}}\,d\rho. \end{cases} \tag{6.50}$$

Here, 3-contact discontinuity does not exist because $\rho_{*_1} = 0 = \rho_{*_2}$, so it is possible to express the solution as

$$(\rho, u, p)(t, x) = \begin{cases} (\rho_-, \nu_-, p_-), & x < x_1^-(t), \\ (\rho_{r_1}, u_{r_1}, p_{r_1}), & x_1^-(t) \le x \le x_1^+(t), \\ (0, u(t,x), 0), & x_1^+(t) < x \le x_3^-(t), \\ (\rho_{r_2}, u_{r_2}, p_{r_2}), & x_3^-(t) < x \le x_3^+(t), \\ (\rho_+, \nu_+, p_+), & x > x_3^+(t), \end{cases} \tag{6.51}$$

where $x_1^-(t)$ and $x_3^+(t)$ are given by (6.49) and

$$\begin{cases} x_1^+(t) = u_{*_1}t + \int_0^t H(y)dy, \\ x_3^-(t) = u_{*_2}t + \int_0^t H(y)dy. \end{cases} \tag{6.52}$$

A vacuum is defined as $(0, u(t,x), 0)$, using $u(x_1^+(t), t) = u_{*_1}$, $u(x_3^-(t), t) = u_{*_2}$. The 1-rarefaction wave $R_1(\rho_-, \nu_-, p_-)$ is connected to the 2-rarefaction wave $R_2(\rho_+, \nu_+, p_+)$ by a vacuum $(0, u(t,x), 0)$ through curves $x_1^+(t)$ and $x_3^-(t)$ as their respective boundaries on the left and right sides.

Case-3(c): $\bar{u} = \bar{\bar{u}}$, using (6.47), we obtain the curves $R_1(\rho_-, \nu_-, p_-)$ and $R_2(\rho_+, \nu_+, p_+)$ meeting at a point (u_*, p_*) along the plane (u, p); here $p_* = 0$ and

$$u_* = \nu_- - \int_{\rho_-}^{0} \frac{1}{\rho}\sqrt{\frac{p_-(\Gamma(1+\bar{b}\rho)^2-\bar{b}\rho^2(\theta+\bar{b}))}{\rho(1-z)(1+\bar{b}\rho_-)}\left(\frac{\rho(1-\theta\rho_-)}{\rho_-(1-\theta\rho)}\right)^{\Gamma}\left(\frac{(1-\theta\rho_-)}{(1-\theta\rho)}\right)^{(\Gamma-1)\bar{b}/\theta}}\,d\rho.$$

It is also noted that $\rho_{*_1} = 0 = \rho_{*_2}$; there will be no 3-contact discontinuity. It follows that the solution is

$$(\rho, u, p)(t, x) = \begin{cases} (\rho_-, \nu_-, p_-), & x < x_1^-(t), \\ (\rho_{r_1}, u_{r_1}, p_{r_1}), & x_1^-(t) \leq x \leq x_1^+(t), \\ (\rho_{r_2}, u_{r_2}, p_{r_2}), & x_1^+(t) < x \leq x_3^+(t), \\ (\rho_+, \nu_+, p_+), & x > x_3^+(t), \end{cases} \tag{6.53}$$

where $x_1^+(t)$, $x_1^-(t)$ and $x_3^+(t)$ are given in (6.52) and (6.49), respectively. Here, it is clear that the $R_1(\rho_-, \nu_-, p_-)$ and $R_2(\rho_+, \nu_+, p_+)$ has a common boundary with $x_1^+(t)$ as $t > 0$, and a vacuum $(0, u_*, 0)$ forms in the Riemann solutions.

Case-4: When the state $(\rho_+, \nu_+, p_+) \in \text{IV}$. We find the Riemann solution as

$$(\rho, u, p)(t, x) = \begin{cases} (\rho_-, \nu_-, p_-), & x < x_1^-(t), \\ (\rho_{r_1}, u_{r_1}, p_{r_1}), & x_1^-(t) \leq x \leq x_1^+(t), \\ (\rho_{*_1}, u_*, p_*), & x_1^+(t) < x < x_2(t), \\ (\rho_{*_2}, u_*, p_*), & x_2(t) < x < x_3(t), \\ (\rho_+, \nu_+, p_+), & x > x_3(t). \end{cases} \tag{6.54}$$

The following equations can be solved to obtain (u_*, p_*)

$$\begin{cases} u_* = \nu_+ + \int_{\rho_+}^{\rho_{*_1}} \frac{1}{\rho} \sqrt{\frac{p_*(\Gamma(1+\bar{b}\rho)^2 - \bar{b}\rho^2(\theta+\bar{b}))}{\rho(1-\theta\rho)(1+\bar{b}\rho)}} d\rho, \\ u_* = \nu_+ + \sqrt{(p_* - p_-)\left(\frac{1}{\rho_+} - \frac{1}{\rho_{*_2}}\right)}, \end{cases} \tag{6.55}$$

and

$$\begin{cases} p_* = \frac{p_+}{1+\bar{b}\rho_+}(1+\bar{b}\rho_{*_1})\left(\frac{\rho_{*_1}(1-\theta\rho_+)}{\rho_+(1-\theta\rho_{*_1})}\right)^{\Gamma}\left(\frac{(1-\theta\rho_+)}{(1-\theta\rho_{*_1})}\right)^{(\Gamma-1)\bar{b}/\theta}, \\ p_* = -\frac{(\Gamma-1)(\rho_{*_2}-\rho_+)+2\rho_{*_2}(1-\theta\rho_+)/(1+\bar{b}\rho_+)}{(\Gamma-1)(\rho_{*_2}-\rho_+)-2\rho_+(1-\theta\rho_{*_2})/(1+\bar{b}\rho_{*_2})}p_+, \end{cases} \tag{6.56}$$

where

$$x_3(t) = \left(u_+ + \sqrt{\frac{(\Gamma-1)(1+\bar{b}\rho_{*_2})(p_*+p_+)}{2\rho_+(1-\theta\rho_{*_2})} + \frac{p_*(1+\bar{b}\rho_+)(1-\theta\rho_{*_2})+p_+\rho_{*_2}(\theta+\bar{b})}{\rho_+(1-\theta\rho_{*_2})(1+\bar{b}\rho_+)}}\right)t + \int_0^t H(y)dy.$$

Here, $x_3(t)$ is known as 2-shock wave curve and $x_1^+(t)$, $x_1^-(t)$ are given in (6.49), and $x_2(t) = u_* t + \int_0^t H(y)dy$ is 3-contact discontinuity.

6.4 Solution of non-homogeneous system to the Riemann problem

Now, we will discuss the solution of the non-homogeneous system (6.3) and (6.4). The solution of the mentioned above systems (6.3)-(6.4) is obtained by applying the corresponding solution of the transformed systems (6.6)-(6.7) using a modification of variables $(\rho, \nu, p)(t, x) = (\rho, u + H(t), p)(t, x)$. There are some cases to the Riemann solution, which we will describe as follows:

Case-A: In the present case, the state $(\rho_+, \nu_+, p_+) \in$ I. The Riemann solution is given as

$$(\rho, u, p)(t, x) = \begin{cases} (\rho_-, \nu_- + H(y), p_-), & x < x_1(t), \\ (\rho_{*_1}, u_* + H(y), p_*), & x_1(t) < x < x_2(t), \\ (\rho_{*_2}, u_* + H(y), p_*), & x_2(t) < x < x_3(t), \\ (\rho_+, \nu_+ + H(y), p_+), & x > x_3(t). \end{cases} \tag{6.57}$$

Fig. 6.2 represents that if state $(\varrho_+, \nu_+, p_+) \in$ I, then the Riemann problem of system (6.3-6.4) admits a solution composed of 1-shock wave S_1, 3-contact discontinuity J, and 2-shock wave S_2, and (6.41) provides the values of the variables.

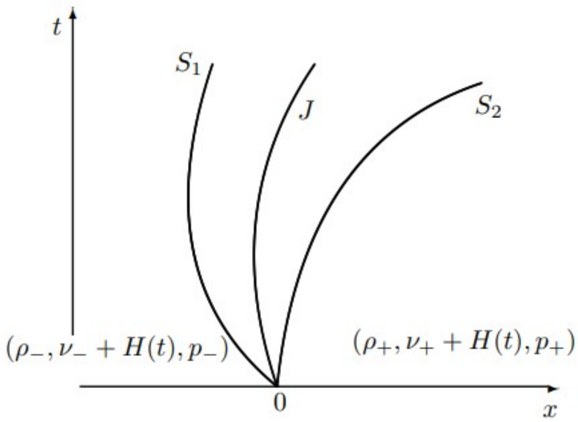

FIGURE 6.2: The structure of solution of a non-homogeneous model (6.3) as a case-A.

Case-*B*: This case presents the $(\rho_+, \nu_+, p_+) \in$ II. The Riemann solution is constructed as

$$(\rho, u, p)(t, x) = \begin{cases} (\rho_-, \nu_- + H(t), p_-), & x < x_1(t), \\ (\rho_{*_1}, u_* + H(t), p_*), & x_1(t) < x < x_2(t), \\ (\rho_{*_2}, u_* + H(t), p_*), & x_2(t) < x < x_3^-(t), \\ (\rho_{r_2}, u_{r_2} + H(t), p_{r_2}), & x_3^-(t) \le x \le x_3^+(t), \\ (\rho_+, \nu_+ + H(t), p_+), & x > x_3^+(t). \end{cases} \tag{6.58}$$

Fig. 6.3 shows that if state $(\varrho_+, \nu_+, p_+) \in$ II, then the Riemann problem of system (6.3-6.4) admits a solution composed of 1-shock wave S_1, 3-contact discontinuity J, and 2-rarefaction wave R_2, and the value of the varriables can be constructed in (6.44).

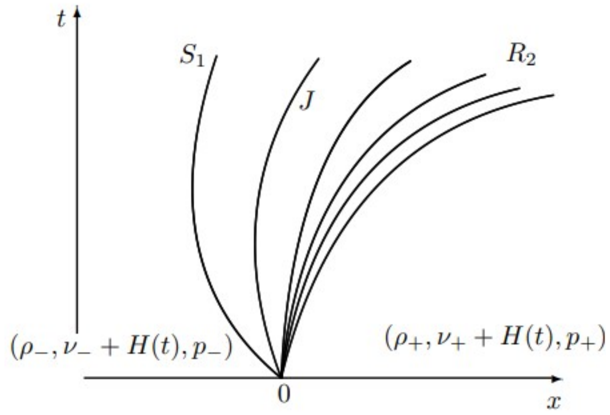

FIGURE 6.3: The structure of solution of a non-homogeneous model (6.3) as a case-B.

Case-C: Consider the state $(\rho_+, \nu_+, p_+) \in$ III. There arise three cases , which can be determined as follows

Case-C_1: $\bar{u} > \bar{\bar{u}}$, the Riemann solution is presented as

$$(\rho, u, p)(t, x) = \begin{cases} (\rho_-, \nu_- + H(t), p_-), & x < x_1^-(t), \\ (\rho_{r_1}, u_{r_1} + H(t), p_{r_1}), & x_1^-(t) \le x \le x_1^+(t), \\ (\rho_{*_1}, u_* + H(t), p_*), & x_1^+(t) < x < x_2(t), \\ (\rho_{*_2}, u_* + H(t), p_*), & x_2(t) < x < x_3^-(t), \\ (\rho_{r_2}, u_{r_2} + H(t), p_{r_2}), & x_3^-(t) \le x \le x_3^+(t), \\ (\rho_+, \nu_+ + H(t), p_+), & x > x_3^+(t). \end{cases} \tag{6.59}$$

From fig. 6.4 it is clear that if state $(\varrho_+, \nu_+, p_+) \in$ III and $\bar{u} > \bar{\bar{u}}$, then the Riemann problem of system (6.3-6.4) admits a solution composed of 1-rarefaction wave R_1, 3-contact discontinuity J, and 2-rarefaction wave R_2, and (6.48) provides the values of the variables.

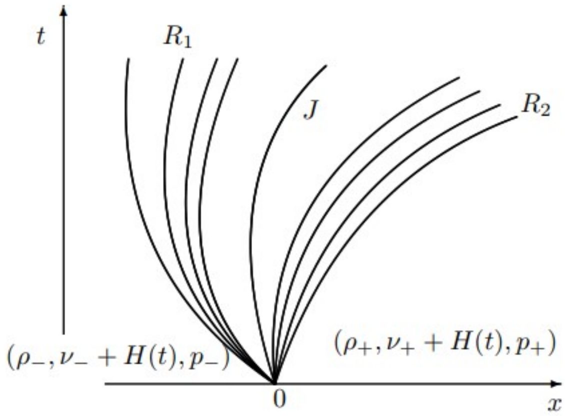

FIGURE 6.4: The structure of solution of a non-homogeneous model (6.3) as a case-(C_1).

Case-C_2: $\bar{u} < \bar{\bar{u}}$, therefore the Riemann solution is determined as

$$(\rho, u, p)(t, x) = \begin{cases} (\rho_-, \nu_- + H(t), p_-), & x < x_1^-(t), \\ (\rho_{r_1}, u_{r_1} + H(t), p_{r_1}), & x_1^-(t) \leq x \leq x_1^+(t), \\ (0, u(t, x) + H(t), 0), & x_1^+(t) < x \leq x_3^-(t), \\ (\rho_{r_2}, u_{r_2} + H(t), p_{r_2}), & x_3^-(t) < x \leq x_3^+(t), \\ (\rho_+, \nu_+ + H(t), p_+), & x > x_3^+(t). \end{cases} \tag{6.60}$$

Fig. 6.5 represents that if state $(\varrho_+, \nu_+, p_+) \in$ III and $\bar{u} < \bar{\bar{u}}$, then the Riemann problem of system (6.3-6.4) admits a solution composed of 1-rarefaction wave R_1, vacuum state, and 2-rarefaction wave R_2, and (6.51) gives the values of the variables.

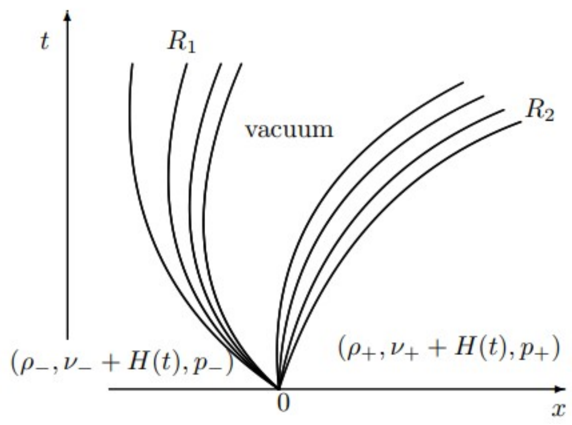

FIGURE 6.5: The structure of solution of a non-homogeneous model (6.3) as a case-(C_2).

Case-C_3: $\bar{u} = \bar{\bar{u}}$, for the present case the Riemann solution is observed as

$$(\rho, u, p)(t, x) = \begin{cases} (\rho_-, \nu_- + H(t), p_-), & x < x_1^-(t), \\ (\rho_{r_1}, u_{r_1} + H(t), p_{r_1}), & x_1^-(t) \leq x \leq x_1^+(t), \\ (\rho_{r_2}, u_{r_2} + H(t), p_{r_2}), & x_1^+(t) < x \leq x_3^+(t), \\ (\rho_+, \nu_+ + H(t), p_+), & x > x_3^+(t). \end{cases} \tag{6.61}$$

From figure 6.6 we observe that if state $(\varrho_+, \nu_+, p_+) \in$ III and $\bar{u} = \bar{\bar{u}}$, then the Riemann problem of system (6.3-6.4) admits a solution composed of 1-rarefaction wave R_1 and 2-rarefaction wave R_2, and (6.53) gives the values of the variables.

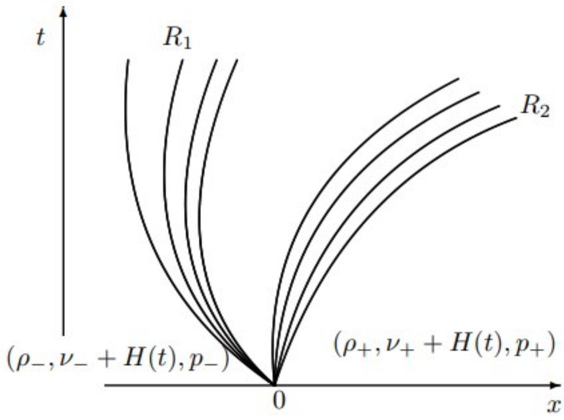

FIGURE 6.6: The structure of solution of a non-homogeneous model (6.3) as a case-(C_3).

Case-D: This case presents the $(\rho_+, \nu_+, p_+) \in$ IV. The Riemann solution is obtained as

$$(\rho, u, p)(t, x) = \begin{cases} (\rho_-, \nu_- + H(t), p_-), & x < x_1^-(t), \\ (\rho_{r_1}, u_{r_1} + H(t), p_{r_1}), & x_1^-(t) \leq x \leq x_1^+(t), \\ (\rho_{*_1}, u_* + H(t), p_*), & x_1^+(t) < x < x_2(t), \\ (\rho_{*_2}, u_* + H(t), p_*), & x_2(t) < x < x_3(t), \\ (\rho_+, \nu_+ + H(t), p_+), & x > x_3(t). \end{cases} \tag{6.62}$$

Figure 6.7 shows that the structure of the solution of the Riemann problem of the system (6.3-6.4), the solution can be represented as 1-rarefaction wave R_1, 3-contact discontinuity J, and 2-shock wave S_2, when state $(\varrho_+, \nu_+, p_+) \in$ IV, and (6.54) gives the values of the variables.

After thoroughly examining the Riemann problem's constructed solutions for (6.3) and (6.4), we conclude that there are only two possible cases (6.2) and (6.3) in which the vacuum is present in the solution. Furthermore, we determined that the RP (6.3)-(6.4) has a solution that includes contact discontinuities, rarefaction, and shock waves if and only if the initial values satisfies the condition $\bar{u} > \bar{\bar{u}}$. If this condition fails, a vacuum exists in the solution. Fig. (6.2-6.7) represents the density and velocity distribution and the formation of shock wave, rarefaction wave and contact discontinuity. It is observed that the addition of non-ideal dusty parameter $\bar{b}$ makes the solution of elementary wave curves more complex. Further, the addition of external force (friction term) makes the system of governing equastions non-homogeneous which is solved by transforming the system into homogeneous one. Also, the friction term causes to bend the solution curve to the parabolic shape which is clear from the Figure (6.2-6.7).

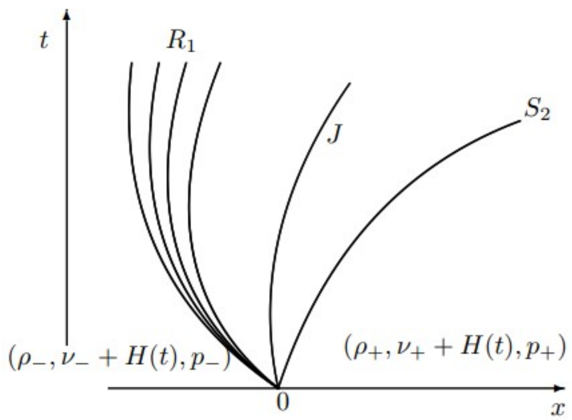

FIGURE 6.7: The structure of a solution of a non-homogeneous model (6.3) as a case-(D).

6.5 Conclusion

This chapter presents the solution to the RP, continuous with time, for the one-dimensional flow of a non-ideal dusty gas with external force. Here, we derive elementary wave curves and analyze their behavior. In the case of a non-homogeneous system of equations, the dust particles present in a non-ideal gas play a crucial role in solving the equations. A study of elementary wave solutions involving rarefaction waves, shock waves, and contact discontinuities is presented. Then, we have analyzed the case where all the elementary waves appear in the RP solution, as well as the case where the vacuum appears in it. All analytical solutions are obtained with the help of these observances. By adding parameter $\bar{b}$ for the non-ideal dusty gas, the elementary wave curves become more complex than in the ideal dusty gas [170]. Our work reveals how dusty gas flow behaves in low-density regions and high-temperature regions. The implication of addition of external force term to the governing equations of non-ideal gas flow is that the self similar solution of the system does not exist. Further, it is observed that the elementary wave curves bend into a parabolic shape due to presence of an external force term.

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