Chapter 6

Adaptive Switching Scheme Design Using Ceiling Function for Discrete-Time Sliding Mode Control

6.1 Introduction

The Chattering and reaching time, the main issues associated with its real-world application, have been actively researched in DSMC. To reduce chattering, extensive studies on switching functions and boundary layers have been conducted by researchers [35]. Reducing reaching time is also considered a challenging problem, as numerous applications require strict time constraints, as in the case of missile applications [43].

Most DSMC-based RLs are designed based on the switching function, leading to the undesirable chattering phenomenon. On the other hand, the DSMC-based equivalent control strategy lacks an approaching process, which results in an inevitably considerable control effort. Both methods result in the sliding mode state being unable to remain on the sliding surface to reach the origin. Consequently, a quasi-sliding-mode domain is formed, and chattering occurs around the origin. To address these problems, various DSMC strategies have been proposed by researchers worldwide [47, 57, 58]. Also, in [44], a modified RL was proposed by the authors, which achieves system state convergence, and the time steps for the switching function to reach the QSMD and the range of the sliding manifold for the closed-loop control system were calculated and simulated.

However, obtaining prior knowledge of the disturbance bounds, which is necessitated by these algorithms, is not always feasible. The adaptive sliding mode is served as a fitting design approach, seeking to harness the advantages of sliding mode design without relying on prior knowledge of disturbance bounds. The technique in [45–47] is the most widely recognized among reaching law-based DSMs employing adaptive gain techniques. Nevertheless, the issues of both over- and under-estimation of the adaptive switching gain are noted in [45]. Motivated by the problem of gain estimation in the absence of a bound of disturbance, a minimum operator and ceiling function-based adaptive RL are proposed.

Let us consider a perturbed discrete-time linear time-invariant system:

$$z(k+1) = Az(k) + bu(k) + \xi(k), \quad k \in \mathbb{Z}^+ \tag{6.1}$$

In the context of system (6.1), the symbol z represents the state vector, which belongs to $\mathbb{R}^n$. The control input, denoted by u, belongs to $\mathbb{R}$, while A is a matrix and b is a vector of suitable dimension. The disturbance vector, $\xi(k)$, belongs to $\mathbb{R}^n$ and represents the external disturbance impact on the system and satisfies the matching condition. The sliding variable is described as

$$\aleph(k) = c^T z(k), \tag{6.2}$$

where the vector $c^T \in \mathbb{R}^{n\times 1}$ is chosen in a way that ensures $c^T b \neq 0$ and (6.2) is intended to guarantee the stability of the system dynamics when confined to the sliding surface $\aleph(k) = 0$.

Definition 6.1 *If the motion of system* (6.1) *satisfies the inequality* $|\aleph(k)| \leq \Delta W_b$ *for all* $k \geq \lceil k_f + k_0 \rceil$ *where* $k_f \in \mathbb{R}^+$ *is the user-specified time and* $k_0 \in \mathbb{R}$ *is the initial time, then the system is said to be in a quasi sliding mode (QSM) within the* ΔW_b *neighborhood of the specified* $\aleph(k) = 0$*. The quasi-sliding-mode domain (QSMD) is the region in which the quasi-sliding mode occurs, and its width is denoted by* ΔW_b*.*

Definition 6.2 *For a discrete-time system* (6.1) *to satisfy the condition of reachability, it must fulfil the following conditions for all time steps* $k \geq 0$

$$|\aleph(k+1)| < |\aleph(k)| - \delta, \text{ if } |\aleph(k)| > \Delta W_b$$
$$|\aleph(k+1)| \leq \Delta W_b, \text{ if } |\aleph(k)| \leq \Delta W_b$$

where $\delta > 0$ *is a small constant.*

The disturbance fulfils the matching condition. The sliding function (6.2) is affected by the disturbance vector $\xi(k)$, and this effect can be described as

$$d(k) = c^T \xi(k). \tag{6.3}$$

Although the disturbance is unknown, it is assumed to be bounded in magnitude. There is a positive constant $\bar{\xi}$ that satisfies

$$|d(k)| \leq \bar{\xi} \;\forall\; k \tag{6.4}$$

In [6], authors proposed an RL given by

$$\aleph(k+1) = (1-\phi)\aleph(k) - \gamma \mathrm{sgn}(\aleph(k)) + d(k), \tag{6.5}$$

where $\phi \in (0, 1)$ and $\gamma > 0$. Provided that the gain γ is selected in a manner such that (s.t.)

$$\gamma > \frac{1+\phi}{1-\phi}\bar{\xi}, \tag{6.6}$$

in such a case, the RL (6.5) ensures QSM within a range $\Delta W_b = \gamma + \bar{\xi}$. The width of the band ΔW_b depends on the switching gain γ

Thus, the challenge is to design γ to effectively handle uncertainties and produce acceptable controller accuracy while minimizing control input consumption.

6.2 DSMC with Adaptive Switching Gain

In [45], authors proposed a discrete sliding mode approach with an adaptive switching gain when bound on uncertainty is unknown. The uncertain system given by (6.1) and the sliding function defined by (6.2) along with

$$\aleph(k+1) = (1-\phi)\aleph(k) - \gamma(k)\mathrm{sgn}(\aleph(k)) + d(k), \tag{6.7}$$

where $\gamma(k)$ is given by

$$\gamma(k) = \gamma(k-1) + \eta \mathrm{sgn}(\aleph(k))\mathrm{sgn}(\aleph(k-1)), \quad \eta > 0 \tag{6.8}$$

When using the adaptation law (AL) (6.8) in conjunction with the RL (6.7), there is always an issue of overestimating and underestimating control input [47]. An over-estimated switching gain leads to unnecessarily large control input, while an under-estimated gain reduces controller accuracy due to the applied gain being lower than required. Also, (6.8) exhibits a constant rate of adaptation.

The limitation of constant gain adaptation in (6.8) was addressed in [46] by implementing an adaptive gain strategy, however, the underestimation problem still prevails, as the adaptive gain needs to account for the sliding function. In [47], the authors proposed a solution using an adaptive gain strategy to overcome the problem of overestimation and underestimation with a time-varying gain strategy having a gain function based on the sliding function.

6.3 Main Result

A novel AL is proposed to achieve better convergence and simultaneously alleviate chattering. The RL (6.7), along with AL (6.9), aim to efficiently guide the system toward the desired state while reducing oscillations or chattering effects. Consider the following RL

$$\aleph(k+1) = (1-\varphi)\aleph(k) - \gamma(k)\mathrm{sgn}(\aleph(k)) + d(k)$$

We have proposed a new gain AL for RL (6.7) as

$$\begin{aligned} \gamma(k) &= \min\left\{|\aleph(k)|^{\beta}, \frac{|\aleph(k)|}{\lceil k_f + k_0 \rceil - k}\right\}, \\ &\text{where } k_f \in \mathbb{R}^+ > 1, \quad k_0 \in \mathbb{R}, \quad (1-\varphi) \in (0,1), \\ &\beta < 0 \quad \text{if } s(k_0) \in B(\aleph(k), 1), \text{otherwise } \beta > 1 \\ &\beta = 1 + \frac{\log(1-\varphi)}{\omega} \quad \text{if } k = \lceil k_f + k_0 \rceil, \\ &\omega = \max\{\log(|N(k)|/s_0), \underline{\omega}\}, \end{aligned} \tag{6.9}$$

where $s_0 \in \mathbb{R}^+$ is a design parameter expressed in the same units as the sliding variable and $\underline{\omega} \in \mathbb{R}^+$ small design parameter.
The motivation behind the formation of $\gamma(k)$ is to balance the control action between ensuring the desired time to reach the sliding surface and handling the problem of overestimation and underestimation.

Theorem 6.3 *Consider the DTS* (6.1) *with the sliding surface defined as* (6.2) *and if we choose RL* (6.7) *along with AL* (6.9)*, irrespective of the initial state of the system, it is ensured that the sliding variable becomes zero in the case of unperturbed system at*

$k^* \geq \lceil k_f + k_0 \rceil$ also under the disturbance condition as given in (6.3) *with Assumption 1, $\aleph(k)$ will enter the region ΔW_b at $k^* \geq \lceil k_f + k_0 \rceil$ steps.*

Proof:

Unperturbed Case: We will first analyze the positive half of state space, such that $0 < s(k_0)$.

We will further break the analysis into two segments, firstly considering $0 < s(k_0) < 1$. Using RL (6.7) we have

$$\aleph(k+1) = (1-\varphi)\aleph(k) - \gamma(k). \tag{6.10}$$

For $0 < s(k_0) < 1$, we have $\beta < 0$ and $k_f > 1$, proposed AL (6.9) will ensure that at $k_0 + 1$ instance,

$$\gamma(k) = \frac{|\aleph(k)|}{\lceil k_f + k_0 \rceil - k} < |\aleph(k)|^{\beta}. \tag{6.11}$$

For $s(k_0) \geq 1$, we have $\beta > 1$ and $k_f > 1$, proposed AL (6.9) will again ensure that at $k_0 + 1$ instance,

$$\gamma(k) = \frac{|\aleph(k)|}{\lceil k_f + k_0 \rceil - k} < |\aleph(k)|^{\beta}. \tag{6.11}$$

using (6.10) and (6.11) we have two scenarios at $k_0 + 1$ instance. If $(1-\varphi)\aleph(k) = \gamma(k)$, then $\aleph(k+1) = 0$, and if $(1-\varphi)\aleph(k) > \gamma(k)$, then $\aleph(k)$ will continue to decrease upto $k \leq \lceil k_f + k_0 \rceil - \lfloor (1-\varphi)^{-1} \rfloor = \lceil k_f + k_0 \rceil - 1$.

At $k = \lceil k_f + k_0 \rceil$,

$$\gamma(k) = |\aleph(k)|^{\beta} \leq \frac{|\aleph(k)|}{\lceil k_f + k_0 \rceil - k} \text{ since } \beta = 1 + \frac{\log(1-\varphi)}{\omega} \tag{6.12}$$

using (6.10) and (6.12), we have

$$\aleph(k+1) = 0 \text{ for } k^* \geq \lceil k_f + k_0 \rceil$$

Similar proof holds for the negative half of state space $s(k_0) < 0$;

Consider $s(k_0) = \aleph(k_0) < 0$. The RL (6.7) yields

$$\aleph(k+1) = (1-\varphi)\,\aleph(k) - \gamma(k)\,\mathrm{sgn}(\aleph(k)) = (1-\varphi)\,\aleph(k) + \gamma(k), \tag{6.13}$$

since $\mathrm{sgn}(\aleph(k)) = -1$ for $\aleph(k) < 0$. As before, two regimes are analyzed.

if $-1 < \aleph(k_0) < 0$, then $\beta < 0$ and $k_f > 1$. The AL (6.9) ensures at $k_0 + 1$ that

$$\gamma(k) = \frac{|\aleph(k)|}{\lceil k_f + k_0 \rceil - k} \; < \; |\aleph(k)|^{\beta}. \tag{6.14}$$

Using (6.13)–(6.19), two possibilities exist at k_0+1. If $\gamma(k) = (1-\varphi)\,|\aleph(k)|$, then $\aleph(k+1) = 0$. If $\gamma(k) < (1-\varphi)\,|\aleph(k)|$, then $\aleph(k+1) = -((1-\varphi)\,|\aleph(k)| - \gamma(k)) < 0$, i.e., $\aleph(k)$ remains negative with strictly reduced magnitude and continues to move toward zero for $k \le \lceil k_f + k_0 \rceil - 1$.

if $\aleph(k_0) \le -1$, then $\beta > 1$ and $k_f > 1$, and again at k_0+1

$$\gamma(k) = \frac{|\aleph(k)|}{\lceil k_f + k_0 \rceil - k} \; < \; |\aleph(k)|^{\beta}. \tag{6.19}$$

The same dichotomy follows from (6.13): equality gives $\aleph(k+1) = 0$; otherwise $\aleph(k+1) < 0$ with a strictly smaller $|\aleph(k+1)|$, and the trajectory keeps approaching zero up to $k \le \lceil k_f + k_0 \rceil - 1$.

At $k = \lceil k_f + k_0 \rceil$, the adaptive law sets

$$\gamma(k) = |\aleph(k)|^{\beta} \; \le \; \frac{|\aleph(k)|}{\lceil k_f + k_0 \rceil - k} \quad \text{since} \quad \beta = 1 + \frac{\log(1-\varphi)}{\omega}, \quad \omega = \log \frac{|\aleph(k)|}{s_0}, \tag{6.15}$$

and substituting (6.15) into (6.13) gives

$$\aleph(k+1) = 0 \quad \text{for} \quad k^* \ge \lceil k_f + k_0 \rceil.$$

Thus, for all initial conditions with $\aleph(k_0) < 0$, the sliding variable reaches the origin no later than $k^* = \lceil k_f + k_0 \rceil$, completing the symmetry of the argument for the negative half of the state space.

Perturbed Case: Now consider again positive half of state space $s(k_0) > 0$. Using RL(6.7) and disturbance condition (6.4)

$$\aleph(k+1) \le (1-\varphi)\aleph(k) - \gamma(k) + \bar{\xi}. \tag{6.16}$$

For $0 < s(k_0) < 1$, we have $\beta < 0$ and $k_f > 1$, proposed AL (6.9) will ensure that at k_0+1 instance,

$$\gamma(k) = \frac{|\aleph(k)|}{\lceil k_f + k_0 \rceil - k} < |\aleph(k)|^{\beta} \tag{6.11}$$

Also, for $s(k_0) \ge 1$, we have $\beta > 1$ and $k_f > 1$, proposed AL (6.9) will again ensure that at k_0+1 instance,

$$\gamma(k) = \frac{|\aleph(k)|}{\lceil k_f + k_0 \rceil - k} < |\aleph(k)|^{\beta}$$

using (6.16) and (6.11) we have two scenarios. If $(1-\varphi)\aleph(k) = \gamma(k)$, then $\aleph(k+1) \le \bar{\xi}$, and if $(1-\varphi)(k) > \gamma(k)$, then $\aleph(k)$ will continue to decrease upto $k < \lceil k_f + k_0 \rceil - 1$.

For $k \le \lceil k_f + k_0 \rceil$

$$\gamma(k) = \frac{|\aleph(k)|}{\lceil k_f + k_0 \rceil - k} < |\aleph(k)|^{\beta}\,, \tag{6.17}$$

using (6.16) and (6.17) we have

$$\aleph(k+1) \leq \bar{\xi} \text{ at } k^* \leq \lceil k_f + k_0 \rceil.$$

The similar proof is for $s(k_0) < 0$;

Let $s(k_0) = \aleph(k_0) < 0$. With the RL (6.7) and disturbance $d(k)$ satisfying the bound in (6.4), i.e., $|d(k)| \leq \bar{\xi}$ (Assumption 1), we have for $\aleph(k) < 0$ (thus $\text{sgn}(\aleph(k)) = -1$)

$$\aleph(k+1) = (1-\varphi)\aleph(k) + \gamma(k) + d(k) \;\; \Rightarrow \;\; \aleph(k+1) \; \geq \; (1-\varphi)\aleph(k) + \gamma(k) - \bar{\xi}. \quad (6.18)$$

As in the unperturbed analysis, consider two magnitude regimes, noting that $k_f > 1$.

if $-1 < \aleph(k_0) < 0$, then $\beta < 0$ and the AL (6.9) gives, at $k_0 + 1$,

$$\gamma(k) = \frac{|\aleph(k)|}{\lceil k_f + k_0 \rceil - k} \; < \; |\aleph(k)|^{\beta}. \quad (6.19)$$

There are two possibilities. If $\gamma(k) = (1-\varphi)|\aleph(k)|$, then from (6.18)

$$\aleph(k+1) \; \geq \; -(1-\varphi)|\aleph(k)| + (1-\varphi)|\aleph(k)| - \bar{\xi} \; = \; -\bar{\xi}.$$

Otherwise, if $\gamma(k) < (1-\varphi)|\aleph(k)|$, then

$$\aleph(k+1) \; = \; -((1-\varphi)|\aleph(k)| - \gamma(k)) + d(k) \; \geq \; -((1-\varphi)|\aleph(k)| - \gamma(k)) - \bar{\xi},$$

so $\aleph(k)$ remains negative with strictly decreasing $|\aleph(k)|$ and keeps moving toward 0 for $k \leq \lceil k_f + k_0 \rceil - 1$.

if $\aleph(k_0) \leq -1$, then $\beta > 1$ and again at $k_0 + 1$ (6.19) holds. The same dichotomy as above implies either immediate entry $\aleph(k+1) \geq -\bar{\xi}$ (when $\gamma(k) = (1-\varphi)|\aleph(k)|$) or a strict reduction in $|\aleph(k)|$ while remaining negative up to $k \leq \lceil k_f + k_0 \rceil - 1$.

Finally, at $k = \lceil k_f + k_0 \rceil$, by the AL (6.9),

$$\gamma(k) = |\aleph(k)|^{\beta} \; \leq \; \frac{|\aleph(k)|}{\lceil k_f + k_0 \rceil - k} \quad \text{since} \quad \beta = 1 + \frac{\log(1-\varphi)}{\omega}, \;\; \omega = \log \frac{|\aleph(k)|}{s_0}. \quad (6.20)$$

Substituting (6.20) into (6.18) yields

$$\aleph(k+1) \; \geq \; -\bar{\xi} \quad \text{for} \quad k^* \leq \lceil k_f + k_0 \rceil.$$

Hence, for all initial conditions with $\aleph(k_0) < 0$, the trajectory enters the band $\Delta W_b = \{\, \aleph : |\aleph| \leq \bar{\xi} \,\}$ no later than $k^* = \lceil k_f + k_0 \rceil$, completing the perturbed-case proof for the negative half of the state space.

Remark 1 Here, φ determines how the control effort is distributed among the sampling instances and the maximum control input needed. Moreover, $\lceil k_f + k_0 \rceil$ is the maximum instance at which the sliding variable enters boundary layer ΔW_b, which will be decided by the user a priori and where $k_f > 1$. Once the sliding variable changes its sign, it remains on the surface in the case of an unperturbed system, and in the case of a perturbed system, it remains in the vicinity of the sliding surface. This adaptation helps to reduce chattering, as this AL does not necessitate that the sliding function will cross the sliding plane in every instance.
Further boundedness of $\gamma(k)$ is studied through lemma 1.

Lemma 6.4 *Consider the RL given by* (6.7) *and the AL given by* (6.9)*, then there exists* $\gamma^* \in \mathbb{R}^+$ *such that:*

$$|\gamma(k)| \leq \gamma^* \quad \forall \quad k_0 \geq 0.$$

Proof:

The AL (6.9) states that if $|\aleph(k)| \leq |s(k-1)|$, then the switching gain $\gamma(k)$ decreases. Therefore, whenever $|\aleph(k)| \leq |s(k-1)|$ then $\gamma(k) \leq \gamma^*$, ensuring boundedness of $\gamma(k)$. When $|\aleph(k)| > |\aleph(k-1)|$, the switching gain γ increases. To guarantee that $\gamma(k)$ remains bounded, we will examine two scenarios.

Case 1: $\text{sign}(\aleph(k)) = \text{sign}(\aleph(k-1)), \aleph(k) > 0$ and $|\aleph(k)| > |\aleph(k-1)|$. In this scenario, the sliding function deviates from the sliding surface. Let k_m instance indicate the beginning of Case 1, and using (6.9) and assuming $\bar{\xi}$ in the positive half of state space, we have $\aleph(k+1) \leq (1-\varphi)\aleph(k) - \gamma(k)\text{sgn}(\aleph(k)) + \bar{\xi}$. Considering $\bar{\xi}$ remains fixed and $\gamma(k)$ rises with each sample, there exists a sampling instance k_n at which $\gamma(k) - \bar{\xi} > 0$. Since $\aleph(k_n) > 0$ and $0 < (1-\varphi) < 1$. Hence, $\aleph(k_n+1) < (1-\varphi)\aleph(k_n) < |\aleph(k_n)|$. Now, we will investigate two scenarios

Case 1a: If $\aleph(k_n+1) > 0$, then we have $|\aleph(k)| < |\aleph(k-1)|$ at $k = k_n + 1$. In this scenario, $\gamma(k)$ will decrease. Let $\gamma_1 \in \mathbb{R}^+$ such that $\gamma(k) < \gamma_1 \ \forall \ k \ \in [k_0, k_n + 1]$.

Case 1b: If $\aleph(k_n+1) < 0$, then at $k = k_n + 1$ we have two possibilities

- If $|\aleph(k)| < |\aleph(k-1)|$, then as mentioned in case 1a, $\gamma(k)$ decreases, and our assumption that $\gamma(k) < \gamma_1$ remains valid as long as the sliding variable continues to decrease under the condition of $|\aleph(k)| < |\aleph(k-1)|$.

- If $|\aleph(k)| > |\aleph(k-1)|$, then $\aleph(k) < 0, \aleph(k-1) > 0$. The behavior of s and the limitation of γ in this scenario have been examined in case (2).

Case 2: $\text{sign}(\aleph(k)) \neq \text{sign}(\aleph(k-1)), \aleph(k-1) > 0$

$\aleph(k) < 0$ and $|\aleph(k)| > |\aleph(k-1)|$. In this scenario, using (6.9) and (6.7), the sliding function will be as follows: $\aleph(k+1) = -(1-\varphi)\aleph(k) + \gamma(k) - d(k) \rightarrow < \gamma(k) < |\aleph(k)|$. Hence, from case (1a), $\gamma(k)$ will decrease, and it will be bounded as $\gamma(k) < \gamma_1 \forall k$.

Case 3: $\text{sign}(\aleph(k)) = \text{sign}(\aleph(k-1)), \aleph(k) < 0$ and $|\aleph(k)| > |\aleph(k-1)|$.

Assuming $\bar{\xi}$ in the negative half of sliding manifold, $\aleph(k+1) \leq -(1-\varphi)\aleph(k) + \gamma(k)\text{sgn}(\aleph(k)) - \bar{\xi}$, let k_p indicate the beginning of Case 3, Since $|\aleph(k)| > |\aleph(k-1)|$, $\gamma(k)$ will increase at $k = k_p$ and will continue to increase upto condition $|\aleph(k)| > |\aleph(k-1)|$ holds. considering $\bar{\xi}$ remains fixed and $\gamma(k)$ rises until $k = k_q$, at which $\gamma(k) - \bar{\xi} > 0$ Since $\aleph(k_q) < 0$ as in Case 1 we will have $\aleph(k_q+1) > -(1-\varphi)\aleph(k_q) > -|\aleph(k_q)|$ and $k = k_q + 1$, the sliding variable stops increasing in the negative side of the sliding surface. This leads us to two scenarios,

Case 3a: if $\aleph(k_q+1) < 0$, then we have $|\aleph(k)| < |\aleph(k-1)|$ at $k = k_q + 1$. In this scenario, $\gamma(k)$ decreases. Let us assume there exists $\gamma_2 \in \mathbb{R}^+$ such that $\gamma(k) < \gamma_2 \ \forall \ k \in [k_p, k_q + 1]$

Case 3b: if $\aleph(k_n+1) > 0$, then at $k = k_q + 1$ we have two possibilities

- If $|\aleph(k)| \leq |\aleph(k-1)|$, then, as mentioned in case 3a $\gamma(k)$ decreases, and our assumption that $\gamma(k) < \gamma_2$ remains valid as long as the sliding variable continues to decrease under the condition of $|\aleph(k)| \leq |\aleph(k-1)|$.

- If $|\aleph(k)| > |\aleph(k-1)|$, then $\aleph(k) > 0$, $\aleph(k-1) < 0$. The behavior of s and the limitation of γ in this scenario have been examined in case (4).

Case 4: $\text{sign}(\aleph(k)) \neq \text{sign}(\aleph(k-1)), \aleph(k-1) < 0 \aleph(k) > 0$ and $|\aleph(k)| > |\aleph(k-1)|$. In this scenario, the sliding function will be $\aleph(k+1) = (1-\varphi)\aleph(k) - \gamma(k) + d(k)$, and the analysis will be similar to that in case 2, and therefore, $\gamma(k)$ is bounded. for Case (3) and Case (4), let $\gamma_2 \ \epsilon \ \mathbb{R}^+$ s.t. $\gamma(k) < \gamma_2$ also, $\gamma(k) \leq (\gamma_1, \gamma_2) = \max(\gamma_1, \gamma_2)$.

Now, using the above results, we can further state the boundness results for the sliding variable.

Theorem 6.5 *The combination of the RL* (6.7) *and the gain AL* (6.9) *results in a solution that is ultimately confined within a global bound, with the switching function being ultimately bounded within a band* ΔW_b.

$$|\aleph(k)| = \Delta W_b \ < (\gamma^* + \bar{\xi}). \tag{6.21}$$

Proof: Consider the Lyapunov candidate function.

$$V(k) = \mu|\aleph(k)|. \tag{6.22}$$

Here, $\mu > 0$. We have:

$$\begin{aligned} \Delta V(k) &= V(k+1) - V(k) \\ &= \mu|\aleph(k+1)| - \mu|\aleph(k)| \end{aligned} \tag{6.23}$$

Consider

$$j(k) = -\gamma(k)sgn(\aleph(k)) + d(k) \tag{6.24}$$

Using (6.23) and (6.7), we have

$$\Delta V(k) = \mu|(1-\varphi)\aleph(k) + j(k)| - \mu|\aleph(k)| \tag{6.25}$$

Now, using (6.25)

$$\begin{aligned} \Delta V(k) &\leq \mu|\aleph(k)| - \varphi\mu|\aleph(k)| + \mu|j(k)| - \mu|\aleph(k)| \\ &\leq -\varphi\mu|\aleph(k)| + \mu|j(k)| \\ &\leq -\varphi V(k) + \mu|j(k)| \end{aligned} \tag{6.26}$$

Now using lemma 1 and from (6.4) we have

$$|j(k)| \leq -\gamma(k)sgn(\aleph(k)) + \bar{\xi} \leq \gamma^* + \bar{\xi} \tag{6.27}$$

Now using (6.26) and (6.27), we have

$$\Delta V(k) \leq -\varphi V(k) + \mu(\gamma^* + \bar{\xi}) \tag{6.28}$$

Hence, The solution of the RL (6.7) is Globally Ultimately Bounded if,

$$\Delta W_b \ \leq \frac{(\gamma^* + \bar{\xi})}{\varphi} < (\gamma^* + \bar{\xi}) \tag{6.29}$$

Also, the ultimate band $|\aleph(k)| \leq \Delta W_b$

Corollary 6.6 *Consider the DTS* (6.1) *under Assumption 1. If we choose the sliding surface* (6.2) *under the disturbance condition as* (6.3) *with RL* (6.7) *and AL* (6.9)

- *It is guaranteed that the state $\aleph(k)$ will enter the region ΔW_b within a maximum of k^* steps, regardless of the initial state of the system.*
- *Once the state $\aleph(k)$ becomes a part of the region ΔW_b, it remains confined within it for the future.*

Proof: Our first aim is to demonstrate that the trajectories of the system will inevitably converge to the region,

$$\Delta W_b = \aleph(k) : \ |\aleph(k)| < \gamma^* + \bar{\xi}$$

regardless of the initial state.

Consider $\aleph(k) > 0$,

$$\aleph(k+1) = (1-\varphi)\aleph(k) - \gamma(k)\mathrm{sgn}(\aleph(k)) + d(k).$$

using (6.29) and (6.4), we have

$$\aleph(k+1) \leq \aleph(k) - \gamma^* + \bar{\xi}. \tag{6.30}$$

Using (6.17), equation (6.30) reduces to

$$\aleph(k+1) \leq \aleph(k).$$

This indicates that the sequence $\aleph(k)$ undergoes a decreasing trend until it intersects the switching function. Let us assume the existence of an instant k_t s.t.

$$\aleph(k_t) > 0 \text{ and } \aleph(k_t+1) < 0.$$

then

$$|\aleph(k_t+1)| \leq |(1-\varphi)\aleph(k_t) - \gamma(k)\mathrm{sgn}(\aleph(k_t)) + \xi(k_t)|$$

$$|\aleph(k_t+1)| \leq -\gamma^* - \bar{\xi}$$

Hence, $\aleph(k_t+1) \ \in \ \Delta W_b$ at the instance $k^* = k_t$, and for the case $\aleph(k) < 0$, the proof is the same.

Now our aim is to prove that when $\aleph(k) \in \Delta W_b$, it inevitably follows that $\aleph(k+1) \in \Delta W_b$, provided that $0 < \aleph(k) \leq (\gamma^* + \bar{\xi})$.

Consider $0 < \aleph(k) < (\gamma^* + \bar{\xi})$, then,

$$\begin{aligned}
|\aleph(k+1)| &= |(1-\varphi)\aleph(k) - \gamma(k)\mathrm{sgn}(\aleph(k)) + d(k)| \\
&\leq \aleph(k) - \gamma^* + d(k) \\
&\leq \gamma^* + \xi - (\gamma^* - \xi) \\
&\leq (\gamma^* + \xi).
\end{aligned}$$

a Similar way, we can prove for the condition $-(\gamma^* + \bar{\xi}) < \aleph(k) \leq 0$.

6.4 Simulation

In this section, the validation of the proposed results is showcased through simulation using two examples and comparing with previous results [46].

Example 6.7 *Consider a second-order spring mass damper system*

$$z(k+1) = \begin{pmatrix} 0 & 1 \\ -1 & -0.13 \end{pmatrix} z(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(k) + d(k) \tag{6.31}$$

Here, $z(k)$ represents the state of the system, and $d(k)$ represents the disturbance signal.

$$d(k) = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix} \times \left(\sin(k) + \sin\left(\frac{k}{3}\right) \right) \tag{6.32}$$

In (6.2), the sliding surface is designed with the parameter $c^T = [1\ 2]^T$ and the initial conditions as specified in Table I. Fig.6.4 and Fig.6.4 displays the outcome of the simulation and its comparison with [46] using parameter $w = 0.8, v = 0.6$. Table II provides numerical evaluations for the absolute sum of control input $u(k)$ and the absolute sum of switching gain $\gamma(k)$ concerning different k_f. Table II shows that the switching gain $\Sigma|\gamma(k)|$ and control input $\Sigma|u(k)|$ concerning k_f decreases and increases, respectively. Also from Fig 1, it is clear that the proposed AL reduces QSMD concerning [46].

Example 6.8 *Consider the discrete dynamical system*

$$z(k+1) = \begin{pmatrix} 1 & 1 & 0.5 \\ 0 & 1 & 1 \\ 0 & -0.5 & 1 \end{pmatrix} z(k) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u(k) + d(k). \tag{6.33}$$

Table 6.1

S.No	Initial condition / Parameter	Value
1	k_f	16
2	k_0	2
3	$z_1(0)$	-5
4	$z_2(0)$	20

Table 6.2

k_0	k_f	$\sum_{k=1}^{k_f} \|u(k)\|$	$\sum_{k=1}^{k_f} \|\gamma(k)\|$
2	4	10.71	9.18
2	8	41.97	6.80
2	12	43.06	4.93
2	16	43.43	3.86

Here, $z_1(k)$, $z_2(k)$, $z_3(k)$ represent the states of the system. $d(k)$ represent discretized disturbances signals.

$$d(k) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times (0.2(-1)^{\lfloor k/10 \rfloor}). \tag{6.34}$$

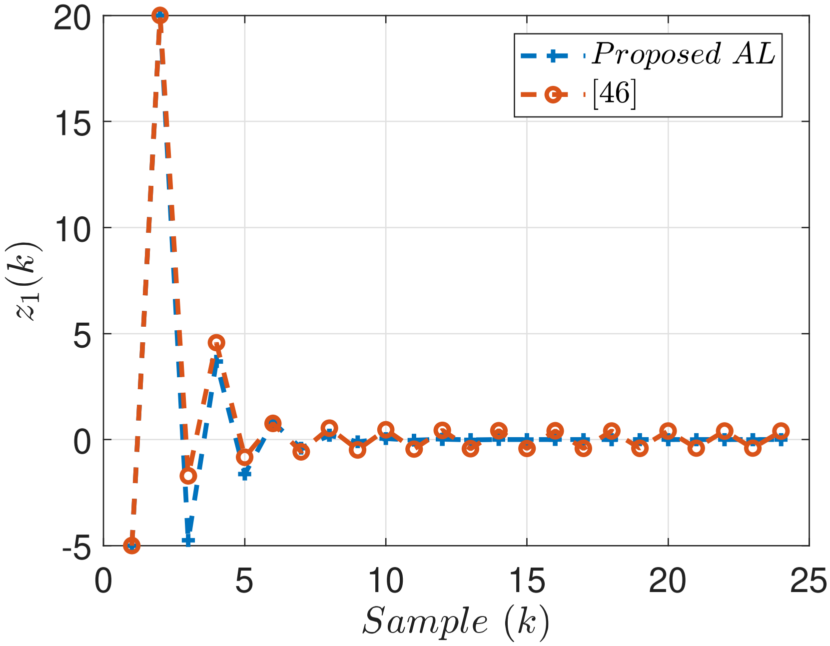
The considered system is representative of a third-order discrete-time system with matched uncertainty. In equation (6.2), the sliding surface is constructed using the parameter $c^T = [1\ 1.5\ 1]^T$ and initial condition as per Table III. The simulation results and its comparison with [46] using the same parameter as defined above, are depicted in Fig.6.4 and Fig.6.4. Tables IV provide a numerical assessment for the absolute sum of the control input $u(k)$ and the absolute sum of the switching gain $\gamma(k)$ at different values of k_f. Obtained results show that the system state, when utilizing the proposed algorithm, stays near the origin. As a result, the proposed adaptive sliding mode control algorithm ensures reduced control efforts, and reduces QSMD concerning [46].

Table 6.3

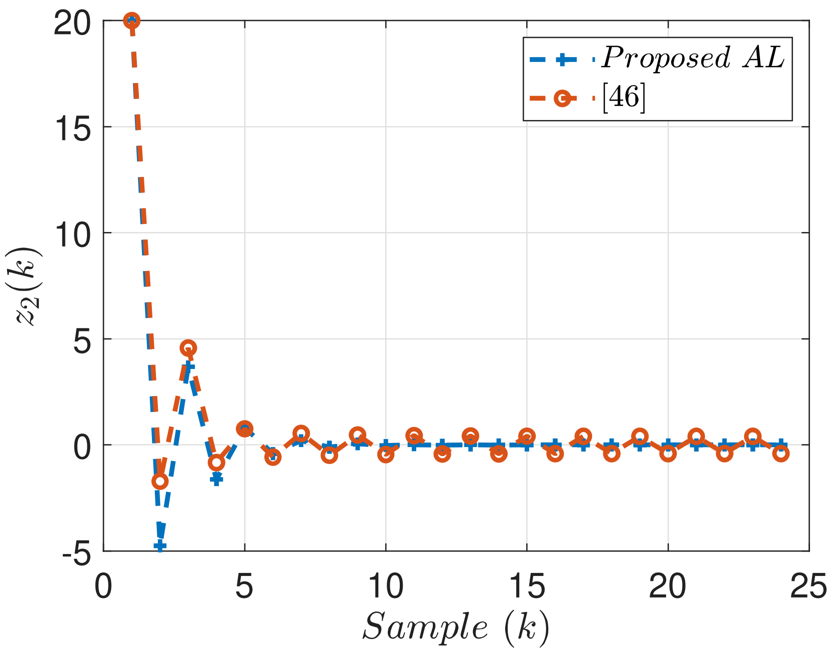
S.No	Initial condition / Parameter	Value
1	k_f	16
2	k_0	1
3	$z_1(0)$	-5
4	$z_2(0)$	20
5	$z_3(0)$	20

Table 6.4

k_0	k_f	$\Sigma_{k=1}^{k_f+1} \|u(k)\|$	$\Sigma_{k=1}^{k_f} \|\gamma(k)\|$
1	4	246.33	15.93
1	8	224.46	9.65
1	10	217.23	6.81
1	16	214.05	5.25



(a) Proposed AL vs [47] for state $z_1(k)$



(b) Proposed AL vs [47] for state $z_2(k)$

Figure 6.1: Proposed AL vs [47]

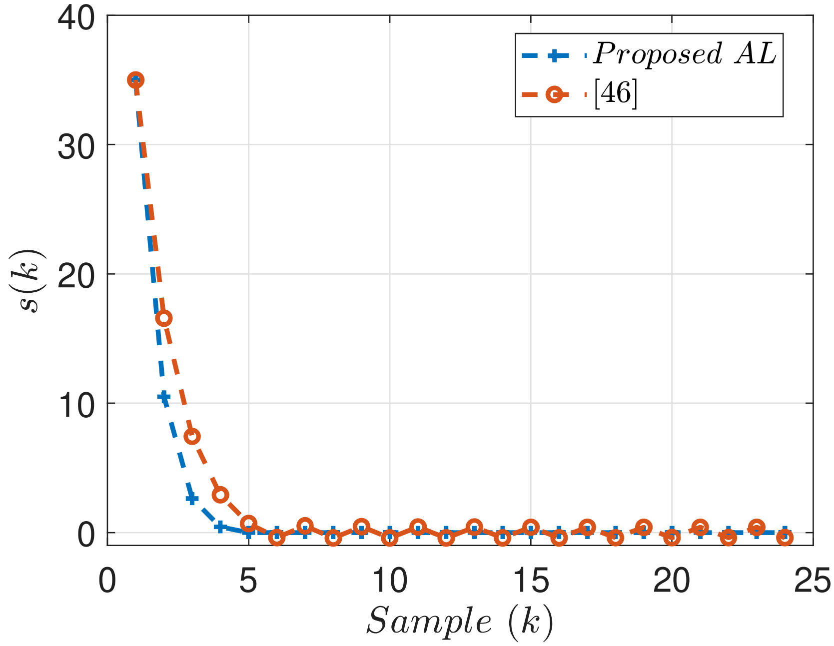


Figure 6.2: Comparison of sliding variable for Proposed AL vs [47]

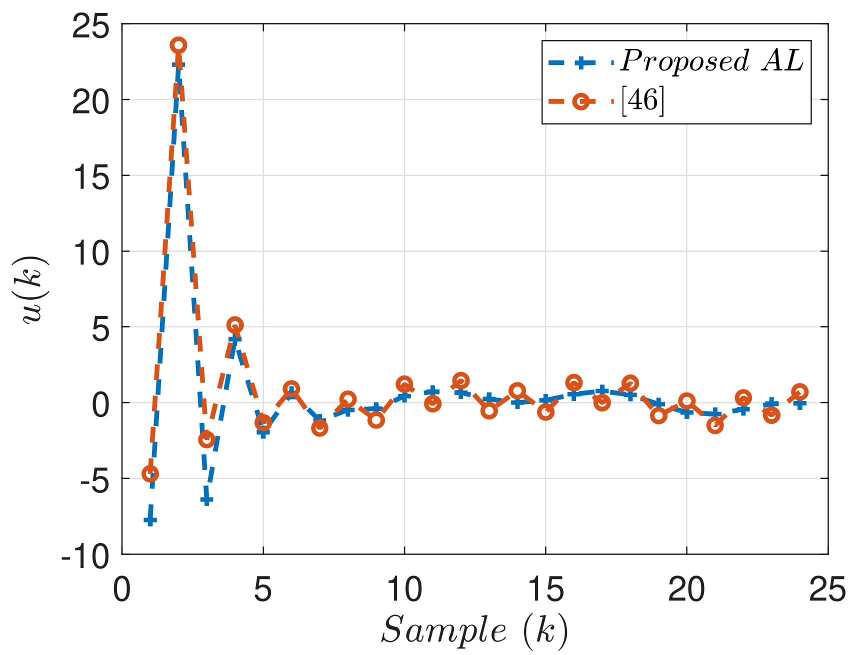
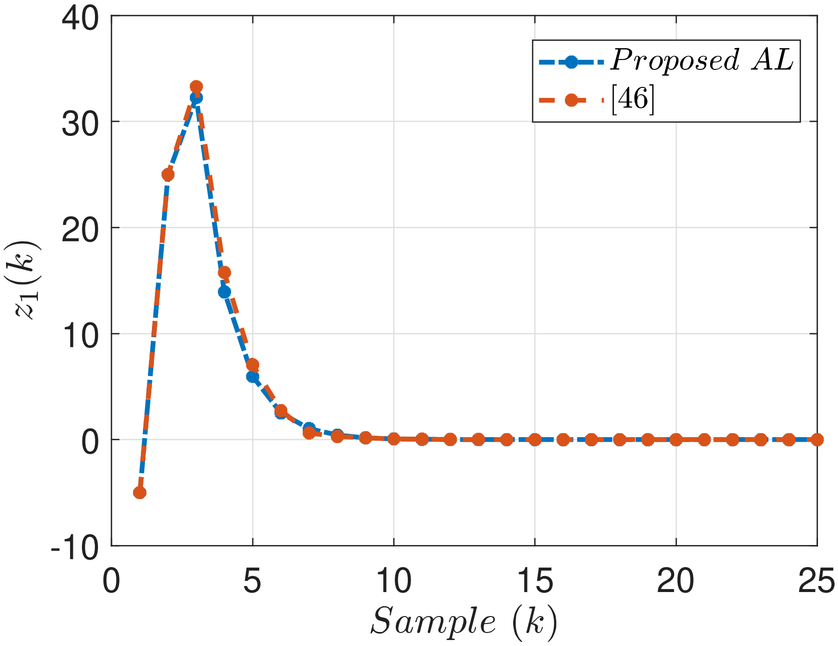
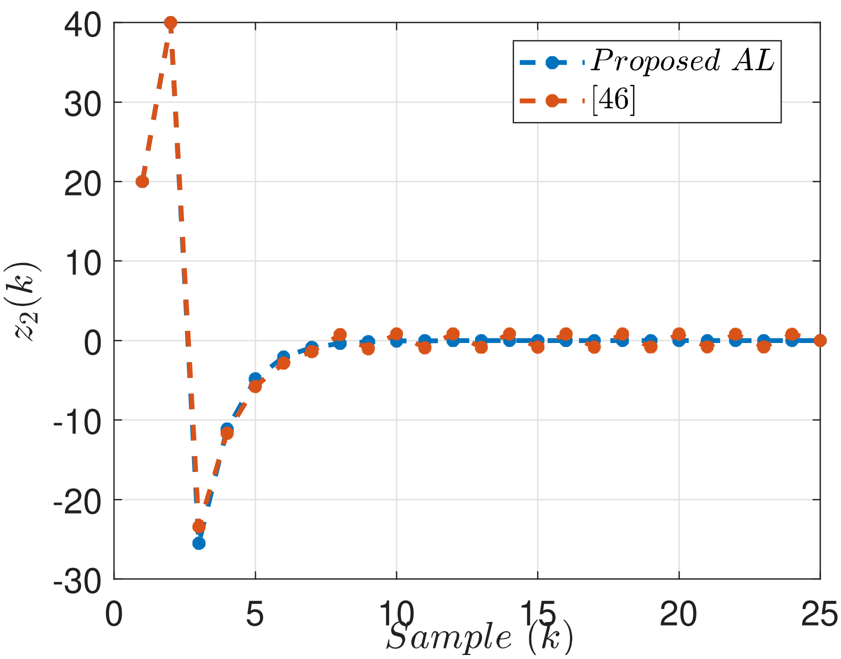


Figure 6.3: Comparison of control performance forProposed AL vs [47]



(a) Proposed AL vs [47] for state $z_1(k)$



(b) Proposed AL vs [47] for state $z_1(k)$

Figure 6.4: Proposed AL vs [47]

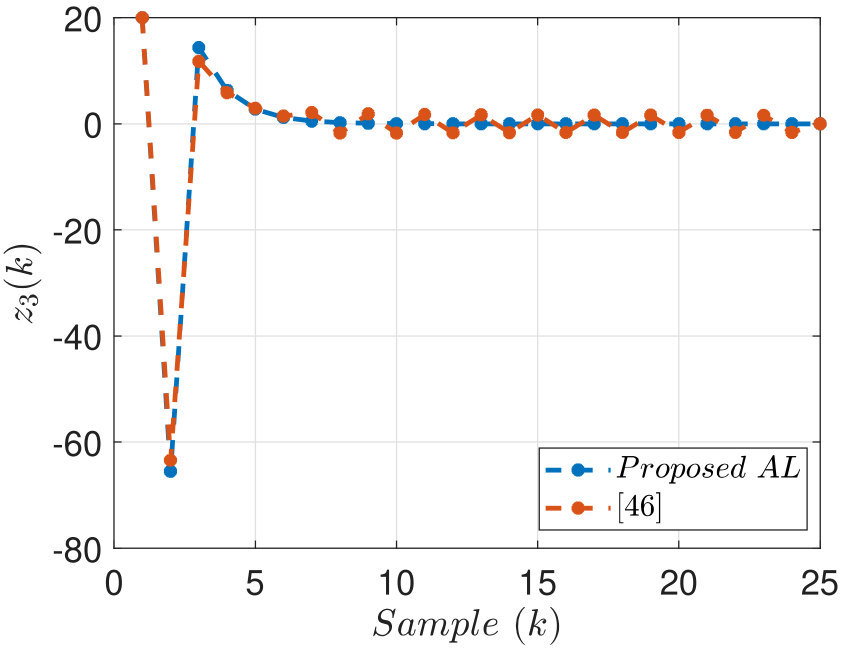


Figure 6.5: Proposed AL vs [47] for state $z_1(k)$

6.5 Conclusions

This chapter presents the design and verification of a discrete-time sliding-mode control system based on an adaptive RL. By incorporating a minimum operator and ceiling function in AL, the proposed adaptive RL gives better control in terms of convergence rate and reaching time. This chapter introduces better AL for gain adaptation concerning [46] in adaptive discrete-time sliding mode. The study also includes a theoretical analysis of the reaching steps and convergence properties under disturbance. Theoretical analysis and simulation results demonstrate that the proposed approach offers reduced QSMD and a better selection of reaching time based on available control effort. This new method also avoids the issues of chattering.

Future work will focus on extending the methodology to systems with more pronounced nonlinearities, higher-dimensional dynamics, lightly excited systems, and unmatched uncertainties. In addition, comparisons with a broader range of state-of-the-art approaches will be conducted to demonstrate wider applicability.