Chapter 2

Solution of variable-order partial integro-differential equation using Legendre wavelet approximation and operational matrices

2.1 Introduction

One of the crucial origins of fresh water is ground water, that fulfills our basic need for drinking water and also water for industries and agriculture. Ground water provides about 97% of the fresh water on the earth that is why it is more important than surface water. Unfortunately, it is getting polluted because of direct or indirect discharge of pollutants into the water bodies. Main factors that are responsible for

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polluting water are industrialization, urbanization and agriculture. India stands in the list of countries affected by water pollution, that is why it is very important to develop a mathematical model to predict the movement of solute in aquifers and how it affects environment and human health. We require excellent understanding of chemical, physical and biological processes those can control solute transportation in ground water. We should definitely consider domain of problems, the parameters used to create groundwater model for field problems and also for the given boundary conditions. Researchers now a days are very much interested in the topic of solute transportation and that is why there are lots of methods found in recent years to solve several different models depicting solute transportation. Presently, many problems with different concepts of science are related with the non-linear equations [78], [79], [80]. During literature survey, various models on nonlinear problems have been found [81], [82], [83], [84], [85]. Many dynamical systems when modeled, generate an integral term consisting of an unknown function. The integro-differential equations (IDEs) and the integral, appear in modeling due to several phenomena of sciences. An integro-differential equation is an equation that contains both integral and derivative of functions. This partial integro-differential equations (PIDEs) are advantageous in various applications, like, quantitative socio-dynamics [86], [87], [88], [89]. Study of PIDEs has an utermost powerful equations, which are fractonal partial integro-differential equations (FPIDEs). In [90] researchers have studied the fractional order thermoelastic problem of porous structure. In last few years, researchers have found the fractional calculus, as an efficient tool which can help one a lot in describing the behaviors of several dynamical systems with more accuracy. The PIDEs can be studied based upon Chebyshev wavelets, one-dimensional Legendre wavelets (LWs), cubic B-spline collocation, and one-dimensional Bernoulli wavelets, etc. Various studies are present in the literature to solve the variable-order differential equation (VODEs) [91] such as optimization method to solve Variable Order

Poisson equation, one-dimensional LWs to solve VO non-linear advection–diffusion equation with variable coefficients [92]. Countless natural systems can be shaped using fractional-order PDE, for example, pollution of groundwater. To model the transportation of pollutant in the surface water, atmosphere and groundwater, one can use the fractional order reaction-advection-diffusion equation (FRADE). The flow equations are non-linear, but advection and diffusion are of primary importance [93], [94], [75], [95]. We are already familiar with the integer order differentiation and integration whereas the integer order differentiation and integration can be generalized with fractional calculus to an arbitrary order.

Recent years, there are a growth of important dynamical problems that depend on time, space, or both and show behaviour of fractional order. In fact, variable order calculus can be more useful for illuminating more complicated dynamical problems. Variable order operators are a new paradigm in science [96], have generalised the variable order case for Riemann–Liouville and Marchaud fractional integration and differentiation and also explained inversion formula. Different authors have explained different definitions for derivatives of variable order each suits the respective aim. Variable order derivative is permitted to vary as a function of t or as a function of x or both. Researchers like [97] have discovered deeper in the concept of variable order derivative. For the Caputo-definition, Riemann-Liouville-definition, Marchaud-definition, and Grünwald-definition, the researchers have suggested definitions based on variable order operators. Operators with variable orders have kernels with variable exponents. This makes finding the analytical solutions for variable order fractional differential equations more challenging. Fortunately, advancement of numerical techniques are at an early stage but can be more effective to find solutions. For the solution of variable order differential equations, a consistent approximation is suggested [98]. Stability and convergence of finite-difference approximation are investigated for the variable-order nonlinear fractional diffusion equation [99]. It is observed that variable order operator is more effective when it comes for solving dynamical problems.

Porous media refers to a medium that contains pores. One of the primary elements for all the living things on earth is water, which is present in two forms on earth (underground, out of which only 2.5 % is fresh) and surface water. Most important source of fresh water is underground water. Contaminated groundwater is very harmful to everyone -humans, wildlife, environment etc. The most of the structures through which contaminated ground water passes are porous type. Many scientists and researchers are working on predicting the movement of contaminated groundwater through several porous media through developing different models.

A lot of work has been done by so many researchers related to fractional order diffusion equation [100], [93] to observe the nature of diffusivity of contaminated water in porous media. But to the best of the authors' knowledge, a very few works related to variable ordered diffusion equation are found in the literature survey. In the present scientific contribution, the authors have considered one dimensional variable ordered non-linear partial differential equation. Main aim of this chapter is to achieve the numerical solution of the VOPIDE with higher accuracy. Firstly, an operational matrix is introduced for variable order derivative and another operational matrix for integration, which have been implemented for one-dimensional Legendre wavelets to obtain the desired results. The VOPIDEs can be then, scaled down into a system of algebraic equations, by employing one dimensional LWs approximations and the operational matrices for variable order derivative and integration. After that Newton–Cotes collocation method is applied on the obtained system of algebraic equations to achieve the numerical solution of the VOPIDE using MATHEMATICA software (version 11.3).

2.2 Solution of the variable order non-linear partial differential equation

In this section a drive has been taken to solve the variable order non-linear reaction-advection-diffusion equation given as

$$
\begin{aligned}
{}_0^cD_t^{\alpha(x,t)}u(x,t) = D_1\frac{\partial^2 u(x,t)}{\partial x^2} + V_1\frac{\partial u(x,t)}{\partial x} + \lambda u(x,t)(1-u(x,t)) \\
+\,\delta\int_o^t u(x,\tau)d\tau, \quad 0 \le \alpha(x,t) \le 1,
\end{aligned} \tag{2.1}
$$

under the initial and boundary conditions as

$$u(x,0) = \xi_1(x), \quad 0 \le x \le 1, \tag{2.2}$$

$$u(0,t) = \xi_2(t), \quad 0 \le t \le 1, \tag{2.3}$$

and

$$u(1,t) = \xi_3(t), \quad 0 \le t \le 1. \tag{2.4}$$

Firstly the shifted Legendre polynomial will be used to approximate the function $u(x,t) \in [0,1] \times [0,1]$ as

$$u(x,t) \approx \Psi_{k,M}^T(t)U\Psi_{k_1,M_1}(x). \tag{2.5}$$

Then its derivatives can be defined as

$${}_0^cD_t^{\alpha(x,t)}u(x,t) \approx ({}_0^cD_t^{\alpha(x,t)}\Psi_{k,M}^T(t))U\Psi_{k_1,M_1}(x) = \Psi_{k,M}^T(t)(Q^{(\alpha(x,t))})^T U\Psi_{k_1,M_1}(x), \tag{2.6}$$

$$\frac{\partial^n u(x,t)}{\partial x^n} \approx \frac{\partial^n(\Psi^T_{k,M}(t)U\Psi_{k_1,M_1}(x))}{\partial x^n} = \Psi^T_{k,M}(t)U\frac{\partial^n \Psi_{k_1,M_1}(x)}{\partial x^n}$$
$$= \Psi^T_{k,M}(t)U(D^n\Psi_{k_1,M_1}(x)), n \in N, \tag{2.7}$$

$$\int_0^t u(x,\tau)d\tau \approx \int_0^t (\Psi^T_{k,M}(\tau)U\Psi_{k_1,M_1}(x))d\tau = (\int_0^t \Psi^T_{k,M}(\tau)d\tau)U\Psi_{k_1,M_1}(x)$$
$$= (\Psi^T_{k,M}(t)P^T)U\Psi_{k_1,M_1}(x). \tag{2.8}$$

Now Substituting the equations (2.6)- (2.8) in equations (2.1) and (2.2), we get

$$\begin{aligned}\Psi^T_{k,M}(t)(R^{-1}F_t^{\alpha(x,t)}R)^T U\Psi_{k_1,M_1}(x) = {} & D_1\Psi^T_{k,M}(t)U(D^2\Psi_{k_1,M_1}(x)) \\ & + V_1\Psi^T_{k,M}(t)U(D\Psi_{k_1,M_1}(x)) \\ & + \lambda\Psi^T_{k,M}(t)U\Psi_{k_1,M_1}(x) \\ & - \lambda[\Psi^T_{k,M}(t)U\Psi_{k_1,M_1}(x)]^2 \\ & + \delta(\Psi^T_{k,M}(t)P^T)U\Psi_{k_1,M_1}(x),\end{aligned} \tag{2.9}$$

and

$$\Psi^T_{k,M}(0)U\Psi_{k_1,M_1}(x) = \psi_1(x), \tag{2.10}$$

where $Q^{\alpha(x,t)} = R^{-1}F_t^{\alpha(x,t)}R$.

Equations (2.9) and (2.10) are rewritten as

$$\begin{aligned}H(x,t) = {} & \Psi^T_{k,M}(t)(R^{-1}F_t^{\alpha(x,t)}R)^T U\Psi_{k_1,M_1}(x) - D_1\Psi^T_{k,M}(t)U(D^2\Psi_{k_1,M_1}(x)) \\ & - V_1\Psi^T_{k,M}(t)U(D\Psi_{k_1,M_1}(x)) - \lambda(\Psi^T_{k,M}(t)U\Psi_{k_1,M_1}(x)) \\ & + \lambda(\Psi^T_{k,M}(t)U\Psi_{k_1,M_1}(x))^2 - \delta((\Psi^T_{k,M}(t)P^T)U\Psi_{k_1,M_1}(x)) \\ & + \Psi^T_{k,M}(0)U\Psi_{k_1,M_1}(x) - \xi_1(x),\end{aligned} \tag{2.11}$$

and the boundary conditions (2.3)- (2.4) become

$$\Psi^T_{k,M}(t)U\Psi_{k_1,M_1}(0) = \xi_2(t), \tag{2.12}$$

$$\Psi^T_{k,M}(t)U\Psi_{k_1,M_1}(1) = \xi_3(t). \tag{2.13}$$

Equation (2.11) is collocated for $(m+1)\times(m+1)$ points at (x_i, t_j) and equations (2.12) and (2.13) are collocated for $(m+1)$ points at t_j. Here x_i's are the roots of shifted Legendre polynomial $P^l_{m-1}(x)$ and t_j's are the roots of shifted Legendre polynomial $P^\tau_{m+1}(t)$. After the collocation of $(m+1)\times(m+1)$ points, a system of non-linear equations with $(m+1)\times(m+1)$ unknowns is obtained which are given as

$$\begin{aligned} H(x_i, t_j) = {} & \Psi^T_{k,M}(t_j)(R^{-1}F_t^{\alpha(x_i,t_j)}R)^T U\Psi_{k_1,M_1}(x_i) - D_1\Psi^T_{k,M}(t_j)U(D^2\Psi_{k_1,M_1}(x_i)) \\ & - V_1\Psi^T_{k,M}(t_j)U(D\Psi_{k_1,M_1}(x_i)) - \lambda\Psi^T_{k,M}(t_j)U\Psi_{k_1,M_1}(x_i) \\ & + \lambda(\Psi^T_{k,M}(t_j)U\Psi_{k_1,M_1}(x_i))^2 - \delta(\Psi^T_{k,M}(t_j)P^T)U\Psi_{k_1,M_1}(x_i) \\ & + \Psi^T_{k,M}(0)U\Psi_{k_1,M_1}(x_i) - \xi_1(x_i), \end{aligned} \tag{2.14}$$

and

$$\Psi^T_{k,M}(t_j)U\Psi_{k_1,M_1}(0) - \xi_2(t_j) = 0, \tag{2.15}$$

$$\Psi^T_{k,M}(t_j)U\Psi_{k_1,M_1}(1) - \xi_3(t_j) = 0. \tag{2.16}$$

The system of non-linear equations are then solved by Newton–Cotes collocation points for $(m+1)\times(m+1)$ unknown entries of the unknown matrix U. The approximate solution $u_{n,m}(x,t)$ given by equation (2.5) can be calculated using simple computation.

2.3 Numerical Application

In this section, the proposed method is applied on the following numerical example whose exact solution is known to validate the accuracy and efficiency of the method.

2.3.1 Example [2]

$$ {}_0^cD_t^{\alpha(x,t)}u(x,t) + x\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} = f(x,t) + \mu\int_0^t u(x,\tau)d\tau, \quad 0 < \alpha(x,t) \leq 1, \tag{2.17} $$

where

$$ f(x,t) = 2t^\alpha + 2x^2 + 2 - \mu\Big(x^2t + \frac{2\Gamma(\alpha+1)t^{2\alpha+1}}{(2\alpha+1)\Gamma(2\alpha+1)}\Big), \tag{2.18} $$

with initial condition

$$ u(x,0) = x^2, x \in [0,1] \tag{2.19} $$

and boundary conditions

$$ u(0,t) = \frac{2\Gamma(\alpha+1)t^{2\alpha}}{\Gamma(2\alpha+1)}, t \in [0,1] \tag{2.20} $$

and

$$ u(1,t) = 1 + \frac{2\Gamma(\alpha+1)t^{2\alpha}}{\Gamma(2\alpha+1)}, t \in [0,1]. \tag{2.21} $$

The exact solution of the problem is given by [2]

$$ u(x,t) = x^2 + \frac{2\Gamma(\alpha+1)t^{2\alpha}}{\Gamma(2\alpha+1)}. \tag{2.22} $$

For the comparison of the approximate solution with the exact solution, let us define the L_2- error as

$$L_2(t) = \sqrt{\int_0^1 |u(x,t) - u_{k,M,k_1,M_1}(x,t)|^2 dx}, \tag{2.23}$$

where $u(x,t)$ and $u_{k,M,k_1,M_1}(x,t)$ represent the exact and approximate solutions, respectively.

The proposed method is applied in the aforementioned example for μ=1, $k = k_1$=1, $M = M_1$=7 and $\alpha = 0.15, 0.35, 0.55, 0.75, 0.95$, just to get the approximate solution and the results of L_2-error are depicted through Table 2.1. Figure 2.1 depicts the absolute error $|u(x,t) - u_{num}(x,t)|$ between exact and approximate solutions *vs.* x at fixed $t = 0.5$, whereas Figure 2.2 depicts the comparison between exact solution and the approximate solution of $u(x,t)$ for a fixed $t = 0.5$.

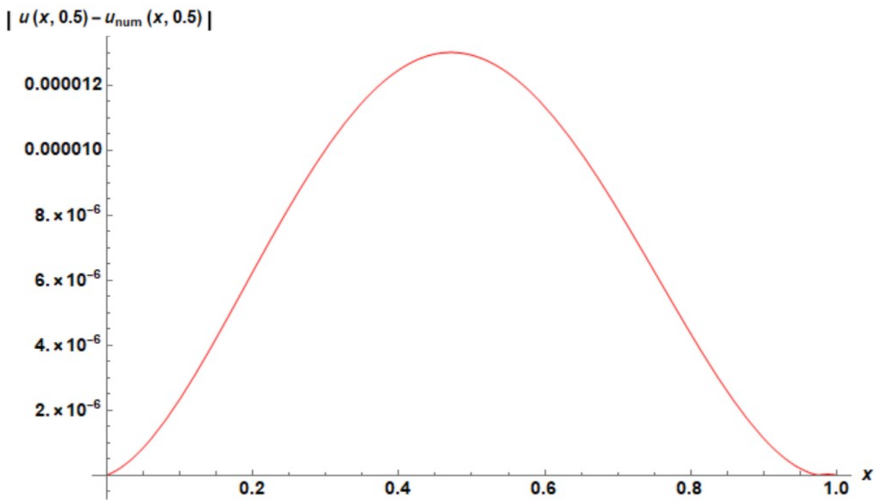

Figure 2.1: Plots of absolute error between exact and approximate solutions *vs.* x at fixed $t = 0.5$.

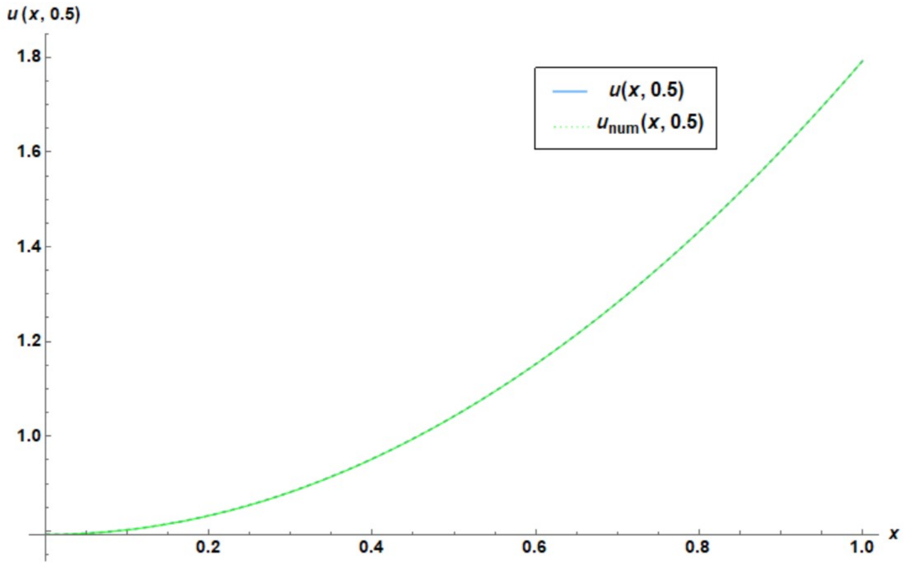

Figure 2.2: Plots of exact $u(x, 0.5)$ and approximate $u_{num}(x, 0.5)$ solutions *vs.* x at fixed $t = 0.5$.

Table 2.1 L_2 - error between exact and approximate solutions at different values of t for $\alpha = 0.15, 0.35, 0.55, 0.75$ and 0.95.

t	$\alpha = 0.15$	$\alpha = 0.35$	$\alpha = 0.55$	$\alpha = 0.75$	$\alpha = 0.95$
0.0625	2.17×10^{-2}	3.09×10^{-3}	6.30×10^{-5}	9.11×10^{-5}	6.63×10^{-6}
0.1875	5.39×10^{-3}	5.81×10^{-4}	5.53×10^{-5}	7.12×10^{-5}	5.07×10^{-6}
0.3125	3.31×10^{-3}	4.57×10^{-4}	1.37×10^{-5}	3.22×10^{-5}	4.77×10^{-6}
0.4375	1.34×10^{-3}	3.86×10^{-4}	5.41×10^{-5}	2.73×10^{-5}	2.07×10^{-6}
0.5625	4.68×10^{-4}	9.69×10^{-5}	4.01×10^{-6}	1.29×10^{-5}	1.02×10^{-6}
0.6875	2.92×10^{-4}	6.81×10^{-5}	2.94×10^{-6}	8.28×10^{-6}	8.23×10^{-7}
0.8125	1.01×10^{-4}	4.54×10^{-5}	1.82×10^{-6}	4.06×10^{-6}	5.62×10^{-7}
0.9375	4.70×10^{-5}	2.27×10^{-5}	1.12×10^{-6}	1.15×10^{-6}	2.48×10^{-7}

2.4 Numerical Results and discussion

After the validation of the efficiency and the accuracy of the derived scheme in the previous section, the proposed method is applied to solve the VOPIDE given in equation (2.1) on domain [0,1] under the following prescribed initial and boundary conditions as

$$u(x, 0) = 1, \quad 0 \leq x \leq 1, \tag{2.24}$$

$$u(0,t) = 0 = w(1,t), \quad 0 \le t \le 1. \tag{2.25}$$

The proposed scheme given in section 2.2 is applied on the problem (2.1) under the conditions (2.24) and (2.25) to obtain a system of equations by Newton–Cotes collocation points. In Figure 2.3, we notice the variations of the solution when the parameters are considered as $\lambda = 1$, $\delta = 1$, $D_1 = 1$, $V_1 = 1$ and $\alpha(x,t) = xt$, $(xt)^2$ and $(x+t)/2$ at fixed $t = 0.5$.

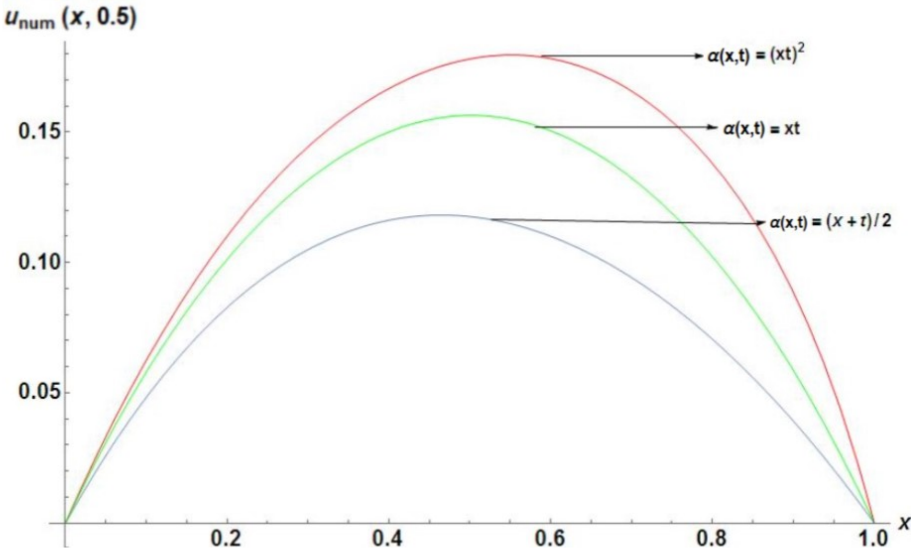

Figure 2.3: The plots of variation of the solution for the values of the parameters $\lambda = 1$, $\delta = 1$, $D_1 = 1$, $V_1 = 1$ and $\alpha(x,t) = xt$, $(xt)^2$ and $(x+t)/2$ at fixed $t = 0.5$.

Figure 2.4 provides the variations of the solution profile when $\lambda = 0$, $\delta = 1$, $D_1 = 1$, $V_1 = 1$ and $\alpha(x,t) = xt$ and $(x+t)/2$ at fixed $t = 0.5$. Whereas from Figure 2.5, it is noticed that the variations of the solution profile when the parameters are fixed as $\lambda = 1$, $\delta = 1$, $D_1 = 1$, $V_1 = 0$ and $\alpha(x,t) = xt$ and $(x+t)/2$ at fixed $t = 0.5$.

Figure 2.6 gives the variations of the solution when we fix $\lambda = 1$, $\delta = 0$, $D_1 = 1$, $V_1 = 1$ and $\alpha(x,t) = xt$ and $(x+t)/2$ at fixed $t = 0.5$ and Figure 2.7 shows

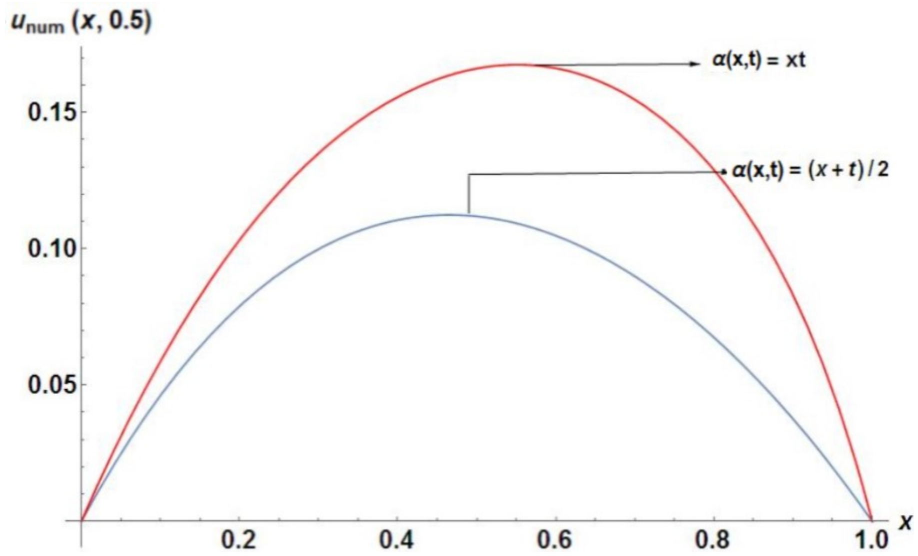

Figure 2.4: Plots of $w(x, 0.5)$ with $\lambda = 0$, $\delta = 1$, $D_1 = 1$, $V_1 = 1$ and $\alpha(x,t) = xt$ and $(x+t)/2$ at fixed $t = 0.5$.

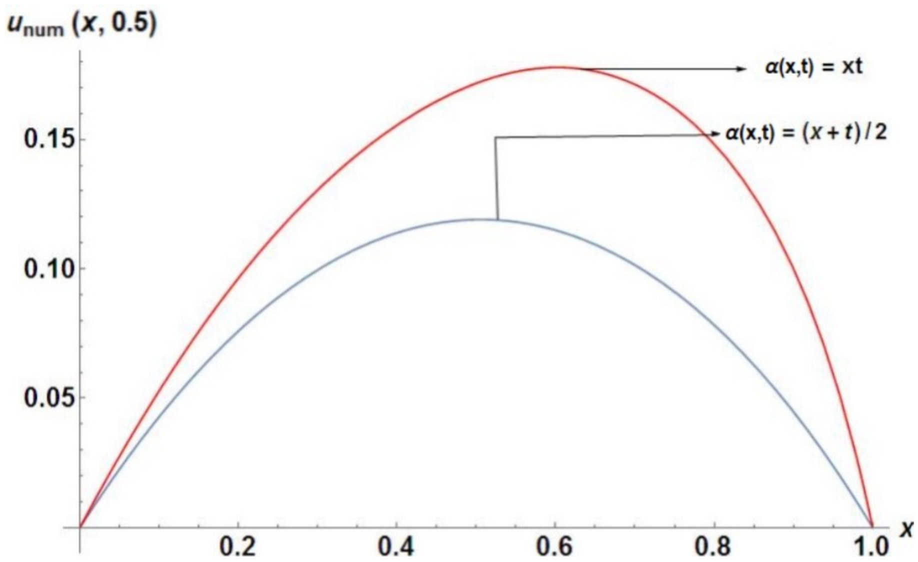

Figure 2.5: Plots of $w(x, 0.5)$ with $\lambda = 1$, $\delta = 1$, $D_1 = 1$, $V_1 = 0$ and $\alpha(x,t) = xt$ and $(x+t)/2$ at fixed $t = 0.5$.

the variations of the solution when $\lambda = 0$, $\delta = 0$, $D_1 = 1$, $V_1 = 0$ and $\alpha(x,t) = xt$ and $(x+t)/2$ at fixed $t = 0.5$.

Figure 2.8 provides the variations of the solution when we fix $\lambda = 0$, $\delta = 0$, $D_1 = 1$, $V_1 = 1$ and $\alpha(x,t) = xt$ and $(x+t)/2$ at fixed $t = 0.5$. Figure 2.9 depicts

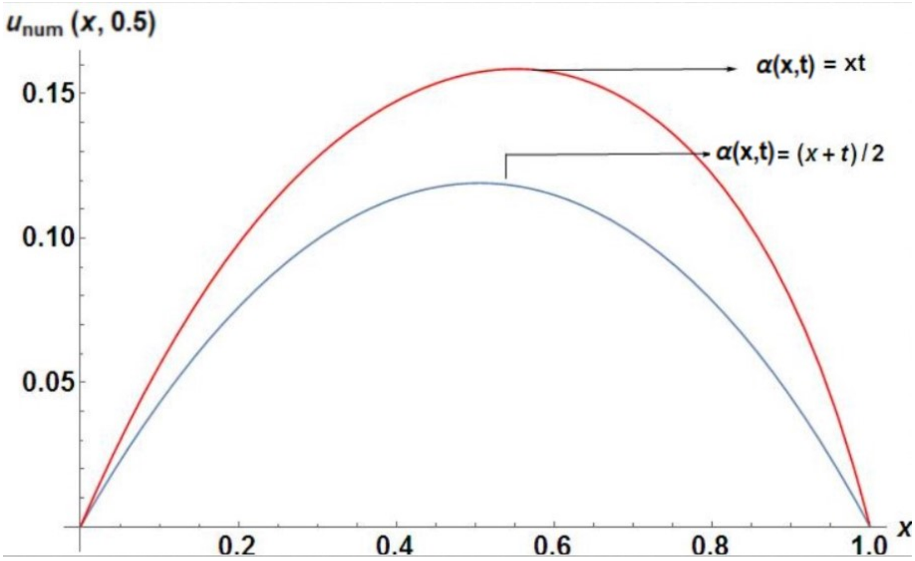

Figure 2.6: Plots of $w(x, 0.5)$ with $\lambda = 1$, $\delta = 0$, $D_1 = 1$, $V_1 = 1$ and $\alpha(x,t) = xt$ and $(x+t)/2$ at fixed $t = 0.5$.

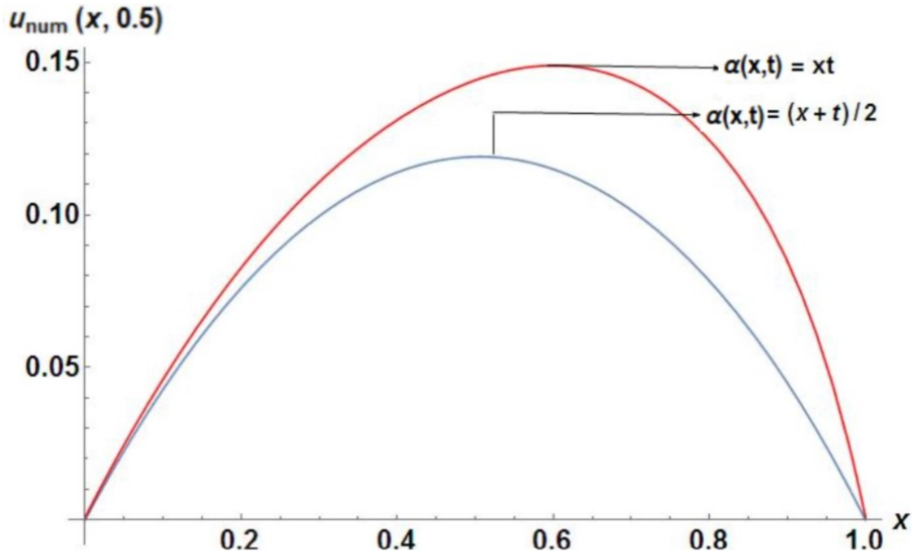

Figure 2.7: Plots of $w(x, 0.5)$ with $\lambda = 0$, $\delta = 0$, $D_1 = 1$, $V_1 = 0$ and $\alpha(x,t) = xt$ and $(x+t)/2$ at fixed $t = 0.5$.

the variations of the solution when $\lambda = 1$, $\delta = 0$, $D_1 = 1$, $V_1 = 0$ and $\alpha(x,t) = xt$ and $(x+t)/2$ at fixed $t = 0.5$.

Figure 2.10 and Figure 2.11 show that variations of $u(x,t)$ *vs.* x for the values of parameters $\lambda = 1$, $\delta = 1$, $D_1 = 1$, $V_1 = 1$ and $\alpha(x,t) = xt$ and $(x+t)/2$ at fixed $t = 0.5$ and $\lambda = 0$, $\delta = 1$, $D_1 = 1$, $V_1 = 0$ and $\alpha(x,t) = xt$ and $(x+t)/2$ at

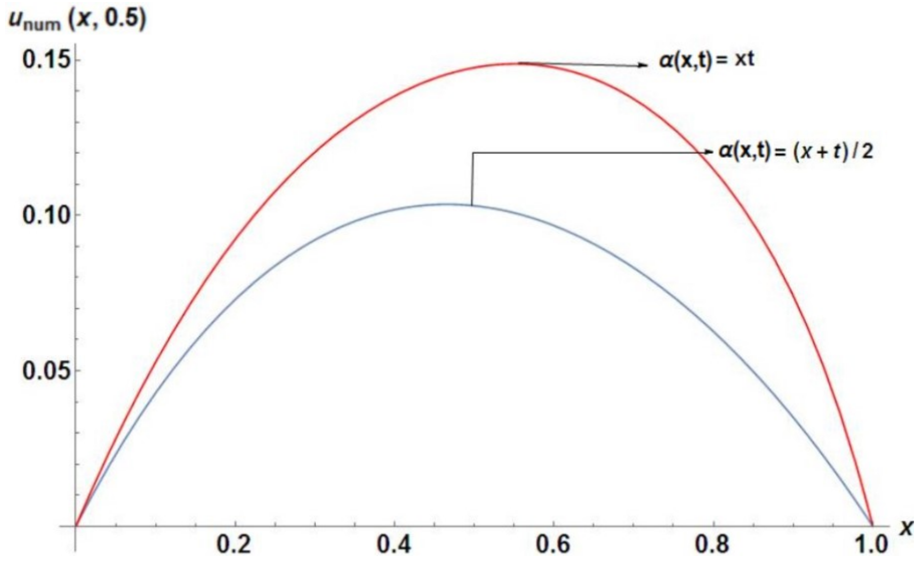

Figure 2.8: Plots of $u(x, 0.5)$ with $\lambda = 0$, $\delta = 0$, $D_1 = 1$,$V_1 = 1$ and $\alpha(x, t) = xt$ and $(x + t)/2$ at fixed $t = 0.5$.

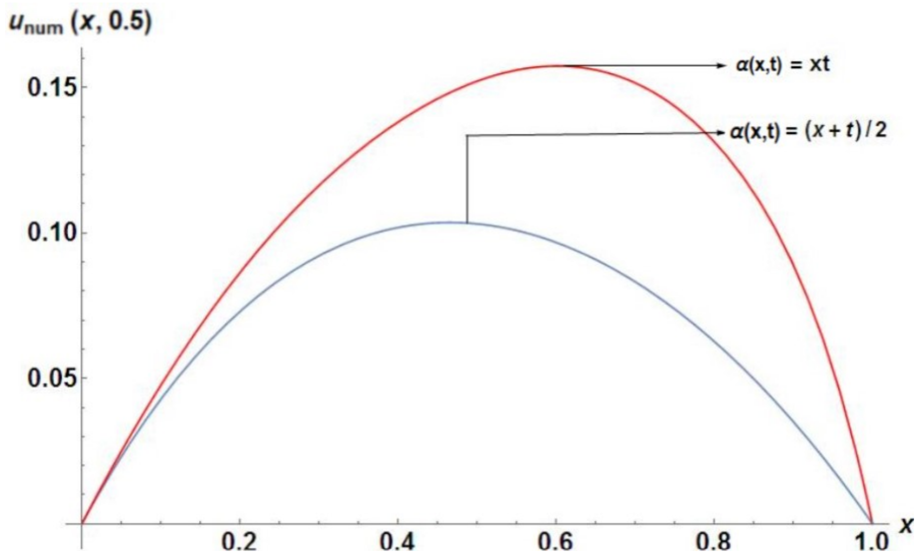

Figure 2.9: Plots of $u(x, 0.5)$ with $\lambda = 1$, $\delta = 0$, $D_1 = 1$,$V_1 = 0$ and $\alpha(x, t) = xt$ and $(x + t)/2$ at fixed $t = 0.5$.

fixed $t = 0.5$, respectively.

It is observed from the Figures 2.3 to 2.11 that the solute concentration diffuses lesser for the case of linear order as compared to that of non-linear order.

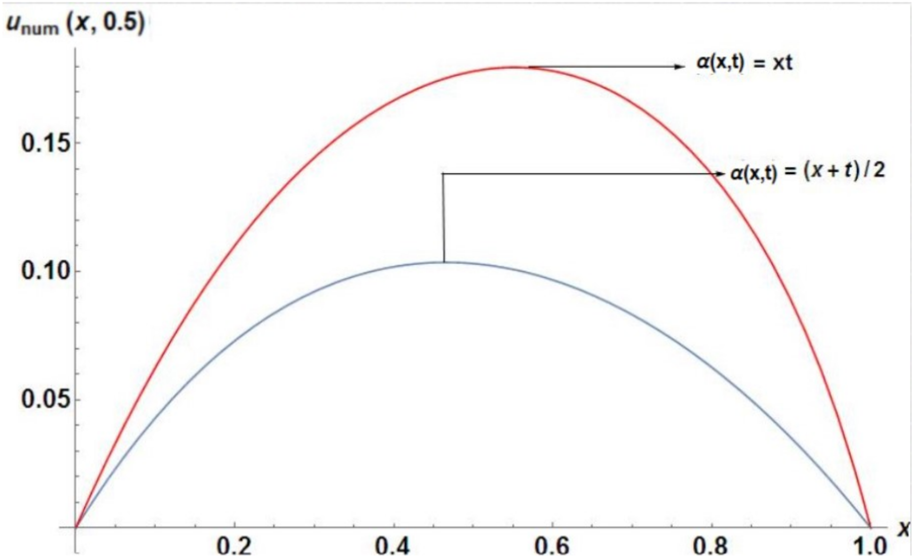

Figure 2.10: Plots of $u(x, 0.5)$ with $\lambda = 1$, $\delta = 1$, $D_1 = 1$,$V_1 = 1$ and $\alpha(x,t) = xt$ and $(x+t)/2$ at fixed $t = 0.5$.

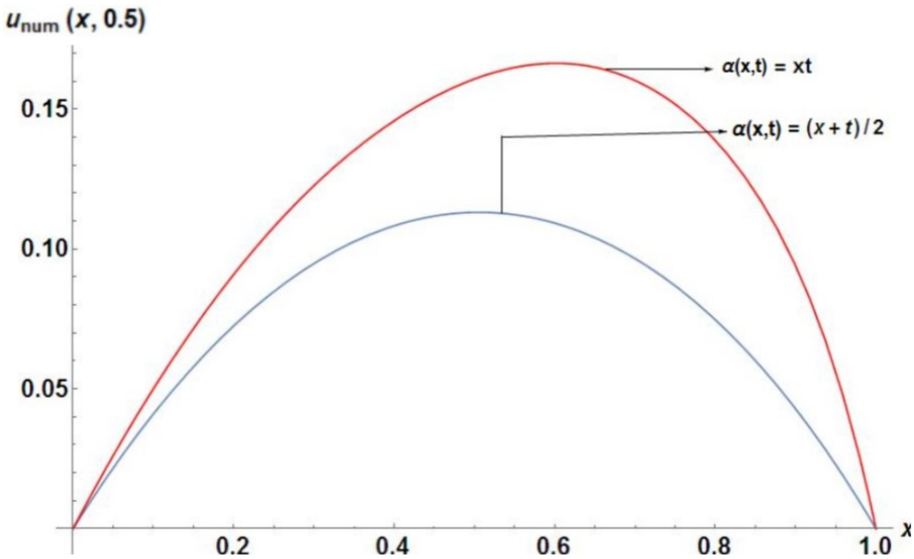

Figure 2.11: Plots of $u(x, 0.5)$ with $\lambda = 0$, $\delta = 1$, $D_1 = 1$,$V_1 = 0$ and $\alpha(x,t) = xt$ and $(x+t)/2$ at fixed $t = 0.5$.

2.5 Conclusion

To solve the non-linear space-time variable-order reaction-advection-diffusion equation, the shifted Legendre collocation method is applied by using operational matrices for integration, partial derivative and variable-order fractional derivative. The efficiency of the proposed model is validated by comparing the results obtained by the proposed method with the existing analytical results through error analysis. The effects of advection term and reaction term on the solution profile for different values of space and time variable order derivatives have been presented graphically. The main point of observation in this chapter is that the diffusivity of solute concentration is directly proportional to the non-linearity of the variable order derivative.

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