Chapter 6

The Riemann Problem for one-dimensional dusty gas dynamics with external forces *

"Each time you fail, you have
eliminated another wrong option.

–Thomas A. Edison.

6.1 Introduction

In this chapter, we studied the Riemann Problem for a non-homogeneous system governing the one-dimensional compressible flow of dusty gas, where the external

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forces are assumed to be continuous function of time. We have examined the effect of dust particles on the density, flow velocity and shock speed, and their consequences on the solution of the Riemann Problem. In recent decades, dusty gases got significant attention from physicist and mathematicians due to its applications in many science and engineering fields. Nowadays various research works has been done on the analysis of the effect of dust particles present in the gas on the behavior of the solution of a given system of gas dynamic equations [141, 168, 148, 147]. Bomb explosions, coal mine explosions, underground explosions, volcanic eruptions, coma collision with the planet, nozzle flow, and a variety of other scientific and industrial areas where dusty gases are used in a wide range of physical processes [109, 15, 108]. The presence of dust particles in the gas is very common and generally occur in most of the flow phenomenon in gas dynamics. The major problem is how to deal with the the situation when the dust particles are present in the gases and what physical changes happen depends on the concentration of the dust particles present in mixture.

In the last seventy years, several methods have been developed for analyzing the elementary wave solution of the Riemann Problem governed by the quasilinear hyperbolic system of PDEs for gas dynamics, magneto gas dynamics, and many other fields. The Riemann problem is an initial value problem with piecewise constant initial data. It's exact solution provides the real physical properties with several wave families and their propagation. Due to the applications of the Riemann Problem (RP), the mathematicians are continuously studying its solution for the various system of PDEs and analyzing the behavior of the solution obtained for the system. In various flows, such as Granular flow, shallow water flow, traffic flow, and many other models, RP is continuously studied by ongoing research. Several analytical and numerical methods have been developed in the past to explore the solution of the Riemann Problem for the system of PDEs [69, 169, 17]. When we deal with the initial value problem for the system of conservation laws, we need a criteria imposed on the solution determined, which ensures physical relevance of the solution. Lax [169] have derived a condition that ensures the physical validation of the solution, known as entropy condition. The basic idea of the Riemann Problem comes from the problem of the motion of a gas in shock tube. The analysis of the solution of the Riemann Problem has been studied by many researchers for homogeneous and non-homogeneous systems in different gas regimes. The governing system for one dimensional compressible flow can be written as [170]

$$\begin{cases} \partial_t \varrho + \partial_x(\varrho u) = 0, \\ \partial_t(\varrho u) + \partial_x(\varrho u^2 + p) = g\varrho, \\ \partial_t(\frac{\varrho u^2}{2} + \varrho e) + \partial_x(\frac{\varrho u^3}{2} + \varrho u e + pu) = g\varrho u, \end{cases} \tag{6.1}$$

where ϱ, u and p denote the density, velocity and pressure of the dusty gas, respectively. Here, x is the spatial coordinate, t is the time and $g \equiv g(t)$ represents the external force, which is a non-zero continuous function of time t. $\partial_t = \frac{\partial}{\partial t}$ represents the partial derivative with respect to t. The internal energy per unit mass of the mixture of gas and small dust particles is denoted by e and is given as [91]

$$e = \frac{(1-Z)p}{(\varGamma - 1)\varrho}. \tag{6.2}$$

Here, $\varGamma$ is Grüneisen coefficient, which is defined as $\varGamma = \gamma(1+\lambda\beta)/(1+\lambda\beta\gamma)$, where $\beta = c_{sp}/c_p$, $\lambda = k_p/(1-k_p)$ and γ is the adiabatic constant, which is the ratio of specific heat at constant pressure to the specific heat at constant volume, c_p represents the specific heat of the gas at constant pressure. c_{sp} denotes the specific heat of the solid particles and k_p stands for the mass fraction of the solid particles in the mixture. The volume fraction Z and the mass fraction k_p are related by the relation $Z = \theta\rho$, $\theta = k_p/\rho_{sp}$, where ρ_{sp} denotes the specific density of the solid particles present in the mixture.

In view of (6.2) and (6.1), the Riemann problem for compressible flow of dusty gas can be written as

$$\begin{cases} \partial_t \varrho + \partial_x(\varrho u) = 0, \\ \partial_t(\varrho u) + \partial_x(\varrho u^2 + p) = g\varrho, \\ \partial_t(\frac{\varrho u^2}{2} + \frac{p}{(\Gamma - 1)}(1 - \theta\varrho)) + \partial_x(\frac{\varrho u^3}{2} + \frac{p}{(\Gamma - 1)}(1 - \theta\varrho)u + pu) = g\varrho u, \end{cases} \tag{6.3}$$

with the initial data

$$(\varrho, u, p)(0, x) = \begin{cases} (\varrho_-, u_-, p_-), & x < 0, \\ (\varrho_+, u_+, p_+), & x > 0, \end{cases} \tag{6.4}$$

where, ϱ_i, p_i are non-zero and $i = \pm$ are used to specify the constant values. For a homogeneous system, i.e., $g(t) = 0$ in (6.1) with the piecewise constant initial data, there are many studies available in the literature. The Riemann Problem for pressureless homogeneous Eular equations is solved by [171, 172, 173], where the delta shock and vacuum is present in the solution. Ambika and Radha [174] have studied the Riemann Problem for one-dimensional van der Waals gas and determined the explicit form of contact discontinuities, shock waves and simple waves. The Riemann Problem for dusty gases is getting significant attention due its wide range of applications in the study of the system of conservation laws by using various analytical and numerical techniques (see [112, 175, 176, 177]). Chaturvedi and Singh [113] have examine the concentration and cavitation phenomenon in the solution of the Riemann Problem for the isentropic zero-pressure dusty gasdynamics. Also, they have discussed the presence of delta shock wave solution and vacuum state in the flow field. In the last few years, some of the authors have generalized the structure of the Riemann solution for the Chaplygin gas and discussed the classical and non-classical waves. Shen [178] investigated the Riemann solution of a given system of equations governing the Chaplygin gas flow with source term, and derived the condition for the existence of delta shock. Shao [179] and Zhang et al. [180] have studied the solution of the RP for isentropic relativistic system and explore the all possibilities in the construction of the solution. Further Shao [181] have discussed the Riemann problem with the initial data containing the Dirac delta function for the isentropic relativistic Chaplygin Euler equations and constructively obtained the global existence of generalized solutions including delta shock waves that explicitly exhibit four kinds of different structures. Pang et al. [176] constructed the Riemann solution for a non-homogeneous system in the presence of external force and examined the van der Waals effect on the solution obtained for different states. In the present scenario, the elementary waves are the basic building block to understand the mathematical theory of non-linear waves described by the system of conservation laws. The non-classical solutions of the Riemann Problem are also getting excellent attention because of their particular form of discontinuity. Chaturvedi and Singh [182] obtained the classical wave solution for the logotropic system in the presence of Coulomb-type friction and examined the behavior of the solution under the effect of the friction. The author in [183] have studied the mixed initial–boundary value problem for first-order quasilinear hyperbolic systems with general nonlinear boundary conditions in the half space and showed some applications to quasilinear hyperbolic systems arising in physics and other disciplines, particularly to the system describing the motion of the relativistic closed string in the Minkowski space.

This study aims to analyze the behavior of the solution of the Riemann Problem under the effect of small dust particles for a non-homogeneous system. In this chapter, we use a transformation to reduce the original system into the simplest conservative form, and then we determine the solution of the reduced system. The entire work of this chapter is divided into the sections as follows: In the second section of this chapter, we obtained the Riemann invariants for the corresponding eigenvectors and analyzed the elementary wave curves associated with the corresponding Riemann invariants. In section 6.3, we determined the solution of the Riemann problem for the reduced homogeneous system of equations under the certain conditions. The fourth section is referred to the solution of the original non-homogeneous system with the given initial data. The conclusion of the entire analysis of this chapter is summarized in the last section 6.5.

Consider a change of state variable v, which is defined as ([184])

$$v = u - G(t) \triangleq u - \int_0^t g(y)dy. \tag{6.5}$$

With the help of (6.5), (6.3) reduces into the following form:

$$\begin{cases} \partial_t \varrho + \partial_x(\varrho(v + G(t))) = 0, \\ \partial_t(\varrho v) + \partial_x(\varrho v(v + G(t)) + p) = 0, \\ \partial_t(\frac{\varrho v^2}{2} + \frac{p}{(\Gamma-1)}(1-\theta\varrho)) + \partial_x((\frac{\varrho v^2}{2} + \frac{p}{(\Gamma-1)}(1-\theta\varrho))(v + G(t)) + pv) = 0, \end{cases} \tag{6.6}$$

and the piecewise constant initial data is given as

$$(\varrho, v, p)(0, x) = \begin{cases} (\varrho_-, u_-, p_-), & x < 0, \\ (\varrho_+, u_+, p_+), & x > 0. \end{cases} \tag{6.7}$$

6.2 Elementary waves

In this section, we shall discuss the elementary waves of corresponding system (6.6). The eigenvalues corresponding to (6.6) are

$$\begin{cases} \mu_1 = v + G(t) - \sqrt{\frac{\Gamma p}{\varrho(1-\theta\varrho)}}, \\ \mu_2 = v + G(t), \\ \mu_3 = v + G(t) + \sqrt{\frac{\Gamma p}{\varrho(1-\theta\varrho)}}, \end{cases} \tag{6.8}$$

and the corresponding right eigenvectors are given as

$$\begin{cases} r_1 = \left(\varrho, -\sqrt{\frac{\Gamma p}{\varrho(1-\theta\varrho)}}, \frac{\Gamma p}{\varrho(1-\theta\varrho)}\right)^T, \\ r_2 = (1,0,0)^T, \\ r_3 = \left(\varrho, \sqrt{\frac{\Gamma p}{\varrho(1-\theta\varrho)}}, \frac{\Gamma p}{\varrho(1-\theta\varrho)}\right)^T. \end{cases}$$

We see that $\nabla\mu_1.r_1 = -\frac{1}{2}\sqrt{\frac{\Gamma p(1+\Gamma)^2}{\varrho(1-\theta\varrho)^3}}$, $\mu_2.r_2 = 0$, $\nabla\mu_3.r_3 = \frac{1}{2}\sqrt{\frac{\Gamma p(1+\Gamma)^2}{\varrho(1-\theta\varrho)^3}}$, which indicates that the first and third characteristic fields are genuinely non-linear and the second characteristic field is linearly degenerate. Here, ∇ is defined as $\nabla = (\partial_\varrho, \partial_v, \partial_p)$. Now, the Riemann invariants with respect to these eigenvectors can be written as

1-Riemann invariant : $\left\{v + \frac{2}{\Gamma-1}\sqrt{\Gamma p\left(\frac{1}{\varrho} - \theta\right)}, p\left(\frac{1}{\varrho} - \theta\right)^\Gamma\right\}$,

2-Riemann invariant : $\{v, p\}$,

3-Riemann invariant : $\left\{v - \frac{2}{\Gamma-1}\sqrt{\Gamma p\left(\frac{1}{\varrho} - \theta\right)}, p\left(\frac{1}{\varrho} - \theta\right)^\Gamma\right\}$.

6.2.1 Continuous solutions

1-Rarefaction wave : Now, the rarefaction wave of equation (6.6), with the left state (ϱ_-, u_-, p_-), associated with the 1-Riemann invariant for $\mu_1(\varrho_-, v_-, p_-) < \mu_1(\varrho, v, p)$, the 1-rarefaction wave curve can obtained as

$$R_1(\varrho_-, u_-, p_-) : \begin{cases} \frac{dx}{dt} = v - \sqrt{\frac{\Gamma p}{\varrho(1-\theta\varrho)}} + G(t), \\ v + \frac{2}{\Gamma-1}\sqrt{\Gamma p\left(\frac{1}{\varrho} - \theta\right)} = u_- + \frac{2}{\Gamma-1}\sqrt{\Gamma p_-\left(\frac{1}{\varrho_-} - \theta\right)}, p > p_-, \\ p\left(\frac{1}{\varrho} - \theta\right)^{\Gamma} = p_-\left(\frac{1}{\varrho_-} - \theta\right)^{\Gamma}. \end{cases} \tag{6.9}$$

Therefore, from equation (6.9), along the 1-rarefaction wave curve, we observe that

$$\begin{cases} \frac{dv}{dp} = -\sqrt{\frac{(1-\theta\varrho)}{\Gamma p\varrho}} < 0, \\ \frac{d^2v}{dp^2} = \frac{\Gamma+1}{2\Gamma p^2}\sqrt{\Gamma p\left(\frac{1}{\varrho} - \theta\right)} > 0, \\ \frac{d\varrho}{dp} = \frac{\varrho(1-\theta\varrho)}{\Gamma p} > 0. \end{cases} \tag{6.10}$$

Further, we notice that as $p \to 0$, $v \to u_- + \frac{2}{\Gamma-1}\sqrt{\Gamma p_-\left(\frac{1}{\varrho_-} - \theta\right)}$ and $\varrho \to 0$. From equations (6.9) and (6.10), we have

$$\begin{cases} \frac{\partial v}{\partial \theta} = \frac{\varrho_-}{2(1-\theta\varrho_-)}(u_- - v) < 0, \\ \frac{\partial \varrho}{\partial \theta} = \frac{\varrho}{(1-\theta\varrho_-)}(\varrho_- - \varrho) > 0. \end{cases} \tag{6.11}$$

Equation (6.11) shows that, in the 1-rarefaction wave with the left state (ϱ_-, u_-, p_-), an increase in the value of the parameter θ, the variable v decreases where as the density ϱ increases as θ increases.

Next, we shall find the value of ϱ_{r_1}, v_{r_1} and p_{r_1} inside the 1-rarefaction wave. On solving the following initial value problem $\frac{dx}{dt} = \mu_1(\varrho_{r_1}, v_{r_1}, p_{r_1}), x(0) = 0$, we obtain

$$\left(v_{r_1} - \sqrt{\frac{\Gamma p_{r_1}}{\varrho_{r_1}(1-\theta\varrho_{r_1})}} \right) t = x - \int_0^t g(y)dy. \tag{6.12}$$

The quantities ϱ_{r_1}, v_{r_1} and p_{r_1} satisfies the following equations

$$\begin{cases} v_{r_1} + \frac{2}{\Gamma-1}\sqrt{\Gamma p_{r_1}\left(\frac{1}{\varrho_{r_1}} - \theta\right)} = u_- + \frac{2}{\Gamma-1}\sqrt{\Gamma p_-\left(\frac{1}{\varrho_-} - \theta\right)}, \\ p_{r_1} = p_-\left(\frac{1}{\varrho_-} - \theta\right)^{\Gamma}\left(\frac{1}{\varrho_{r_1}} - \theta\right)^{-\Gamma}. \end{cases} \tag{6.13}$$

From equation (6.12) and (6.13), we have

$$\begin{aligned} &\frac{1+\Gamma-2\theta\varrho_{r_1}}{\Gamma-1}\sqrt{\frac{\Gamma p_-}{\varrho_{r_1}(1-\theta\varrho_{r_1})}\left(\frac{\varrho_{r_1}}{1-\theta\varrho_{r_1}}\left(\frac{1}{\varrho_- - \theta}\right)\right)^{\Gamma}} \\ &-u_- - \frac{2}{\Gamma-1}\sqrt{\Gamma p_-\left(\frac{1}{\varrho_-} - \theta\right)} + \frac{x}{t} - \frac{1}{t}\int_0^t g(y)dy = 0. \end{aligned} \tag{6.14}$$

From the above equation (6.14), one can uniquely determine the value of $\varrho_{r_1} = \varrho_{r_1}(t,x)$. Now, in view of (6.12), (6.13) and (6.14), ϱ_{r_1}, v_{r_1} and p_{r_1} can be obtained as

$$\begin{cases} \varrho_{r_1} = \varrho_{r_1}(t,x), \\ v_{r_1} = \sqrt{\frac{\Gamma p_{r_1}}{\varrho_{r_1}(1-\theta\varrho_{r_1})}} + \frac{x}{t} - \frac{1}{t}\int_0^t g(y)dy, \\ p_{r_1} = p_-\left(\frac{1}{\varrho_-} - \theta\right)^{\Gamma}\left(\frac{1}{\varrho_{r_1}} - \theta\right)^{-\Gamma}. \end{cases} \tag{6.15}$$

3-Rarefaction wave : The 3-rarefaction wave curve with the right state (ϱ_+, u_+, p_+) can be written as

$$R_3(\varrho_+, u_+, p_+) : \begin{cases} \frac{dx}{dt} = v + \sqrt{\frac{\Gamma p}{\varrho(1-\theta\varrho)}} + G(t), \\ v + \frac{2}{\Gamma-1}\sqrt{\Gamma p\left(\frac{1}{\varrho} - \theta\right)} = u_+ - \frac{2}{\Gamma-1}\sqrt{\Gamma p_+\left(\frac{1}{\varrho_+} - \theta\right)}, p < p_+, \\ p\left(\frac{1}{\varrho} - \theta\right)^{\Gamma} = p_+\left(\frac{1}{\varrho_+} - \theta\right)^{\Gamma}. \end{cases} \tag{6.16}$$

Along the 3-rarefaction wave curve, we have

$$\begin{cases} \frac{dv}{dp} = \sqrt{\frac{(1-\theta\varrho)}{\Gamma p\varrho}} > 0, \\ \frac{d^2v}{dp^2} = -\frac{\Gamma+1}{2\Gamma p^2}\sqrt{\Gamma p\left(\frac{1}{\varrho}-\theta\right)} < 0, \\ \frac{d\varrho}{dp} = \frac{\varrho(1-\theta\varrho)}{\Gamma p} > 0. \end{cases} \tag{6.17}$$

Further, we notice that as $p \to 0$, $v \to u_+ - \frac{2}{\Gamma-1}\sqrt{\Gamma p_+\left(\frac{1}{\varrho_+}-\theta\right)}$ and $\varrho \to 0$.

$$\begin{cases} \frac{\partial v}{\partial\theta} = \frac{\varrho_+}{2(1-\theta\varrho_+)}(u_+ - v) > 0, \\ \frac{\partial\varrho}{\partial\theta} = \frac{\varrho}{(1-\theta\varrho_+)}(\varrho_+ - \varrho) > 0. \end{cases} \tag{6.18}$$

Thus, for the 3-rarefaction wave with the right state (ϱ_+, u_+, p_+), v and ϱ both are the increasing function of θ. Inside the 3-rarefaction wave, ϱ_{r_3}, v_{r_3} and p_{r_3} can be obtained as

$$\begin{cases} \varrho_{r_3} = \varrho_{r_3}(t,x), \\ v_{r_3} = -\sqrt{\frac{\Gamma p_{r_3}}{\varrho_{r_3}(1-\theta\varrho_{r_3})}} + \frac{x}{t} - \frac{1}{t}\int_0^t g(y)dy, \\ p_{r_3} = p_+\left(\frac{1}{\varrho_+}-\theta\right)^{\Gamma}\left(\frac{1}{\varrho_{r_3}}-\theta\right)^{-\Gamma}. \end{cases} \tag{6.19}$$

Here, $\varrho_{r_3} = \varrho_{r_3}(t,x)$ can be determined from the following equation

$$\begin{aligned} &\frac{1+\Gamma-2\theta\varrho_{r_3}}{\Gamma-1}\sqrt{\frac{\Gamma p_+}{\varrho_{r_3}(1-\theta\varrho_{r_3})}\left(\frac{\varrho_{r_3}}{1-\theta\varrho_{r_3}}\left(\frac{1}{\varrho_+-\theta}\right)\right)^{\Gamma}} \\ &+u_+ - \frac{2}{\Gamma-1}\sqrt{\Gamma p_+\left(\frac{1}{\varrho_+}-\theta\right)} - \frac{x}{t} + \frac{1}{t}\int_0^t g(y)dy = 0. \end{aligned} \tag{6.20}$$

Vacuum state solution : Other than the 1-rarefaction and 3-rarefaction waves, there exists a vacuum, which is a continuous solution of (6.6) given as

$$(\varrho, v, p) = (0, v(t,x), 0), \tag{6.21}$$

where $v(t,x)$ is a smooth arbitrary function.

6.2.2 Discontinuous solutions

The R-H condition for a discontinuous solution of (6.6) is given as

$$\begin{cases} -\sigma[\varrho] + [\varrho(v+G(t))] = 0, \\ -\sigma[\varrho v] + [\varrho v(v+G(t)) + p] = 0, \\ -\sigma[\frac{\varrho v^2}{2} + \frac{p}{(\Gamma-1)}(1-\theta\varrho)] + [(\frac{\varrho v^2}{2} + \frac{p}{(\Gamma-1)}(1-\theta\varrho))(v+G(t)) + pv] = 0. \end{cases} \tag{6.22}$$

Here, $[n] = n - n_-$ is defined as the jump in the function n through the discontinuity $x = x(t)$, where $n_- = n(t, x(t)-0)$ and $n = n(t, x(t)+0)$. $\sigma = x'(t)$ represents the speed of the propagating discontinuity. On solving equation (6.22), we get

$$\begin{cases} \sigma_1 = v_- + G(t) - \sqrt{\frac{\Gamma-1}{\varrho_-(1-\theta\varrho)}\left(\frac{p+p_-}{2} + \frac{p(1-\theta\varrho)+p_-\theta\varrho}{\Gamma-1}\right)}, \\ \sqrt{\frac{\Gamma-1}{\varrho_-(1-\theta\varrho)}\left(\frac{p+p_-}{2} + \frac{p(1-\theta\varrho)+p_-\theta\varrho}{\Gamma-1}\right)}[\varrho] + \varrho[v] = 0, \\ \varrho_-\sqrt{\frac{\Gamma-1}{\varrho_-(1-\theta\varrho)}\left(\frac{p+p_-}{2} + \frac{p(1-\theta\varrho)+p_-\theta\varrho}{\Gamma-1}\right)}[v] + [p] = 0, \end{cases} \tag{6.23}$$

$$\begin{cases} \sigma_2 = v_- + G(t), \\ [v] = 0, [p] = 0, [\varrho] \neq 0, \end{cases} \tag{6.24}$$

$$\begin{cases} \sigma_3 = v_- + G(t) + \sqrt{\frac{\Gamma-1}{\varrho_-(1-\theta\varrho)}\left(\frac{p+p_-}{2} + \frac{p(1-\theta\varrho)+p_-\theta\varrho}{\Gamma-1}\right)}, \\ -\sqrt{\frac{\Gamma-1}{\varrho_-(1-\theta\varrho)}\left(\frac{p+p_-}{2} + \frac{p(1-\theta\varrho)+p_-\theta\varrho}{\Gamma-1}\right)}[\varrho] + \varrho[v] = 0, \\ -\varrho_-\sqrt{\frac{\Gamma-1}{\varrho_-(1-\theta\varrho)}\left(\frac{p+p_-}{2} + \frac{p(1-\theta\varrho)+p_-\theta\varrho}{\Gamma-1}\right)}[v] + [p] = 0. \end{cases} \tag{6.25}$$

1-Shock wave :

The Lax entropy condition with the left state (ϱ_-, u_-, p_-) in terms of (6.22) is $\mu_1(\varrho, v, p) < \sigma_1 < \mu_1(\varrho_-, u_-, p_-)$, $\sigma_1 < \mu_2(\varrho, v, p)$.

The 1-shock wave curve is given as

$$S_1(\varrho_-, u_-, p_-) : \begin{cases} \sigma_1 = u_- + G(t) - \sqrt{\frac{(\Gamma+1)p+(\Gamma-1)p_-}{2\varrho_-(1-\theta\varrho_-)}}, \\ \varrho = \varrho_- \frac{(\Gamma+1)p+(\Gamma-1)p_-}{(\Gamma-1+2\theta\varrho_-)p+(\Gamma+1-2\theta\varrho_-)p_-}, \quad p > p_-, \\ v = u_- - \sqrt{\frac{2(1-\theta\varrho_-)}{\varrho_-((\Gamma+1)p+(\Gamma-1)p_-)}}(p - p_-). \end{cases} \tag{6.26}$$

On the 1-shock wave curve $S_1(\varrho_-, u_-, p_-)$, we have

$$\begin{cases} \frac{dv}{dp} = -\sqrt{\frac{1-\theta\varrho_-}{2\varrho_-}} \frac{(\Gamma+1)p+(3\Gamma-1)p_-}{((\Gamma+1)p+(\Gamma-1)p_-)^{3/2}} < 0, \\ \frac{d^2v}{dp^2} = (\Gamma+1)\sqrt{\frac{1-\theta\varrho_-}{8\varrho_-}} \frac{(\Gamma+1)p+(7\Gamma-1)p_-}{((\Gamma+1)p+(\Gamma-1)p_-)^{5/2}} > 0, \\ \frac{d\varrho}{dp} = \frac{4\Gamma p_- \varrho_-(1-\theta\varrho_-)}{((\Gamma-1+2\theta\varrho_-)p+(\Gamma+1-2\theta\varrho_-)p_-)^2} > 0. \end{cases} \tag{6.27}$$

Further, we notice that as $p \to +\infty$, $v \to -\infty$ and $\varrho \to \frac{\varrho_-(\Gamma+1)}{\Gamma+2\theta\varrho_- -1} < \frac{1}{\theta}$. From equations (6.26) and (6.27), we obtain

$$\begin{cases} \frac{\partial\varrho}{\partial\theta} = \frac{2\varrho_-^2(p_- - p)((\Gamma+1)p+(\Gamma-1)p_-)}{((\Gamma-1+2\theta\varrho_-)p+(\Gamma+1-2\theta\varrho_-)p_-)^2} < 0, \\ \frac{\partial v}{\partial\theta} = \frac{p-p_-}{\sqrt{2(1-\theta\varrho_-)}}\sqrt{\frac{\varrho_-}{(\Gamma+1)p+(\Gamma-1)p_-}} > 0, \\ \frac{\partial\sigma_1}{\partial\theta} = -\frac{\varrho_-}{2(1-\theta\varrho_-)}\sqrt{\frac{(\Gamma+1)p+(\Gamma-1)p_-}{2\varrho_-(1-\theta\varrho_-)}} < 0. \end{cases} \tag{6.28}$$

Thus, we see that, for the 1-shock wave, an increase in the value of the parameter θ causes to decrease in the speed of the shock wave and the density ϱ, whereas an increase in the value of θ causes to increase v, with the left state (ϱ_-, u_-, p_-).

2-Contact discontinuity :

The 2-Contact discontinuity with the left state (ϱ_-, u_-, p_-), in terms of (6.24) can be written as

$$J_2(\varrho_-, u_-, p_-) : \begin{cases} \sigma_2 = u_- + G(t), \\ p = p_-, \\ v = u_-, \\ \varrho \neq \varrho_- \end{cases} \tag{6.29}$$

3-Shock wave :

The Lax entropy condition with the right state (ϱ_+, u_+, p_+) in terms of (6.25) is $\mu_3(\varrho_+, u_+, p_+) < \sigma_3 < \mu_3(\varrho, u, p)$, $\sigma_3 > \mu_2(\varrho, v, p)$.

Now, the 3-shock wave curve can be written as

$$S_3(\varrho_+, u_+, p_+) : \begin{cases} \sigma_3 = u_+ + G(t) + \sqrt{\frac{(\Gamma+1)p+(\Gamma-1)p_+}{2\varrho_+(1-\theta\varrho_+)}}, \\ \varrho = \varrho_+ \frac{(\Gamma+1)p+(\Gamma-1)p_+}{(\Gamma-1+2\theta\varrho_+)p+(\Gamma+1-2\theta\varrho_+)p_+}, \quad p > p_+, \\ v = u_+ - \sqrt{\frac{2(1-\theta\varrho_+)}{\varrho_+((\Gamma+1)p+(\Gamma-1)p_+)}}(p - p_+). \end{cases} \tag{6.30}$$

On the 3-shock wave curve $S_3(\varrho_+, u_+, p_+)$, we have

$$\begin{cases} \frac{dv}{dp} = \sqrt{\frac{1-\theta\varrho_+}{2\varrho_+}} \frac{(\Gamma+1)p+(3\Gamma-1)p_+}{((\Gamma+1)p+(\Gamma-1)p_+)^{3/2}} > 0, \\ \frac{d^2v}{dp^2} = -(\Gamma+1)\sqrt{\frac{1-\theta\varrho_+}{8\varrho_+}} \frac{(\Gamma+1)p+(7\Gamma-1)p_+}{((\Gamma+1)p+(\Gamma-1)p_+)^{5/2}} < 0, \\ \frac{d\varrho}{dp} = \frac{4\Gamma p_+\varrho_+(1-\theta\varrho_+)}{((\Gamma-1+2\theta\varrho_+)p+(\Gamma+1-2\theta\varrho_+)p_+)^2} > 0. \end{cases} \tag{6.31}$$

Moreover, we observe that as $p \to +\infty$, $v \to +\infty$ and $\varrho \to \frac{\varrho_+(\Gamma+1)}{(\Gamma+2\theta\varrho_+-1)} < \frac{1}{\theta}$. With the help of (6.30) and (6.31), we obtain

$$\begin{cases} \frac{\partial\varrho}{\partial\theta} = \frac{2\varrho_+^2(p_+-p)((\Gamma+1)p+(\Gamma-1)p_+)}{((\Gamma-1+2\theta\varrho_+)p+(\Gamma+1-2\theta\varrho_+)p_+)^2} < 0, \\ \frac{\partial v}{\partial\theta} = \frac{p_+-p}{\sqrt{2(1-\theta\varrho_+)}}\sqrt{\frac{\varrho_+}{(\Gamma+1)p+(\Gamma-1)p_+}} < 0, \\ \frac{\partial\sigma_3}{\partial\theta} = \frac{\varrho_+}{2(1-\theta\varrho_+)}\sqrt{\frac{(\Gamma+1)p+(\Gamma-1)p_+}{2\varrho_+(1-\theta\varrho_+)}} > 0. \end{cases} \tag{6.32}$$

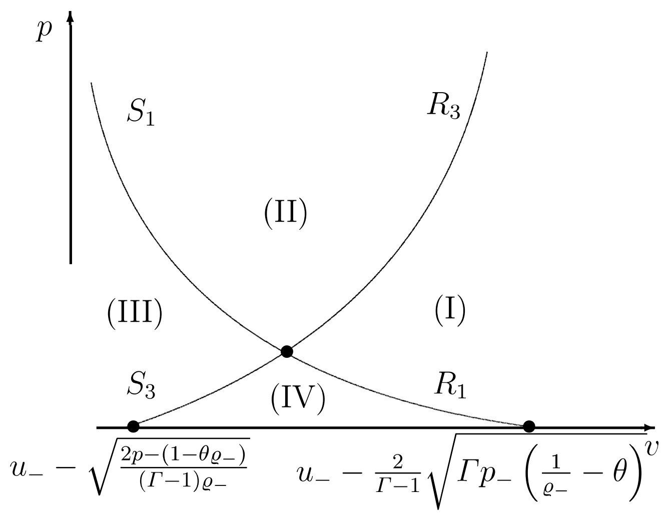

FIGURE 6.1: Projections of the elementary wave curves R, S onto the (v, p) phase plane for the model (6.6) and (6.7) with $p > 0$.

From (6.32), it can be seen that for the 3-shock wave, as θ increases, the speed of the discontinuity increases, whereas the variables ϱ and v decreases with an increase in θ.

6.3 Riemann solutions of the modified homogeneous system (6.6) and (6.7)

Fig.6.1 depicts the projection of the curves $S_1(\varrho_-, u_-, p_-)$, $S_3(\varrho_+, u_+, p_+)$, $R_1(\varrho_-, u_-, p_-)$ and $R_3(\varrho_+, u_+, p_+)$, which divides the entire (v, p)-plane into four different regions I, II, III and IV.

Now, on the space (ϱ, v, p) with $\varrho > 0$ and $p > 0$, $R_3(\varrho_-, u_-, p_-)$, $S_3(\varrho_-, u_-, p_-)$ can be written as

$$R_3(\varrho_-,u_-,p_-):\begin{cases}\frac{dx}{dt}=v+\sqrt{\frac{\Gamma p}{\varrho(1-\theta\varrho)}}+G(t),\\ v-\frac{2}{\Gamma-1}\sqrt{\Gamma p\left(\frac{1}{\varrho}-\theta\right)}=u_--\frac{2}{\Gamma-1}\sqrt{\Gamma p_-\left(\frac{1}{\varrho_-}-\theta\right)},\ \ p>p_-,\\ p\left(\frac{1}{\varrho}-\theta\right)^{\Gamma}=p_-\left(\frac{1}{\varrho_-}-\theta\right)^{\Gamma},\end{cases}\tag{6.33}$$

$$S_3(\varrho_-,u_-,p_-):\begin{cases}\sigma_3=u_-+G(t)+\sqrt{\frac{(\Gamma+1)p+(\Gamma-1)p_-}{2\varrho_-(1-\theta\varrho_-)}},\\ \varrho=\varrho_-\frac{(\Gamma+1)p+(\Gamma-1)p_-}{(\Gamma-1+2\theta\varrho_-)p+(\Gamma+1-2\theta\varrho_-)p_-},\ \ p<p_-,\\ v=u_-+\sqrt{\frac{2(1-\theta\varrho_-)}{\varrho_-((\Gamma+1)p+(\Gamma-1)p_-)}}(p-p_-).\end{cases}\tag{6.34}$$

We will construct the solutions to the Riemann problem for the modified homogeneous system (6.6) with the constant initial data (6.7), and discuss all possible cases.

Case I: Let us consider that the projection of the state (ϱ_+,u_+,p_+) lies in region I. Three cases arise, which we shall discuss separately as follows:

Case I(i): $u_-+\frac{2}{\Gamma-1}\sqrt{\Gamma p_-\left(\frac{1}{\varrho_-}-\theta\right)}>u_+-\frac{2}{\Gamma-1}\sqrt{\Gamma p_+\left(\frac{1}{\varrho_+}-\theta\right)}$. The intersection of the curves $R_1(\varrho_-,u_-,p_-)$ and $R_3(\varrho_+,u_+,p_+)$ in the (v,p)-plane is denoted as (v^*,p^*), and is given as

$$\begin{cases}v_*=u_-+\frac{2}{\Gamma-1}\sqrt{\Gamma p_-\varrho_-^{-1}(1-\theta\varrho_-)}-\frac{2}{\Gamma-1}\sqrt{\Gamma p_-^{1/\Gamma}\varrho_-^{-1}(1-\theta\varrho_-)}\,p_*^{(\Gamma-1)/2\Gamma},\\ p_*=\left(\frac{\Gamma-1}{2}\frac{u_--u_++\frac{2}{\Gamma-1}\left(\sqrt{\Gamma p_-\varrho_-^{-1}(1-\theta\varrho_-)}+\sqrt{\Gamma p_+\varrho_+^{-1}(1-\theta\varrho_+)}\right)}{\sqrt{\Gamma p_-^{1/\Gamma}\varrho_-^{-1}(1-\theta\varrho_-)}+\sqrt{\Gamma p_+^{1/\Gamma}\varrho_+^{-1}(1-\theta\varrho_+)}}\right)^{2\Gamma/(\Gamma-1)}.\end{cases}\tag{6.35}$$

Next, we find the intersection of 2-contact discontinuity $J_2(\varrho_1,v_*,p_*)$ with the curves $R_1(\varrho_-,u_-,p_-)$ and $R_3(\varrho_+,u_+,p_+)$ at the points (ϱ_{*_1},v_*,p_*), (ϱ_{*_2},v_*,p_*), respectively. ϱ_{*_1} and ϱ_{*_2} can be calculated, which is obtained as

$$\begin{cases}\varrho_{*_1}=\frac{\varrho_-}{1+\theta\varrho_-(1+p_*p_-^{-1})^{1/\Gamma}}(p_*p_-^{-1})^{1/\Gamma},\\ \varrho_{*_2}=\frac{\varrho_+}{1+\theta\varrho_+(1+p_*p_+^{-1})^{1/\Gamma}}(p_*p_+^{-1})^{1/\Gamma}.\end{cases}\tag{6.36}$$

In this case, we can construct the Riemann solution as follows:

$$(\varrho, v, p)(t,x) = \begin{cases} (\varrho_-, u_-, p_-), & x < x_1^-(t), \\ (\varrho_{r_1}, v_{r_1}, p_{r_1}), & x_1^-(t) \le x \le x_1^+(t), \\ (\varrho_{*_1}, v_*, p_*), & x_1^+(t) < x < x_2(t), \\ (\varrho_{*_2}, v_*, p_*), & x_2(t) < x < x_3^-(t), \\ (\varrho_{r_3}, v_{r_3}, p_{r_3}), & x_3^-(t) \le x \le x_3^+(t), \\ (\varrho_+, u_+, p_+), & x > x_3^+(t), \end{cases} \tag{6.37}$$

where, $(\varrho_{r_1}, v_{r_1}, p_{r_1})$ is 1-rarefaction wave, which can be obtained from (6.15) and $(\varrho_{r_3}, v_{r_3}, p_{r_3})$ is 3-rarefaction wave, which is obtained from (6.19). $x_1^-(t)$, $x_1^+(t)$, $x_3^-(t)$ and $x_3^+(t)$ is given as

$$\begin{cases} x_1^-(t) = \left(u_- - \sqrt{\frac{\Gamma p-}{\varrho_-(1-\theta\varrho_-)}}\right) t + \int_0^t g(y)dy, \\ x_1^+(t) = \left(v_* - \sqrt{\frac{\Gamma p*}{\varrho_{*_1}(1-\theta\varrho_{*_1})}}\right) t + \int_0^t g(y)dy, \\ x_3^-(t) = \left(v_* + \sqrt{\frac{\Gamma p*}{\varrho_{*_2}(1-\theta\varrho_{*_2})}}\right) t + \int_0^t g(y)dy, \\ x_3^+(t) = \left(u_+ + \sqrt{\frac{\Gamma p+}{\varrho_+(1-\theta\varrho_+)}}\right) t + \int_0^t g(y)dy, \end{cases} \tag{6.38}$$

and the 2-contact discontinuity is $x_2(t) = v_* t + \int_0^t g(y)dy$.

Case I(ii): $u_- + \frac{2}{\Gamma-1}\sqrt{\Gamma p_-\left(\frac{1}{\varrho_-} - \theta\right)} = u_+ - \frac{2}{\Gamma-1}\sqrt{\Gamma p_+\left(\frac{1}{\varrho_+} - \theta\right)}$. From (6.35) and (6.36), we observe that on the phase plane (v,p), the curves $R_1(\varrho_-, u_-, p_-)$ and $R_3(\varrho_+, u_+, p_+)$ meets at a point (v_*, p_*), where $p_* = 0$ and $v_* = u_- + \frac{2}{\Gamma-1}\sqrt{\Gamma p_-\left(\frac{1}{\varrho_-} - \theta\right)}$. Further, we notice that, $\varrho_{*_1} = \varrho_{*_2} = 0$, i.e., the 2-contact discontinuity will not exist.

Therefore, in this case, the solution can be constructed as

$$(\varrho, v, p)(t,x) = \begin{cases} (\varrho_-, u_-, p_-), & x < x_1^-(t), \\ (\varrho_{r_1}, v_{r_1}, p_{r_1}), & x_1^-(t) \leq x \leq x_1^+(t), \\ (\varrho_{r_3}, v_{r_3}, p_{r_3}), & x_1^+(t) < x \leq x_3^+(t), \\ (\varrho_+, u_+, p_+), & x > x_3^+(t), \end{cases} \tag{6.39}$$

where, $x_1^-(t)$ and $x_3^+(t)$ is given in (6.38) and $x_1^+(t) = v_* t + \int_0^t g(y)dy$. In this case, we see that $x_1^+(t)$ for $t > 0$ is a common boundary of $R_1(\varrho_-, u_-, p_-)$ and $R_3(\varrho_+, u_+, p_+)$, where a vacuum $(0, v_*, 0)$ develops in the solution.

Case I(iii): $u_- + \frac{2}{\Gamma - 1}\sqrt{\Gamma p_- \left(\frac{1}{\varrho_-} - \theta\right)} < u_+ - \frac{2}{\Gamma - 1}\sqrt{\Gamma p_+ \left(\frac{1}{\varrho_+} - \theta\right)}$. In this case, the curves $R_1(\varrho_-, u_-, p_-)$ and $R_3(\varrho_+, u_+, p_+)$ do not intersect, and meets the v-axis at points $(v_{*_1}, 0)$ and $(v_{*_2}, 0)$, respectively. Here, v_{*_1} and v_{*_2} are

$$\begin{cases} v_{*_1} = u_- + \frac{2}{\Gamma - 1}\sqrt{\Gamma p_- \left(\frac{1}{\varrho_-} - \theta\right)}, \\ v_{*_2} = u_+ - \frac{2}{\Gamma - 1}\sqrt{\Gamma p_+ \left(\frac{1}{\varrho_+} - \theta\right)}. \end{cases}$$

Since $\varrho_{*_1} = \varrho_{*_2} = 0$, therefore the 2-contact discontinuity does not occur in this case. Hence, the solution can be constructed as follows,

$$(\varrho, v, p)(t,x) = \begin{cases} (\varrho_-, u_-, p_-), & x < x_1^-(t), \\ (\varrho_{r_1}, v_{r_1}, p_{r_1}), & x_1^-(t) \leq x \leq x_1^+(t), \\ (0, v(t,x), 0), & x_1^+(t) < x \leq x_3^-(t), \\ (\varrho_{r_3}, v_{r_3}, p_{r_3}), & x_3^-(t) < x \leq x_3^+(t), \\ (\varrho_+, u_+, p_+), & x > x_3^+(t). \end{cases} \tag{6.40}$$

Here, $x_1^-(t)$ and $x_3^+(t)$ is given by (6.38) and $x_1^+(t)$, $x_3^-(t)$ are

$$\begin{cases} x_1^+(t) = v_{*_1}t + \int_0^t g(y)dy, \\ x_3^-(t) = v_{*_2}t + \int_0^t g(y)dy, \end{cases}$$

and $(0, v(t,x), 0)$ is vacuum with $v(x_1^+(t), t) = v_{*_1}$, $v(x_3^-(t), t) = v_{*_2}$. In this case, it is observed that a vacuum $(0, v(t,x), 0)$ connects the 1-rarefaction wave $R_1(\varrho_-, u_-, p_-)$ and 3-rarefaction wave $R_3(\varrho_+, u_+, p_+)$ through the curves $x_1^+(t)$ and $x_3^-(t)$ as its left side and right side boundaries, respectively.

Case II: We consider that the state (ϱ_+, u_+, p_+) lies in the II region. In this case, solution can be constructed as

$$(\varrho, v, p)(t,x) = \begin{cases} (\varrho_-, u_-, p_-), \ \ x < x_1(t), \\ (\varrho_{*_1}, v_*, p_*), \ \ x_1(t) < x < x_2(t), \\ (\varrho_{*_2}, v_*, p_*), \ \ x_2(t) < x < x_3^-(t), \\ (\varrho_{r_3}, v_{r_3}, p_{r_3}), \ \ x_3^-(t) \leq x \leq x_3^+(t), \\ (\varrho_+, u_+, p_+), \ \ x > x_3^+(t), \end{cases} \tag{6.41}$$

where,

$$\varrho_{*_1} = \varrho_- \frac{(\Gamma+1)p_* + (\Gamma-1)p_-}{(\Gamma - 1 + 2\theta\varrho_-)p_* + (\Gamma + 1 - 2\theta\varrho_-)p_-},$$

$$\varrho_{*_2} = \frac{\varrho_+}{1 + \theta\varrho_+(1 + p_*p_+^{-1})^{1/\Gamma}}(p_*p_+^{-1})^{1/\Gamma}.$$

$$x_3^-(t) = \left(v_* + \sqrt{\frac{\Gamma p_*}{\varrho_{*_2}(1-\theta\varrho_{*_2})}}\right)t + \int_0^t g(y)dy,$$

$$x_3^+(t) = \left(u_+ + \sqrt{\frac{\Gamma p_+}{\varrho_+(1-\theta\varrho_+)}}\right)t + \int_0^t g(y)dy,$$

and (v_*, p_*) can be determined by using the the following equations

$$v_* = \frac{2}{\Gamma-1}\sqrt{\Gamma p_+^{1/\Gamma}\varrho_+^{-1}(1-\theta\varrho_+)p_*^{(\Gamma-1)/\Gamma}} + u_+ - \frac{2}{\Gamma-1}\sqrt{\Gamma p_+\left(\varrho_+^{-1}(1-\theta\varrho_+)\right)},$$

$$v_* = u_- - \sqrt{\frac{2(1-\theta\varrho_-)}{\varrho_-((\Gamma+1)p_* + (\Gamma-1)p_-)}}(p_* - p_-).$$

$x_2(t) = u_- - \sqrt{\frac{(\Gamma+1)p_*+(\Gamma-1)p_-}{2\varrho_-(1-\theta\varrho_-)}} + \int_0^t g(y)dy$ denotes the 1-shock wave and $x_2(t) = v_*t + \int_0^t g(y)dy$ is the 2-contact discontinuity.

Case III: In this case, the state (ϱ_+, u_+, p_+) lies in the III region. The solution can be written as

$$(\varrho, v, p)(t, x) = \begin{cases} (\varrho_-, u_-, p_-), & x < x_1(t), \\ (\varrho_{*_1}, v_*, p_*), & x_1(t) < x < x_2(t), \\ (\varrho_{*_2}, v_*, p_*), & x_2(t) < x < x_3(t), \\ (\varrho_+, u_+, p_+), & x > x_3(t), \end{cases} \tag{6.42}$$

Here, (v_*, p_*) can be determined by the following equations

$$\begin{cases} v_* = u_+ + \sqrt{\frac{2(1-\theta\varrho_+)}{\varrho_+((\Gamma+1)p_*+(\Gamma-1)p_+)}}(p_* - p_+), \\ v_* = u_- - \sqrt{\frac{2(1-\theta\varrho_-)}{\varrho_-((\Gamma+1)p_*+(\Gamma-1)p_-)}}(p_* - p_-). \end{cases}$$

and ϱ_{*_1}, ϱ_{*_2} is given as

$$\begin{cases} \varrho_{*_1} = \varrho_-\frac{(\Gamma+1)p_*+(\Gamma-1)p_-}{(\Gamma-1+2\theta\varrho_-)p_*+(\Gamma+1-2\theta\varrho_-)p_-}, \\ \varrho_{*_2} = \varrho_+\frac{(\Gamma+1)p_*+(\Gamma-1)p_+}{(\Gamma-1+2\theta\varrho_+)p_*+(\Gamma+1-2\theta\varrho_+)p_+}, \end{cases}$$

$x_1(t) = \left(u_- - \sqrt{\frac{\varrho_-((\Gamma+1)p_*+(\Gamma-1)p_-)}{2\varrho_-(1-\theta\varrho_-)}}\right)t + \int_0^t g(y)dy$ denotes the 1-shock wave, $x_2(t) = v_*t + \int_0^t g(y)dy$ is the 2-contact discontinuity and $x_3(t) = \left(u+ + \sqrt{\frac{(\Gamma+1)p_*+(\Gamma-1)p_+}{2\varrho_+(1-\theta\varrho_+)}}\right)t + \int_0^t g(y)dy$ is the 3-shock wave.

Case IV: In this case, the projection of the state (ϱ_+, u_+, p_+) belongs to the IV region. The solution can be constructed as given below,

$$(\varrho, v, p)(t,x) = \begin{cases} (\varrho_-, u_-, p_-), \;\; x < x_1^-(t), \\ (\varrho_{r_1}, v_{r_1}, p_{r_1}), \;\; x_1^-(t) \le x \le x_1^+(t), \\ (\varrho_{*_1}, v_*, p_*), \;\; x_1^+(t) < x < x_2(t), \\ (\varrho_{*_2}, v_*, p_*), \;\; x_2(t) < x < x_3(t), \\ (\varrho_+, u_+, p_+), \;\; x > x_3(t), \end{cases} \tag{6.43}$$

where

$$\varrho_{*_1} = \frac{\varrho_-}{1 + \theta\varrho_-(1 + p_*p_-^{-1})^{1/\Gamma}}(p_*p_-^{-1})^{1/\Gamma},$$

$$\varrho_{*_2} = \varrho_+ \frac{(\Gamma+1)p_* + (\Gamma-1)p_+}{(\Gamma - 1 + 2\theta\varrho_+)p_* + (\Gamma + 1 - 2\theta\varrho_+)p_+},$$

$$x_1^-(t) = \left(u_- - \sqrt{\frac{\Gamma p-}{\varrho_-(1-\theta\varrho_-)}}\right)t + \int_0^t g(y)dy,$$

$$x_1^+(t) = \left(v_* - \sqrt{\frac{\Gamma p*}{\varrho_{*_1}(1-\theta\varrho_{*_1})}}\right)t + \int_0^t g(y)dy,$$

and (v_*, p_*) can be determined by solving the following equations

$$\begin{cases} v_* = u_+ + \sqrt{\frac{2(1-\theta\varrho_+)}{\varrho_+((\Gamma+1)p_*+(\Gamma-1)p_+)}}(p_* - p_+), \\ v_* = -\frac{2}{\Gamma-1}\sqrt{\Gamma p_-^{1/\Gamma}\varrho_-^{-1}(1-\theta\varrho_-)p_*^{(\Gamma-1)/\Gamma}} + u_- + \frac{2}{\Gamma-1}\sqrt{\Gamma p_-\left(\varrho_-^{-1}(1-\theta\varrho_-)\right)}. \end{cases}$$

The 3-shock shock curve is denoted by x_3, and is given as

$$x_3 = \left(u_+ - \sqrt{\frac{((\Gamma+1)p_* + (\Gamma-1)p_+)}{2\varrho_+(1-\theta\varrho_+)}}\right)t + \int_0^t g(y)dy,$$

and the 2-contact discontinuity is

$$x_2 = v_* t + \int_0^t g(y)dy.$$

6.4 Riemann solutions of the system (6.3) and (6.4)

In this section, we will discuss the solution of the Riemann problem to the original system (6.3) and (6.4). By using the change of state variables $(\varrho, u, p)(t, x) = (\varrho, v + G(t), p)(t, x)$, solution of the given system (6.3) and (6.4) can be determined by using the corresponding solution of the modified system (6.6) and (6.7). Here, we will discuss six different cases, which we describe one by one as follows:

Case (a): When the state (ϱ_+, u_+, p_+) lies in region I with $u_- + \frac{2}{\Gamma-1}\sqrt{\Gamma p_-\left(\frac{1}{\varrho_-} - \theta\right)} > u_+ - \frac{2}{\Gamma-1}\sqrt{\Gamma p_+\left(\frac{1}{\varrho_+} - \theta\right)}$, the solution can be written as

$$(\varrho, v, p)(t, x) = \begin{cases} (\varrho_-, u_- + G(t), p_-), & x < x_1^-(t), \\ (\varrho_{r_1}, v_{r_1} + G(t), p_{r_1}), & x_1^-(t) \leq x \leq x_1^+(t), \\ (\varrho_{*_1}, v_* + G(t), p_*), & x_1^+(t) < x < x_2(t), \\ (\varrho_{*_2}, v_* + G(t), p_*), & x_2(t) < x < x_3^-(t), \\ (\varrho_{r_3}, v_{r_3} + G(t), p_{r_3}), & x_3^-(t) \leq x \leq x_3^+(t), \\ (\varrho_+, u_+ + G(t), p_+), & x > x_3^+(t). \end{cases} \tag{6.44}$$

In this case, the solution is presented in Fig.6.2, and the values of the variables are given in (6.37).

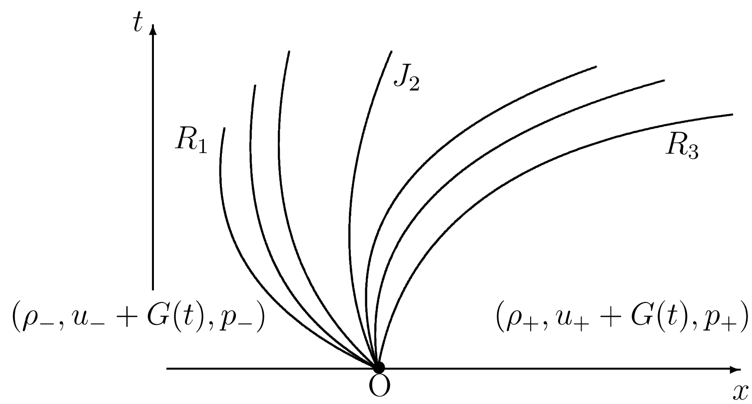

FIGURE 6.2: Solution structure of system (6.3) and (6.4) for case (a).

Case (b): When the state (ϱ_+, u_+, p_+) lies in region I with $u_- + \frac{2}{\Gamma-1}\sqrt{\Gamma p_- \left(\frac{1}{\varrho_-} - \theta\right)} = u_+ - \frac{2}{\Gamma-1}\sqrt{\Gamma p_+ \left(\frac{1}{\varrho_+} - \theta\right)}$, the solution can be written as

$$(\varrho, v, p)(t,x) = \begin{cases} (\varrho_-, u_- + G(t), p_-), & x < x_1^-(t), \\ (\varrho_{r_1}, v_{r_1} + G(t), p_{r_1}), & x_1^-(t) \le x \le x_1^+(t), \\ (\varrho_{r_3}, v_{r_3} + G(t), p_{r_3}), & x_1^+(t) < x \le x_3^+(t), \\ (\varrho_+, u_+ + G(t), p_+), & x > x_3^+(t). \end{cases} \tag{6.45}$$

In this case, the solution is presented in Fig.6.3, and the values of the variables are given in (6.39).

t

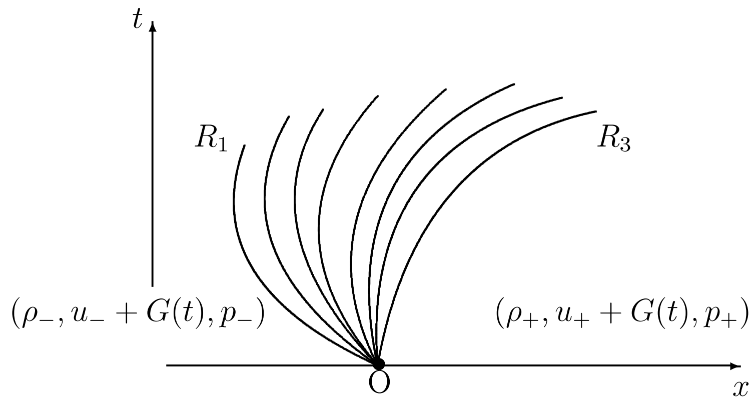

FIGURE 6.3: Solution structure of system (6.3) and (6.4) for case (b).

Case (c): When the state (ϱ_+, u_+, p_+) lies in region I with $u_- + \frac{2}{\Gamma-1}\sqrt{\Gamma p_-\left(\frac{1}{\varrho_-} - \theta\right)} < u_+ - \frac{2}{\Gamma-1}\sqrt{\Gamma p_+\left(\frac{1}{\varrho_+} - \theta\right)}$, the solution can be constructed as follows,

$$(\varrho, v, p)(t,x) = \begin{cases} (\varrho_-, u_- + G(t), p_-), & x < x_1^-(t), \\ (\varrho_{r_1}, v_{r_1} + G(t), p_{r_1}), & x_1^-(t) \leq x \leq x_1^+(t), \\ (0, v(t,x) + G(t), 0), & x_1^+(t) < x \leq x_3^-(t), \\ (\varrho_{r_3}, v_{r_3} + G(t), p_{r_3}), & x_3^-(t) < x \leq x_3^+(t), \\ (\varrho_+, u_+ + G(t), p_+), & x > x_3^+(t). \end{cases} \tag{6.46}$$

In this case, the solution is shown in Fig.6.4, and the values of the variables can be obtained from (6.40).

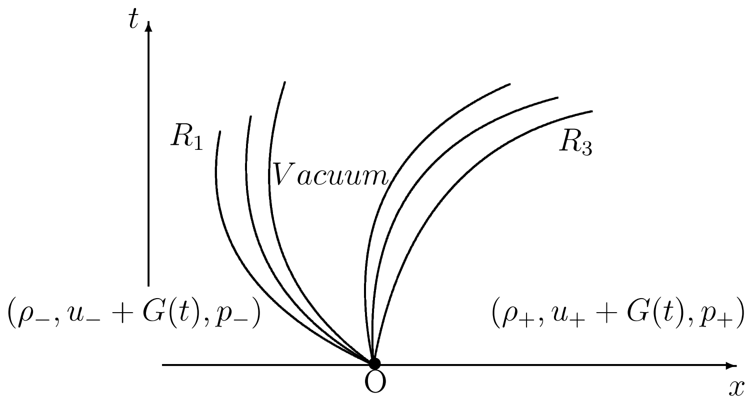

FIGURE 6.4: Solution structure of system (6.3) and (6.4) for case (c).

Case (d): When the state (ϱ_+, u_+, p_+) belongs to the region II of the phase plane (v, p), the solution can be presented as

$$(\varrho, v, p)(t, x) = \begin{cases} (\varrho_-, u_- + G(t), p_-), \;\; x < x_1(t), \\ (\varrho_{*_1}, v_* + G(t), p_*), \;\; x_1(t) < x < x_2(t), \\ (\varrho_{*_2}, v_* + G(t), p_*), \;\; x_2(t) < x < x_3^-(t), \\ (\varrho_{r_3}, v_{r_3} + G(t), p_{r_3}), \;\; x_3^-(t) \leq x \leq x_3^+(t), \\ (\varrho_+, u_+ + G(t), p_+), \;\; x > x_3^+(t), \end{cases} \tag{6.47}$$

In this case, the solution is shown in Fig.6.5, and the values of the variables can be obtained from (6.41).

t

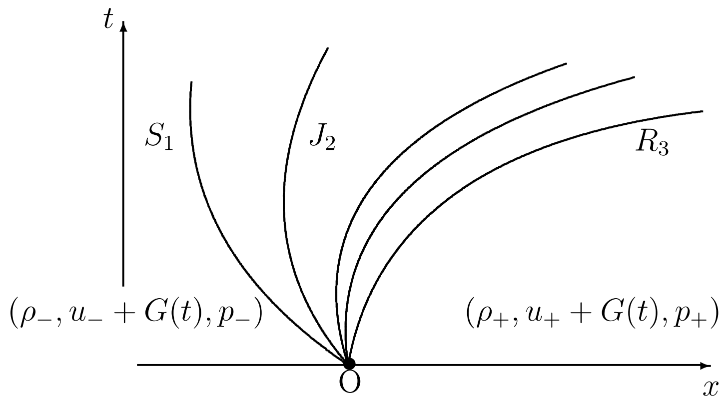

FIGURE 6.5: Solution structure of system (6.3) and (6.4) for case (d).

Case (e): When the state (ϱ_+, u_+, p_+) belongs to the region III of the phase plane (v, p), the solution can be obtained as

$$(\varrho, v, p)(t, x) = \begin{cases} (\varrho_-, u_- + G(t), p_-), & x < x_1(t), \\ (\varrho_{*_1}, v_* + G(t), p_*), & x_1(t) < x < x_2(t), \\ (\varrho_{*_2}, v_* + G(t), p_*), & x_2(t) < x < x_3(t), \\ (\varrho_+, u_+ + G(t), p_+), & x > x_3(t). \end{cases} \tag{6.48}$$

The solution is presented in Fig.6.6, and the values of the variables are given in (6.42).

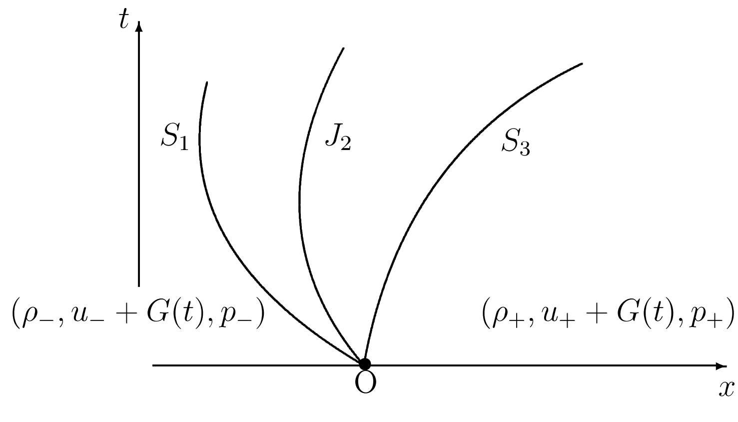

FIGURE 6.6: Solution structure of system (6.3) and (6.4) for case (e).

Case (f): When the state (ϱ_+, u_+, p_+) belongs to the region IV of the phase plane (v, p), the solution can be written as

$$(\varrho, v, p)(t, x) = \begin{cases} (\varrho_-, u_- + G(t), p_-), & x < x_1^-(t), \\ (\varrho_{r_1}, v_{r_1} + G(t), p_{r_1}), & x_1^-(t) \leq x \leq x_1^+(t), \\ (\varrho_{*_1}, v_* + G(t), p_*), & x_1^+(t) < x < x_2(t), \\ (\varrho_{*_2}, v_* + G(t), p_*), & x_2(t) < x < x_3(t), \\ (\varrho_+, u_+ + G(t), p_+), & x > x_3(t), \end{cases} \tag{6.49}$$

The solution is presented in Fig.6.7, and the values of the variables are given in (6.43).

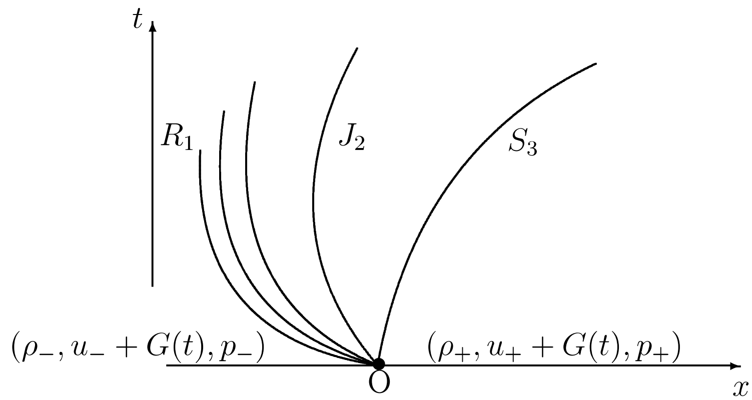

FIGURE 6.7: Solution structure of system (6.3) and (6.4) for case (f).

After the entire analysis of the solution of the Riemann problem constructed for the (6.3) and (6.4), we arrived at a result that only two cases are possible (b),(c), where the vacuum exists in the solution. Moreover, we observed that the solution of the Riemann Problems (6.3) and (6.4) consist the rarefaction waves, shock waves and contact discontinuity if and only if the given initial data satisfy the condition

$$u_- + \frac{2}{\Gamma - 1}\sqrt{\Gamma p_- \left(\frac{1}{\varrho_-} - \theta\right)} > u_+ - \frac{2}{\Gamma - 1}\sqrt{\Gamma p_+ \left(\frac{1}{\varrho_+} - \theta\right)}, \tag{6.50}$$

if the above condition fails, then there exists a vacuum in the solution.

6.5 Conclusion

In this chapter, we constructed the solution of the Riemann problem for one-dimensional flow of dusty gas with the presence of external force, which is a continuous function of the time. The presence of dust particles plays a crucial role in the construction of the solution of the considered non-homogeneous system of equations. The elementary wave solutions such as rarefaction wave, shock wave and contact discontinuity are presented. Further, we have discuss the case, where all the elementary waves present in the solution of the Riemann Problem and also, we described the case, where the vacuum appears in the solution. It is observed that an increase in the value of the parameter θ causes to increase the density and decrease the velocity for 1-rarefaction wave with a given left state, whereas in case of 3-rarefaction wave through a given right state, the velocity and density both increases with respect to an increase in θ. In addition, we noticed that if we increase the value of θ, the shock wave speed and the density both simultaneously decreases for 1-shock wave with the left state. In contrast, velocity increases with respect to an increase in the parameter θ for 1-shock wave. In case of 3-shock wave through a given right state, if we increase the value of θ, the velocity and density both decreases whereas the shock speed increases. We observed that, in all kind of solutions discussed in this chapter, no one is self-similar because of the presence of external force appears in the system considered for the flow of dusty gas. Moreover, due to presence of external force, the elementary wave curves bend and takes the parabolic shape.

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