Chapter 6

Numerical solution of Volterra integral equation using λ- Bernstein operators

6.1 Introduction

The Volterra integral equations of the second kind (2ndVIEs), is expressed as follows:

$$\Psi(x) = \Phi(x) + \mu \int_0^x \kappa(x, \xi)\Psi(\xi)d\xi. \tag{6.1}$$

Similarly, the Volterra integral equations of the first kind (1stVIEs) take the following form:

$$\Phi(x) = \mu \int_0^x \kappa(x, \xi)\Psi(\xi)d\xi, \tag{6.2}$$

where $\Psi(x)$ is an unknown function and $\Phi(x)$ is a known real-valued function and $\kappa(x, \xi)$ is a given bivariate kernel function, and μ is a given non-zero parameter.

Maleknejad et al. [83] have proposed a novel method for solving first and second-kind Volterra integral equations using the Bernstein approximation method. In their study, they have provided a convergence analysis for these methods based on the Voronovskaya-type theorem related to classical Bernstein approximation. Further, their subsequent works [84] and [98] have introduced numerical approaches for solving fractional Volterra integral equations of both kinds. It is worth noting that the operators used in that technique, i.e., the Bernstein operators are popular for their

simplicity and accuracy in approximating continuous functions. Moreover, the literature indicates that numerous researchers have employed Bernstein approximation techniques to address various types of Fredholm and Volterra integral equations. For further details, we refer to [30, 110, 111] and the references cited therein.

However, the λ-Bernstein operators introduced by Cai et al. [40] are of particular interest to us. These operators offer significant advantages due to their modeling flexibility, which is controlled by the parameter λ. For specific values of λ, these operators can converge more rapidly than traditional Bernstein operators for certain functions. Consequently, our objective is to solve equations (6.1) and (6.2) numerically using these operators and to illustrate their effectiveness in the solution process. For a continuous function Ψ on $[0,1]$, the λ-Bernstein operator and the Bezier basis functions are given in (2.22) and (2.21) respectively.

From now onwards, we denote the space of all continuously a-times differentiable functions on $[0,1]$ by $\mathfrak{C}^a[0,1]$, where $a \in \mathbb{N}$. This space is a normed linear spaces with respect to the sup norm: $\|\Psi\| = \sup\{\Psi(x) : x \in [0,1]\}$. Here, we provide key details about the convergence of the operators (2.22), which will be important for the following discussions.

Theorem 6.1. [40] *For $\lambda \in [-1,1]$ and any continuous function Ψ on $[0,1]$, the sequence of polynomials $P_{m,\lambda}(\Psi)$ converges uniformly to Ψ.*

The above theorem ensures that for any $\epsilon > 0$ and any continuous function Ψ on $[0,1]$, $\exists\ m \in \mathbb{N}$ such that

$$\|P_{m,\lambda}(\Psi) - \Psi\| < \epsilon.$$

Also, for $\Psi \in \mathfrak{C}^2[0,1]$, we can deduce that $|P_{m,\lambda}(\Psi) - \Psi| \leq \frac{\|\Psi''\|}{2m} x(1-x)$, which is an asymptotic pointwise error bound for the operators (2.22). Then, using the fact

that for any $x \in X$, $x(1-x) \leq \frac{1}{8}$, we can get the relation

$$|P_{m,\lambda}(\Psi) - \Psi| \leq \frac{\|\Psi''\|}{8m}. \tag{6.3}$$

This chapter aims to advance the numerical computational methods for solving Volterra integral equations of the first and the second kind by employing the λ-Bernstein approximation method. To achieve this, we develop a numerical scheme for solving first and second-kind Volterra integral equations. To validate the proposed algorithm, we present both theoretical and numerical results.

The chapter is organized as follows: Sect. 6.2 details the algorithms for solving integral equations using the λ-Bernstein approximation method. Sect. 6.3 provides an error analysis of these methods, utilizing the Voronovskaja-type theorem for λ-Bernstein operators and some numerical examples are discussed in Sect. 6.4.

6.2 Discretization and Algorithms

This section deals with the numerical implementation of the operators (2.22) to obtain the approximate solution of (6.1) and (6.2) using the λ-Bernstein approximation method.

6.2.1 Algorithm to solve the 2ndVIEs

This part is dedicated to designing an algorithm in a one-dimensional setting to solve the 2ndVIEs using the λ-Bernstein approximation method. To computationally solve the 2ndVIEs, we first need to approximate the unknown function $\Psi(x)$, which is to

be determined, as follows:

$$\Psi(x) \approx P_{m,\lambda}(\Psi; x) = \sum_{j=0}^{m} \tilde{p}_{m,j}(\lambda; x)\Psi\left(\frac{j}{m}\right). \tag{6.4}$$

We apply (6.4) to (6.1) to obtain

$$\begin{aligned} &\sum_{j=0}^{m} \tilde{p}_{m,j}(\lambda; x)\Psi\left(\frac{j}{m}\right) = \Phi(x) + \mu \int_0^x \kappa(x,\xi) \sum_{j=0}^{m} \tilde{p}_{m,j}(\lambda; \xi)\Psi\left(\frac{j}{m}\right) d\xi \\ \Longrightarrow &\sum_{j=0}^{m} \Psi\left(\frac{j}{m}\right) \left[\tilde{p}_{m,j}(\lambda; x) - \mu \int_0^x \kappa(x,\xi)\tilde{p}_{m,j}(\lambda; \xi) d\xi\right] = \Phi(x), \quad 0 \le x \le 1. \end{aligned} \tag{6.5}$$

The method involves computing the approximate values of $\Psi\left(\frac{j}{m}\right)$ for j=0,1,. . . ,m. These values are then substituted into equation (6.4) to ultimately obtain an approximate solution to equation (6.2). This is achieved by converting equation (6.5) into a linear system of equations.

Before proceeding further, we replace x with $x_k = \frac{k}{m} + \delta,\ k = 0, 1, ..., m-1$ and set $x_m = 1 - \delta$ to avoid the singularities, where δ is a very small positive real number. It is also important to note that the approximate solutions of $\Psi\left(\frac{j}{m}\right)$ will depend on the values of m and λ. Therefore, we denote $\Psi\left(\frac{j}{m}\right)$ as $\Psi_{m,\lambda}\left(\frac{j}{m}\right)$, for $j = 0, 1, ..., m$. Consequently, we obtain

$$\sum_{j=0}^{m} \Psi_{m,\lambda}\left(\frac{j}{m}\right) \left[\tilde{p}_{m,j}(\lambda; x_k) - \mu \int_0^{x_k} \kappa(x_k,\xi)\tilde{p}_{m,j}(\lambda; \xi) d\xi\right] = \Phi(x_k), \quad k = 0, 1, ..., m.$$

This then boils down to a linear system of equations of the form: $[\mathbf{V}][\mathbf{Y}] = [\mathbf{U}]$, where

$$[\mathbf{V}] = \left[\tilde{p}_{m,j}(\lambda; x_k) - \mu \int_0^{x_k} \kappa(x_k, \xi)\tilde{p}_{m,j}(\lambda; \xi)d\xi\right]_{(m+1)\times(m+1)}, \qquad k, j = 0, 1, ..., m, \tag{6.6}$$

$$[\mathbf{U}] = \begin{bmatrix} \Phi(x_0) \\ \Phi(x_1) \\ \vdots \\ \Phi(x_m) \end{bmatrix}_{(m+1)\times 1}, \qquad [\mathbf{Y}] = \begin{bmatrix} \Psi_{m,\lambda}(0) \\ \Psi_{m,\lambda}(1/m) \\ \vdots \\ \Psi_{m,\lambda}(1) \end{bmatrix}_{(m+1)\times 1}.$$

One should note the fact that the matrix equation $[\mathbf{V}][\mathbf{Y}] = [\mathbf{U}]$ is solvable if and only if $[\mathbf{V}]$ is invertible. Therefore, we must first calculate the matrices $[\mathbf{V}]$ and $[\mathbf{U}]$. Once these are determined, we can easily evaluate $[\mathbf{Y}] = [\mathbf{V^{-1}}][\mathbf{U}]$:

Algorithm 1 Approximate solution of 2ndVIE using λ-Bernstein operators

Input: $x_k, k = 0, 1, \ldots, m$

1: **for** $i \leftarrow 0$ to m **do**
2: **for** $j \leftarrow 0$ to m **do**
3: Compute $[\mathbf{V}]_{(m+1)\times(m+1)}$
4: Compute $[\mathbf{Y}] = [\mathbf{V^{-1}}][\mathbf{U}]$.

Output: Compute $\sum_{j=0}^{m} \tilde{p}_{m,j}(\lambda; x)\Psi_{m,\lambda}\left(\frac{j}{m}\right)$ using $[\mathbf{Y}]$.

6.2.2 Algorithm to solve the 1stVIEs

This part aims to construct an algorithm to solve 1stVIEs using the λ-Bernstein approximation method in a one-dimensional setting. To computationally solve the 1stVIEs, we first apply (6.4) to (6.2) to obtain

$$\Phi(x) = \mu \int_0^x \kappa(x, \xi) \sum_{j=0}^{m} \tilde{p}_{m,j}(\lambda; \xi)\Psi\left(\frac{j}{m}\right) d\xi$$

$$\implies \sum_{j=0}^{m} \Psi\left(\frac{j}{m}\right) \left[\mu \int_0^x \kappa(x,\xi)\tilde{p}_{m,j}(\lambda;\xi)d\xi\right] = \Phi(x), \quad 0 \le x \le 1. \tag{6.7}$$

The method is to compute the approximate values of $\Psi\left(\frac{j}{m}\right),\ j = 0, 1, ..., m$ by converting (6.7) to a linear system of equations. As explained in the previous subsection, we replace x by $x_k = \frac{k}{m} + \delta,\ k = 0, 1, ..., m-1$ and $x_m = 1 - \delta$ and present $\Psi\left(\frac{j}{m}\right)$ by $\Psi_{m,\lambda}\left(\frac{j}{m}\right)$, for $j = 0, 1, ..., m$. Consequently, we obtain

$$\sum_{j=0}^{m} \Psi_{m,\lambda}\left(\frac{j}{m}\right) \left[\mu \int_0^{x_k} \kappa(x_k,\xi)\tilde{p}_{m,j}(\lambda;\xi)d\xi\right] = \Phi(x_k), \quad k = 0, 1, ..., m. \tag{6.8}$$

This simplifies to a system of linear equations in the following form: $[\mathbf{F}][\mathbf{Y}] = [\mathbf{U}]$, where,

$$[\mathbf{F}] = \left[\mu \int_0^{x_k} \kappa(x_k,\xi)\tilde{p}_{m,j}(\lambda;\xi)d\xi\right]_{(m+1)\times(m+1)}, \qquad k, j = 0, 1, ..., m, \tag{6.9}$$

$$[\mathbf{U}] = \begin{bmatrix} \Phi(x_0) \\ \Phi(x_1) \\ \vdots \\ \Phi(x_m) \end{bmatrix}_{(m+1)\times 1}, \qquad [\mathbf{Y}] = \begin{bmatrix} \Psi_{m,\lambda}(0) \\ \Psi_{m,\lambda}(1/m) \\ \vdots \\ \Psi_{m,\lambda}(1) \end{bmatrix}_{(m+1)\times 1}.$$

It is important to note that the matrix equation $[\mathbf{F}][\mathbf{Y}] = [\mathbf{U}]$ is solvable if and only if $[\mathbf{F}]$ is invertible. Therefore, we first need to compute the matrices $[\mathbf{F}]$ and $[\mathbf{U}]$. Once we have these, we can easily determine $[\mathbf{Y}]$ by evaluating $[\mathbf{Y}] = [\mathbf{F^{-1}}][\mathbf{U}]$:

Algorithm 2 Approximate solution of 1stVIE using λ-Bernstein operators

Input: $x_k, k = 0, 1, \ldots, m$

1: **for** $i \leftarrow 0$ to m **do**
2: **for** $j \leftarrow 0$ to m **do**
3: Compute $[\mathbf{F}]_{(m+1)\times(m+1)}$
4: Compute $[\mathbf{Y}] = [\mathbf{F^{-1}}][\mathbf{U}]$.

Output: Compute $\sum_{j=0}^{m} \tilde{p}_{m,j}(\lambda; x)\Psi_{m,\lambda}\left(\frac{j}{m}\right)$ using $[\mathbf{Y}]$.

6.3 Error analysis

We now turn our attention to analyzing the error of the proposed solution technique to validate it theoretically. Concurrently, we need to determine an upper bound for $\sup_{x_k\in[0,1]} |\Psi(x_k) - P_{m,\lambda}(\Psi_{m,\lambda}; x_k)|$, which must tend to zero as $m \to \infty$.

6.3.1 Error estimation for 2ndVIEs

The following theorem demonstrates that the numerical solution of the 2ndVIEs converge to the exact solution as m increases.

Theorem 6.2. *Suppose that $k(x, \xi)$ is continuous on $[0, 1] \times [0, 1]$ and the solution of (6.1) belong to $C^a[0, 1]$ for some $a > 2$. If the matrix V calculated in (6.6) is non-singular, then*

$$\sup_{x_k\in[0,1]} |\Psi(x_k) - P_{m,\lambda}(\Psi_{m,\lambda}; x_k)| \leq \frac{1}{8m}\|\Psi''\|\Big[1 + (1 + \tau)\|V^{-1}\|\Big],$$

where $\tau = \sup_{x,\xi\in[0,1]} |\mu||\kappa(x, \xi)|$, $x_k = k/m$, $k = 0, 1, ..., m$ and $P_{m,\lambda}(\Psi_{m,\lambda}; x)$ is the proposed approximate solution.

Proof. First, by utilizing the triangle inequality, we have

$$\begin{aligned}
&\sup_{x_k\in[0,1]} |\Psi(x_k) - P_{m,\lambda}(\Psi_{m,\lambda}; x_k)| \\
&= \sup_{x_k\in[0,1]} |\Psi(x_k) - P_{m,\lambda}(\Psi; x_k) + P_{m,\lambda}(\Psi; x_k) - P_{m,\lambda}(\Psi_{m,\lambda}; x_k)| \\
&\leq \sup_{x_k\in[0,1]} |\Psi(x_k) - P_{m,\lambda}(\Psi; x_k)| + \sup_{x_k\in[0,1]} |P_{m,\lambda}(\Psi; x_k) - P_{m,\lambda}(\Psi_{m,\lambda}; x_k)| . \qquad (6.10)
\end{aligned}$$

From (6.3), it is clear that

$$\sup_{x_k\in[0,1]} |\Psi_{m,\lambda}(x_k) - P_{m,\lambda}(\Psi; x_k)| \leq \frac{\|\Psi''\|}{8m}. \qquad (6.11)$$

Therefore, it is only necessary to find a bound for $\sup_{x_k\in[0,1]} |P_{m,\lambda}(\Psi; x_k) - P_{m,\lambda}(\Psi_{m,\lambda}; x_k)|$. By substituting $\Psi(x)$ with $P_{m,\lambda}(\Psi; x)$ in equation (6.1), we obtain a new function, which we will denote as $\tilde{\Phi}(x)$ in place of $\Phi(x)$. Thus, we have

$$\tilde{\Phi}(x) = P_{m,\lambda}(\Psi; x) - \mu \int_0^{x_k} \kappa(x, \xi) P_{m,\lambda}(\Psi; \xi) d\xi. \qquad (6.12)$$

Now, (6.1) and (6.12) yield

$$\Phi(x) - \tilde{\Phi}(x) = \Psi(x) - P_{m,\lambda}(\Psi; x) - \mu \int_0^{x_k} \kappa(x, \xi) \left(\Psi(\xi) - P_{m,\lambda}(\Psi; \xi)\right) d\xi.$$

Taking the supremum on both sides of the above equation over $x_k \in [0, 1]$, and using the relation (6.3), we obtain

$$\begin{aligned}
&\sup_{x_k\in[0,1]} |\Phi(x_k) - \tilde{\Phi}(x_k)| \\
&= \sup_{x_k\in[0,1]} \left|\Psi(x_k) - P_{m,\lambda}(\Psi; x_k) - \mu \int_0^{x_k} \kappa(x_k, \xi) \left(\Psi(\xi) - P_{m,\lambda}(\Psi; \xi)\right) d\xi\right| \\
&\leq \sup_{x_k\in[0,1]} |\Psi(x_k) - P_{m,\lambda}(\Psi; x_k)| + \sup_{x_k\in[0,1]} \left|\mu \int_0^{x_k} \kappa(x_k, \xi) \left(\Psi(\xi) - P_{m,\lambda}(\Psi; \xi)\right) d\xi\right|
\end{aligned}$$

$$\begin{aligned}&\leq \frac{\|\Psi''\|}{8m} + \frac{\tau}{8m}\|\Psi''\| \\ &= \frac{1}{8m}(1+\tau)\|\Psi''\|. \end{aligned} \tag{6.13}$$

However, for $x_k \in [0,1]$, the relationships in (6.1) and (6.12), yield

$$P_{m,\lambda}(\Psi; x_k) - P_{m,\lambda}(\Psi_{m,\lambda}; x_k) = V^{-1}[\Phi(x_k) - \tilde{\Phi}(x_k)].$$

We then apply (6.13) to deduce that

$$\begin{aligned}\sup_{x_k\in[0,1]} |P_{m,\lambda}(\Psi; x_k) - P_{m,\lambda}(\Psi_{m,\lambda}; x_k)| &\leq \|V^{-1}\| \sup_{x_k\in[0,1]} |[\Phi(x_k) - \tilde{\Phi}(x_k)]| \\ &\leq \frac{1}{8m}(1+\tau)\|V^{-1}\|\|\Psi''\|, \end{aligned} \tag{6.14}$$

which, together with equations (6.10), (6.11), and (6.14), yields the desired assertion.

□

Let's denote $cond(A)$ by the condition number of a matrix A. Then, we have the following lemma for the matrix V.

Lemma 6.3. *In the previous theorem, assume that* $\|V - I\| = \eta_2 < 1$, *where the* $\|.\|$ *represents the maximum norm of rows and* I *is the identity matrix of order* $(m+1)$. *Then*

$$\|V^{-1}\| \leq \frac{1}{1-\eta_2} \quad \text{and} \quad cond(V) \leq \frac{1+\eta_1}{1-\eta_2},$$

where $\eta_1 = \max_k |\mu \int_0^{x_k} \kappa(x_k, \xi) d\xi|$.

Proof. With the help of the relation (6.6), we have

$$\|V\| = \max_k \sum_{j=0}^{m} \left| \tilde{p}_{m,j}(\lambda; x_k) - \mu \int_0^{x_k} \kappa(x_k, \xi) \tilde{p}_{m,j}(\lambda; \xi) d\xi \right|$$

$$
\begin{aligned}
&= \max_k \left| 1 - \mu \int_0^{x_k} \kappa(x_k, \xi) d\xi \right| \\
&\leq 1 + \eta_1. \qquad \text{(using the triangle inequality)}
\end{aligned}
$$

Next, we calculate the upper bound for $\|V^{-1}\|$. We have the assumption $\|V - I\| = \eta_2 < 1$, and hence with the aid of the geometric series theorem, we have

$$\|V^{-1}\| = \|(I + (V - I))^{-1}\| \leq \frac{1}{1 - \|V - I\|} = \frac{1}{1 - \eta_2}.$$

Therefore,

$$cond(V) = \|V\|\|V^{-1}\| \leq \frac{1 + \eta_1}{1 - \eta_2}.$$

This completes the proof. □

6.3.2 Error estimation 1stVIEs

The following result is true for the 1stVIEs.

Theorem 6.4. *Suppose that $k(x, \xi)$ is continuous on $[0, 1] \times [0, 1]$ and the solution of (6.2) belong to $C^a[0, 1]$ for some $a > 2$. If the matrix F calculated in (6.9) is invertible, then*

$$\sup_{x_k \in [0,1]} |\Psi(x_k) - P_{m,\lambda}(\Psi_{m,\lambda}; x_k)| \leq \frac{1}{8m} \|\Psi''\| \Big[1 + \alpha \|F^{-1}\| \Big],$$

where $\alpha = \sup_{x,\xi \in [0,1]} |\mu||\kappa(x, \xi)|$, $x_k = k/m$, $k = 0, 1, ..., m$ and $P_{m,\lambda}(\Psi_{m,\lambda}; x)$ is the proposed approximate solution.

Proof. First, by applying the triangle inequality, we obtain

$$\begin{aligned}
&\sup_{x_k\in[0,1]} |\Psi(x_k) - P_{m,\lambda}(\Psi_{m,\lambda}; x_k)| \\
&= \sup_{x_k\in[0,1]} |\Psi(x_k) - P_{m,\lambda}(\Psi; x_k) + P_{m,\lambda}(\Psi; x_k) - P_{m,\lambda}(\Psi_{m,\lambda}; x_k)| \\
&\leq \sup_{x_k\in[0,1]} |\Psi(x_k) - P_{m,\lambda}(\Psi; x_k)| + \sup_{x_k\in[0,1]} |P_{m,\lambda}(\Psi; x_k) - P_{m,\lambda}(\Psi_{m,\lambda}; x_k)| . \qquad (6.15)
\end{aligned}$$

From (6.3), it is clear that

$$\sup_{x_k\in[0,1]} |\Psi_{m,\lambda}(x_k) - P_{m,\lambda}(\Psi; x_k)| \leq \frac{\|\Psi''\|}{8m}. \qquad (6.16)$$

So, it is only required to determine a bound for $\sup_{x_k\in[0,1]} |P_{m,\lambda}(\Psi; x_k) - P_{m,\lambda}(\Psi_{m,\lambda}; x_k)|$. By replacing $\Psi(x)$ with $P_{m,\lambda}(\Psi; x)$ in equation (6.2), we get a new function in place of $\Phi(x)$, denoted as $\hat{\Phi}(x)$, i.e.,

$$\hat{\Phi}(x) = \mu \int_0^{x_k} \kappa(x, \xi) P_{m,\lambda}(\Psi; \xi) d\xi. \qquad (6.17)$$

Now, from (6.2) and (6.17), we obtain

$$\Phi(x) - \hat{\Phi}(x) = \mu \int_0^{x_k} \kappa(x, \xi) \left(\Psi(\xi) - P_{m,\lambda}(\Psi; \xi)\right) d\xi.$$

Taking supremum on both sides of the above equation over $t_k \in [0, 1]$, we obtain the following with the help of the relation (6.3):

$$\begin{aligned}
\sup_{x_k\in[0,1]} |\Phi(x_k) - \hat{\Phi}(x_k)| &= \sup_{x_k\in[0,1]} \left| \mu \int_0^{x_k} \kappa(x_k, \xi) \left(\Psi(\xi) - P_{m,\lambda}(\Psi; \xi)\right) d\xi \right| \\
&\leq \sup_{x_k\in[0,1]} \mu \int_0^{x_k} |\kappa(x_k, \xi)| \left|\Psi(\xi) - P_{m,\lambda}(\Psi; \xi)\right| d\xi \\
&\leq \frac{\alpha}{8m} \|\Psi''\|. \qquad (6.18)
\end{aligned}$$

For $x_k \in [0,1]$, the relations (6.2) and (6.17) yield

$$P_{m,\lambda}(\Psi; x_k) - P_{m,\lambda}(\Psi_{m,\lambda}; x_k) = F^{-1}[\Phi(x_k) - \hat{\Psi}(x_k)].$$

Then, we use (6.18) to deduce that

$$\begin{aligned}\sup_{x_k\in[0,1]} |P_{m,\lambda}(\Psi; x_k) - P_{m,\lambda}(\Psi_{m,\lambda}; x_k)| \leq & \|F^{-1}\| \sup_{t_k\in[0,1]} |[\Phi(x_k) - \hat{\Phi}(x_k)]| \\ \leq & \frac{\alpha}{8m}\|F^{-1}\|\|\Psi''\|, \end{aligned} \tag{6.19}$$

which when combined with equations (6.15) and (6.16), result in the desired assertion. □

Lemma 6.5. *In the previous theorem, assume that* $\|F - I\| = \beta_2 < 1$, *then*

$$\|F^{-1}\| \leq \frac{1}{1-\beta_2} \quad \text{and} \quad cond(F) \leq \frac{\beta_1}{1-\beta_2},$$

where $\beta_1 = \max_k |\mu \int_0^{x_k} \kappa(x_k, \xi) d\xi|$.

Proof. With the help of the relation (6.9), we have

$$\begin{aligned}\|F\| = & \max_k \sum_{j=0}^{m} \left| \mu \int_0^{x_k} \kappa(x_k, \xi)\tilde{p}_{m,j}(\lambda; \xi) d\xi \right| \\ \leq & \beta_1. \qquad \text{(using the triangle inequality)}\end{aligned}$$

Next, we calculate the upper bound for $\|F^{-1}\|$. We have the assumption $\|F - I\| = \beta_2 < 1$, and hence with the aid of the geometric series theorem, we have

$$\|F^{-1}\| = \|(I + (F - I))^{-1}\| \leq \frac{1}{1 - \|F - I\|} = \frac{1}{1-\beta_2}.$$

Therefore,

$$cond(F) = \|F\|\|F^{-1}\| \leq \frac{\beta_1}{1-\beta_2}.$$

This completes the proof. □

6.4 Numerical Illustrations

In this section, we demonstrate the efficiency of our proposed technique and the modeling flexibility of the shape parameter λ present in the Bézier bases (2.21). We present some numerical examples with their root mean square error (RMS-error). For $x_k = k/m,\ k = 0, 1, ..., m$, we define the root mean square error as follows

$$\|e_{m,\lambda}\|' = \left(\int_0^1 e_{m,\lambda}^2(x)dx\right)^{\frac{1}{2}} \approx \left(\frac{1}{m}\sum_{k=0}^{m} e_{m,\lambda}^2(x_k)\right)^{\frac{1}{2}},$$

where, $e_{m,\lambda}(x) = \Psi(x) - P_{m,\lambda}(\Psi_{m,\lambda}; x)$. All the calculations and graphs have been done using MATLAB 2024a. For all the examples below, we have considered $\delta = 0.01$.

6.4.1 Test problems

In this part, we consider three test problems to demonstrate the convergence of our proposed technique for different values of λ. The first example addresses second-order Volterra integral equations (2ndVIEs), while the second example focuses on first-order Volterra integral equations (1stVIEs).

Example 6.6. Consider the 2ndVIE:

$$\Psi(x) = 1 + \int_0^x (\xi - x)\Psi(\xi)d\xi, \quad x \in [0, 1].$$

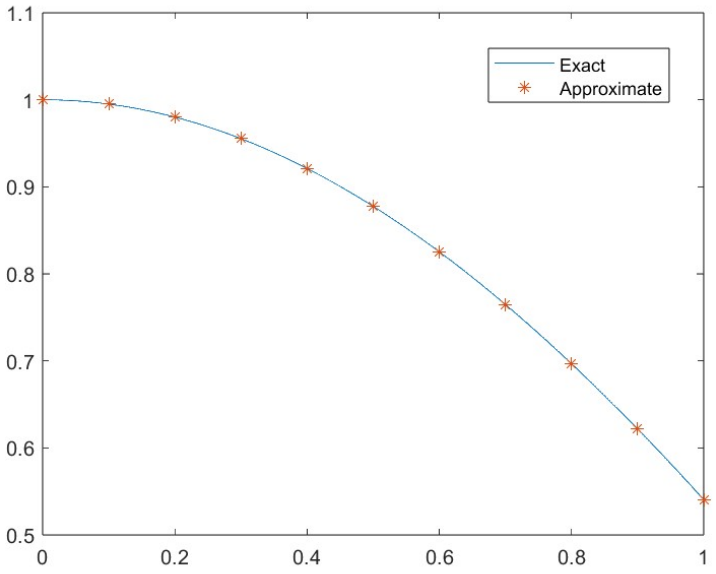

FIGURE 6.1: Result for Example 6.6 with $m = 10$ and $\lambda = 1$

The exact solution of the above equation is $\Psi(x) = cos(x)$. Table 6.1 provides the error for the grid points $m = 6, 7, 8, 9, 10$ at $\lambda = -0.5, 0.5$ and 1. Figure 6.1 shows the comparison between the approximate and exact solutions for $m = 10$ and $\lambda = 1$.

m	$\lambda = -0.5$	$\lambda = 0.5$	$\lambda = 1$
6	1.8902E-06	3.7340E-07	6.3444E-08
7	8.2921E-07	1.5396E-07	6.0814E-09
8	1.6763E-07	2.5879E-08	1.2887E-10
9	6.9704E-08	1.0362E-08	9.3610E-12
10	1.6434E-08	2.1661E-09	1.6908E-13

TABLE 6.1: Error calculations for Example 6.6

Example 6.7. Consider the 1stVIE:

$$sin(x) = \int_0^x e^{x-\xi}\Psi(\xi)d\xi, \quad x \in [0, 1].$$

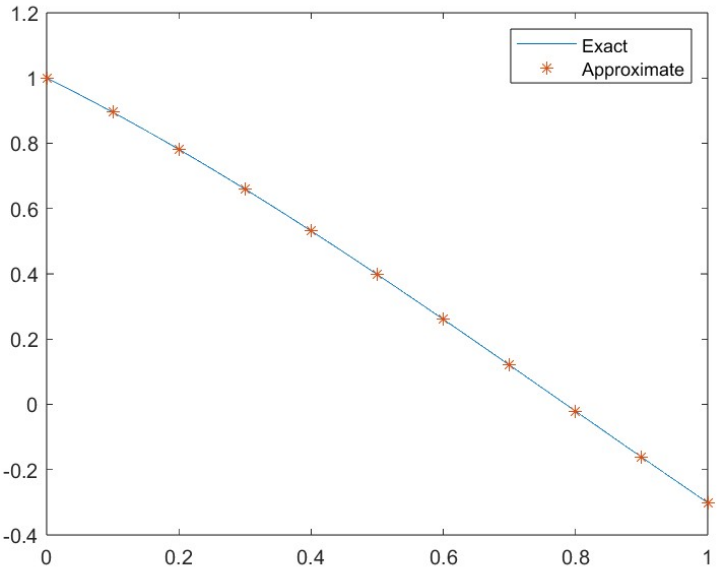

FIGURE 6.2: Result for Example 6.7 with $m = 10$ and $\lambda = 1$

The precise solution to the above equation is $\Psi(t) = cos(t) - sin(t)$. Table 6.2 provides the error for the grid points $m = 6, 7, 8, 9, 10$ at $\lambda = -0.5, 0.1$ and 1. Figure 6.2 illustrates the approximate and exact solution for $m = 10$ and $\lambda = 1$.

m	$\lambda = -0.5$	$\lambda = 0.1$	$\lambda = 1$
6	5.1529E-05	4.7215E-06	1.7781E-06
7	2.8688E-06	2.4817E-07	2.0523E-08
8	3.3875E-06	2.6388E-07	5.1469E-08
9	9.9890E-08	7.7068E-07	6.3290E-07
10	2.6243E-05	7.3777E-06	9.5651E-06

TABLE 6.2: Error calculations for Example 6.7

6.4.2 Comparision with Bernstein approximation technique

In this section, we present examples demonstrating that the approximation using operators (2.22) exhibits lower error compared to using the conventional Bernstein operators for certain parameter choices of λ.

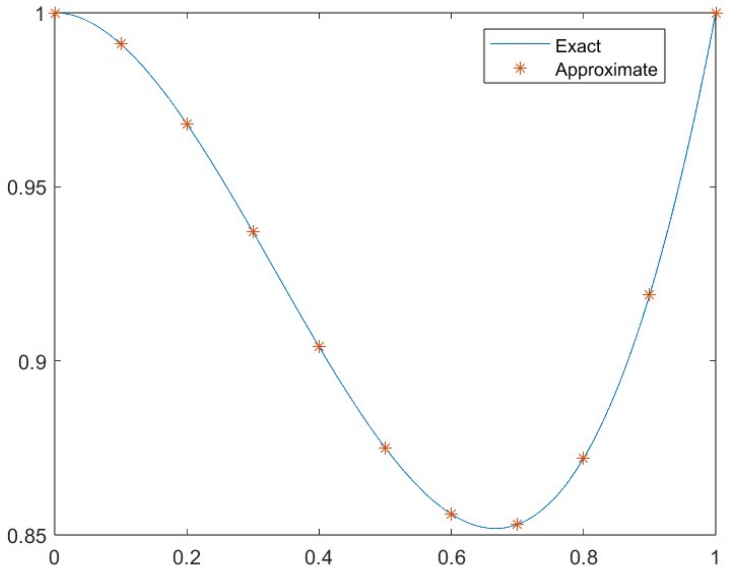

FIGURE 6.3: Result for Example 6.8 with $m = 10$ and $\lambda = 1$

Example 6.8. Consider the following 1stVIE (an Abel's Integral equation):

$$\frac{2}{105}\sqrt{x}(105 - 56x^2 + 48x^3) = \int_0^x \frac{1}{\sqrt{x-\xi}}\Psi(\xi)d\xi, \quad x \in [0,1].$$

The exact solution of the above equation is $\Psi(x) = x^3 - x^2 + 1$. We can observe that the error is less for $\lambda = 1$ than $\lambda = 0$ for $m = 10, 11, 12$, which is shown in Table 6.3. Figure 6.3 illustrates the approximate and exact solution for $m = 10$ and $\lambda = 1$

m	$\lambda = 0$	$\lambda = 1$
9	2.8683E-14	5.6252E-14
10	1.7337E-13	8.9593E-14
11	1.6441E-13	1.1106E-13
12	6.6560E-13	1.9652E-13

TABLE 6.3: Error calculations for Example 6.8

Example 6.9. Consider the following 2ndVIE:

$$\Psi(x) = 1 + x^2 + \int_0^x \frac{1+x^2}{1+\xi^2}\Psi(\xi)d\xi, \quad x \in [0,1].$$

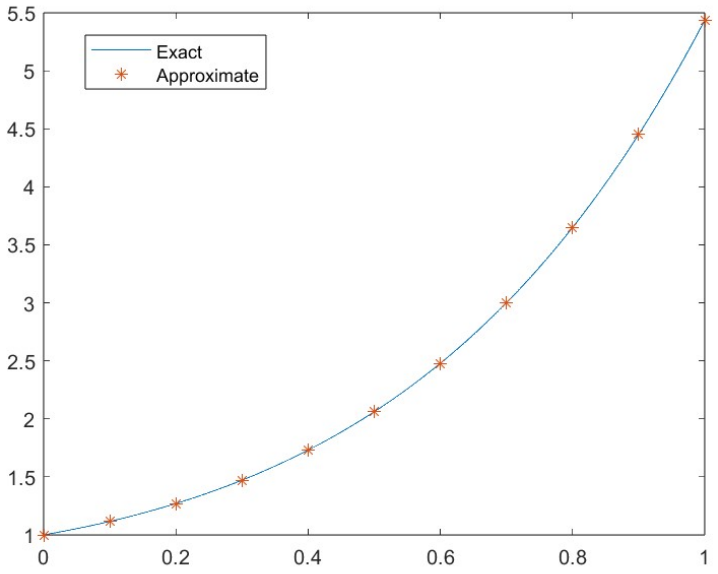

FIGURE 6.4: Result for Example 6.9 with $m = 10$ and $\lambda = 0.00001$

The exact solution is $\Psi(x) = e^x(1 + x^2)$. We can observe that the error is less for $\lambda = 0.00001$ than $\lambda = 0$ for $m = 7, 8, 9, 10$, which is shown in Table 6.4. Figure 6.4 illustrates the approximate and exact solution for $m = 10$ and $\lambda = 0.00001$.

m	$\lambda = 0$	$\lambda = 0.00001$
7	7.9991E-08	7.9872E-08
8	5.5912E-09	5.5482E-09
9	1.9265E-10	1.8425E-10
10	9.2233E-12	5.5276E-12

TABLE 6.4: Error calculations for Example 6.9

Example 6.10. Consider the following weakly singular VIE associated with some heat conduction problems [53]:

$$\Psi(x) = \Phi(x) + \int_0^x \frac{\xi^{\nu-1}}{x^\nu}\Psi(\xi)d\xi, \quad x \in [0, 1].$$

Suppose that $\nu = 1.5$ and Φ be such that the exact solution is $\Psi(x) = x^{6.5} + x^5$. After calculation, from [52], we can deduce that $\Phi(x) = \frac{7}{8}x^{6.5} + \frac{5.5}{6.5}x^5$. We can clearly

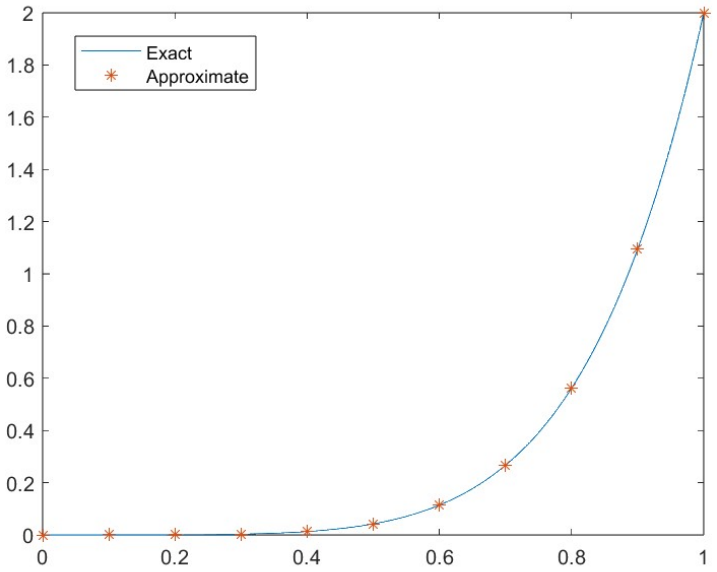

FIGURE 6.5: Result for Example 6.10 with $m = 10$ and $\lambda = 1$

observe that the error is less for $\lambda = 1$ than that of $\lambda = 0$ for $m = 8, 9, 10, 11, 12$, which is shown in Table 6.5. Figure 6.5 illustrates the approximate and exact solution for $m = 10$ and $\lambda = 1$.

m	$\lambda = 0$	$\lambda = 1$
8	5.6139E-07	3.8856E-07
9	1.2806E-07	9.1323E-08
10	3.8839E-08	3.4569E-08
11	1.3983E-08	1.3219E-08
12	5.7658E-09	5.5828E-09

TABLE 6.5: Error calculations for Example 6.10

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