MATHEMATICAL MODELING ON THERMOELASTICITY AND PORO-THERMOELASTICITY

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Doctor of Philosophy

by

Komal Jangid

DEPARTMENT OF MATHEMATICAL SCIENCES
INDIAN INSTITUTE OF TECHNOLOGY
(BANARAS HINDU UNIVERSITY)
VARANASI - 221005
INDIA

Roll No. 18121513

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CHAPTER 6

SUMMARY OF THE THESIS AND FUTURE SCOPE

6.1 Summary

The lack of any elasticity term in the heat conduction equation under uncoupled thermoelasticity appears implausible because the strain due to mechanical stress of an elastic body produces change in the temperature field. The classical coupled theory of thermoelasticity was developed with the purpose of resolving this drawback; however, this theory predicts thermal waves to propagate with an infinite speed. Although this behavior adequately explains thermoelastic systems with low heat flux or long time intervals, but it does not justify the situations that involve high heat flux or fast transient loadings in a body. Fast transient processes are used in a wide range of fields, such as building design, aircraft engines, nuclear engineering, acoustic engineering, chemical engineering, and aircraft engines. Therefore, with the development of technology and intensive research done both in mathematical and physical aspects, the area of thermoelasticity has made tremendous advancements over the past few decades. The present thesis is concerned with the theoretical and numerical investigations on some recently developed generalized thermoelasticity theories. Detailed history about the subject and relevant literature review is presented in Chapter 1 of the thesis. The main work conducted under this thesis can be categorized into three main parts, each of which explores different aspects of generalized thermoelasticity theories by solving some unsolved problems. The key aspects of the current study are outlined in the following three parts:

I. On Moore-Gibson-Thompson thermoelasticity theory:

Chapter 2 and **Chapter 3** make up the first part of the thesis, which analyzes the Moore-Gibson-Thompson (MGT) thermoelasticity theory (Quintanilla (2019)) by considering various thermoelastic problems. **Subchapter 2.1** and **Subchapter 2.2** derive the domain of influence theorems for two different types of thermoelastic processes under this newly developed theory. In Subchapter 2.1, the domain of influence theorem is derived in the context of potential and temperature, whereas Subchapter 2.2 deals with the problem of natural stress-heat flux problem and establishes the domain of influence theorem in terms of stress and heat-flux. It has been mathematically demonstrated that when the condition $K^*\tau_q < K$ holds, the disturbance generated by any thermoelastic loading vanishes outside of a bounded domain. Therefore, the hyperbolicity of the MGT model is proved by this theorem for both the cases. Moreover, the propagation speed is shown to be dependent on the phase lag parameter τ_q, K, K^*, and some other thermoelastic parameters. It is also noted that the propagation speed of thermoelastic disturbance is infinite when the phase-lag parameter τ_q becomes zero which implies the classical case of thermoelasticity. Thus, the results of the present theorem can be used as special cases to figure out the results of the LS theory and the classical thermoelasticity theory.

Following the acquisition of above theoretical findings, **Chapter 3** proceeds to further explore the MGT model by investigating two problems on thermoelastc interactions. **Subchapter 3.1** attempts to examine harmonic plane wave propagation under the MGT thermoelasticity theory for an isotropic, unbounded, and homogeneous thermoelastic medium. In order to conduct a comparative analysis between the outcomes of plane waves under the MGT model and the outcomes predicted by other established theories, the investigation focuses on the unified governing equations of four distinct models: MGT, LS, GN-III, and Biot models. The dispersion relation equation is derived and its solution for the longitudinal plane wave propagation is obtained analytically. The asymptotic expressions for a number of significant wave field properties, including phase velocity, specific loss, and penetration depth for elastic and thermal mode waves, are then obtained for the limiting instances of very high and low frequency values. Applying Whitham's stability criterion it reveals that plane waves propagating under the MGT model are stable. It has been found that all four models exhibit a similar pattern of phase velocity for elastic mode wave in the case of high-frequency. On the other hand, an infinite velocity for the thermal mode wave is predicted in the cases of the Biot and GN-III models, whereas this wave field under the MGT model approaches its limiting value after attaining a local minimum at $\omega \approx 0.3$, like the case of LS model. The behavior of the specific loss for the elastic mode wave under the MGT model has similarities to that of the GN-III model. On the contrary, the specific loss of the thermal mode wave decreases and tends to a finite limiting value after attaining a maximum value. This limited value under the MGT model matches the limited value in the LS model. Further, penetration depth for elastic mode wave is observed to have similar nature under all four models. However, as frequency increases, the limiting values of penetration depth for thermal mode wave are observed to be different under MGT and LS models as compared to the Biot and GN-III models. Therefore, it can be inferred that the influence of the material parameter K^* on wave fields is notable within a small range of frequency values, while the impact of the thermal relaxation parameter on wave components becomes more prominent as the frequency increases. Also similar to the situation with the LS model, the MGT model forecasts the realistic nature of wave propagation in the medium and admits finite speeds of elastic wave as well as of thermal wave.

Subchapter 3.2 aims at analyzing the thermo-mechanical responses on biological tissue during laser irradiation by using the Legendre wavelet method. This subchapter starts the discussion about basic governing equations in the context of MGT model of bioheat transfer considering temperature-dependent material properties. The governing equations contain nonlinear terms because of the variable material properties; hence, the Kirchhoff transformation is utilized in order to linearize the governing equations. Further, the discretization of the time domain and space domain is accomplished by the use of the finite difference approach and the Legendre wavelet method, respectively. To show the efficiency of the proposed method, pointwise convergence studies have been carried out. It is further shown that the current numerical procedure is accurate and effective. The main advantage of the present numerical method is that it yields satisfactory results while making use of a less complex framework and requiring fewer collocation points. The impacts of variable thermal conductivity and relaxation parameters on all the field variables are studied. It is seen that the temperature dependency parameter has a large influence on displacement and stress as compared to the temperature field. Similar to the case of variable thermal conductivity, the relaxation time parameter influences all the field variables. The findings indicate that the outcomes of the MGT model agree with those of the LS model. The fact that all of the field variables only display non-zero values within a restricted region and vanish beyond the location of tumor-tissue interface which is the evidence of the finite speed of thermoelastic wave propagation in the context of the present thermoelastic model.

II. Study on modified temperature-rate dependent two-temperature thermoelasticity:

Chapter 4 is included in the second part of the thesis and it focuses on the temperature-rate dependent two-temperature thermoelasticity theory proposed by Shivay and Mukhopadhyay (2019). Youssef (2006) presented the governing equations for the temperature-rate dependent two-temperature (TRDTT) thermoelasticity theory. In their study, Shivay and Mukhopadhyay (2019) utilized the principles of generalized thermodynamics and proposed a modified theory of thermoelasticity known as MTRDTT. The generalized two-temperature relation obtained in view of this modified TRDTT thermoelasticity theory reduces to the two-temperature relation provided by Youssef in a specific situation. **Subchapter 4.1** begins with the discussion about the considered thermoelasticity for an anisotropic and homogeneous medium. In this subchapter, some important theorems on this new thermoelasticity theory are established. Firstly, an alternative formulation of the mixed initial-boundary value problem is presented. In this formulation, initial conditions are incorporated into the field equations by using convolution. With the use of this theorem, a convolution-type variational theorem is proved. Later, a reciprocal principle for this theory is derived. A key role played by these theorems is in solving problems using the finite element methods and boundary element methods.

Subchapter 4.2 extends the investigation of the MTRDTT theory by solving a different thermoelastic problem. Thermoelastic interactions in a linear, isotropic, and homogeneous unbounded solid in presence of a continuous line heat source are investigated in this subchapter. The unified form of two-temperature relation under the MTRDTT and TRDTT theories is taken in order to compare the results of both theories. To solve the problem, Laplace transform and Hankel transform techniques are applied. The analytical solutions in the space-time domain are then derived by employing the short-time approximation and the inverse Laplace transform. The problem has been further illustrated with numerical findings that were carried out for copper material in order to highlight the variations of the field variables for both the models. From the solutions, it is seen that the temperature-rate-dependent two-temperature models do not admit the finite wave speed of the thermal wave. A discontinuity can be seen in all of the field variables at the elastic wavefront, with the sole exception of the conductive temperature, which remains continuous in the contexts of both the models. It is observed that as the value of two-temperature parameter increases, there is a corresponding increase in the values of all the field variables at the wavefront position, as predicted by the MTRDTT model. Differences between the MTRDTT and TRDTT models are observed for the displacement, thermodynamic temperature, and circumferential stress fields that decrease with increasing value of the two-temperature parameter. Therefore, the two-temperature parameter is shown to exert a significant influence on all the field variables.

III. On generalized poro-thermoelasticity theory:

Chapter 5 illustrates the last part of the thesis which is devoted to the investigation of some problems under generalized poro-thermoelasticity theory. In the framework of Lord-Shulman (LS) thermoelasticity theory, Youssef (2007) developed the governing equations of a thermoelastic porous medium. This theory is also known as LS poro-thermoelasticity theory. **Subchapter 5.1** establishes some theoretical results based on this poro-thermoelasticity theory. After introducing all the basic governing equations and the constitutive relations for an isotropic and homogeneous porous material, an alternative formulation of the mixed initial-boundary value problem is presented for this theory. Then, the convolution-type variational principle is established taking into account of this alternative formulation. This particular type of variational principle has the benefit of recovering all the governing equations as well as the appropriate initial conditions and boundary conditions. Finally, a continuous dependence result for solutions is established, which is shown to be dependent on both the initial data and the supply terms (heat source and body force).

In **Subchapter 5.2**, thermoelastic interactions inside a fluid-saturated porous medium are investigated. A problem of fluid-saturated porous layer with one-dimensional disturbances propagating along the x-axis is considered. To solve this coupled poro-thermoelastic problem, a numerical method based on the Legendre wavelet is proposed. The material properties that depend on temperature are taken into account in order to achieve a more accurate thermoelastic response. In the solution methodology, the time domain is first discretized by using a finite-difference approach. Subsequently, the Legendre wavelet method is employed to approximate the space domain. By utilizing the collocation points, a system of algebraic equations is constructed and solved. The field variables of solid and fluid phases for the present problem are obtained and displayed graphically. It is believed that the present method can be utilized as an alternative numerical technique for solving any coupled thermoelastic and porothermoelastic problem satisfactorily. The observations mark the significant effects of temperature-dependent thermal conductivity, thermal relaxation parameters, and porosity on all the field variables. As can be seen, ignoring the temperature-dependent thermal conductivity will lead to higher temperatures and lower stresses. The absolute values of all the field variables are also seen to increase when the value of porosity rises. It is noteworthy to notice that the effective region of influence for the classical porothermoelasticity theory is observed to be greater due to the infinite speed behavior of classical theory as opposed to the present theory which admits finite speed of heat signals.

6.2 Future Scope

The theory of thermoelasticity finds diverse applications in the fields of engineering, science, and also in medical treatment. The relevance of the topic is demonstrated by the gradual evolution and application of thermoelasticity theories to the numerous thermoelastic problems found in the literature. The accurate assessment of temperature perception and stress distribution inside biological tissues is of utmost importance in ensuring high-quality clinical care. The utilization of a non-Fourier heat transfer model is crucial in order to accurately forecast the thermal history of tissue during laser-assisted thermal therapy. Moreover, in the case of a tissue occupying a significantly small domain, it may be suitable to employ a one-dimensional study in order to understand the thermo-biological reaction when subjected to external heating. However, the nonhomogeneous and non-uniform structure of the human body necessitates the use of two-dimensional analysis in order to accurately forecast temperature scenarios in living tissues. Therefore, it is worth studying the two-dimensional and higher dimensional problem in this direction to achieve more accuracy in clinical care. Limited work is reported in the literature in this direction. Hence, there is scope to analyze thermal and mechanical behavior during thermal treatment using two and higher-dimensional generalized thermoelasticity models of bioheat transfer. Numerical techniques such as finite element method, finite difference method, and other numerical methods are often utilized to deal with the coupled thermo-mechanical problems as these coupled thermoelastic issues lack an analytical closed-form solution. In the present thesis, a numerical approach based on the Legendre wavelet collocation method is used to solve some coupled thermoelasticity and porothermoelasticity problems. The development of numerical methods to address the coupled dynamical thermomechanics problems that produce reliable and effective solutions is required. Therefore, a further attempt is worth to be pursued to create more efficient and accurate numerical approaches as well as to improve the existing numerical tools to solve coupled thermo-mechanical problems. Thus, the current thesis is concluded by considering the aforementioned points for further research.