CHAPTER - 1

INTRODUCTION

Chapter 1

Introduction

1.1 Review of Wavelets

Mathematics is subdivided in many specialized areas, most of which have attained such a technical sophistication that they become rather inaccessible for the non-specialist. Sometimes, but not so often, a new concept breaks through which reverts this tendency, because it uses techniques from, and is relevant for many different areas of mathematics and has, moreover, many applications of very different nature. An example of such a concept is given by wavelets. Truly, the range of pure and applied fields touched by wavelets is extremely wide. They are widely used in Fourier analysis, approximation theory, functional analysis, operator theory, group representation theory, theory of fractals, numerical analysis, signal analysis, image compression, computer science, electrical engineering, physics etc.

The concept of wavelet analysis has been in place in one form or another since the beginning of the twentieth century. The Littlewool-Paley technique and Calderon-Zygmund theory in Harmonic analysis and digital filter bank theory in signal processing may be considered as fore runners of wavelet analysis. However, in its present form, wavelets attracted attention in the 1980s through the work of several researchers from various disciplines [Stromberg (1982), Morlet (1982, 1983), Grossmann (1984), Meyer (1985, 1986, 1990, 1992, 1992 b), Battle (1987, 1988, 1989, 1992), Lemarie (1988, 1990, 1991), Coifman (1991), Daubechies (1988, 1988 b, 1990, 1990 b), Mallat (1989, 1989 b, 1989 c, 1991), Chui (1992, 1992 b, 1992 c)] to name a few.

The name wavelet was coined approximately 25 years ago (Morlet *et al.* (1982, 1983), Grossmann and Morlet (1984)); in the last twenty years interest in them has grown at an explosive rate. There are several reasons for their present success. On the other hand, the concept of wavelets can be viewed as a synthesis of ideas which originated during the last thirty or forty years in engineering (sub band coding), physics (coherent states, renormalization group), and pure mathematics (study of Calderon-Zygmund operators). As a consequence of these interdisciplinary origins, wavelets appeal to scientists and engineers of many different backgrounds. On the other hand, wavelets are a fairly simple mathematical tool with a great variety of possible applications. Already they have led to exciting applications in signal analysis (sound, images) (some references are Kronland-Martinet, Morlet and Grossmann (1987), Mallat (1989b, 1989c) and numerical analysis(fast algorithms for integral transforms in Beylkin, Coifman, and Rokhlin (1991)); many other applications are being studied. This wide applicability also contributes to the interest they generate.

The wavelet analysis is useful for problems in many applied disciplines as well as within mathematics itself tells us that there is something special about it: Wavelet analysis provides a systematic new way to represent and analyze multiscale structures. The prevalence of multiscale structures in nature and in engineering is one reason that wavelets are broadly valuable. Wavelet analysis is also a far-reaching generalization of orthogonal bases of functions whose particular new contribution is a systematic way to represent functions on unbounded domains by linear combinations of orthogonal basis functions that are compactly supported and overlapped. This specific virtue of the orthogonal wavelet basis is the reason behind the success of the various algorithms. These are the kinds of the basis functions that are potentially realizable by physical devices.

1.2 The Legendre's Wavelets

Wavelets are a class of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation and translation parameters a and b vary continuously, the following family of continuous wavelets are obtained

$$\psi_{ab}(t) = |a|^{-1/2}\psi\left(\frac{t-b}{a}\right) \quad , \quad a,b \in \mathbb{R},\ a \neq 0 .$$

When the parameters a and b are restricted to discrete values as $a = 2^{-k}, b = n2^{-k}$, then, we have the following family of discrete wavelets

$$\psi_{kn}(t) = 2^{k/2}\psi(2^k t - n), \quad k,n \in Z,$$

where the function ψ, the mother wavelet, satisfies $\int_R \psi(t)dt = 0$.

We are interested in the case where ψ_{kn} constitutes an orthonormal basis of L^2(R). A systematic way to do this is by means of multiresolution analysis (MRA).

We define the Legendre wavelets ψ_{nm} as follows

$$\psi_{nm}(t) = \begin{cases} \sqrt{2m+1}\, 2^{k/2}\, P_m\left(2^k t - \hat{n}\right), \ for\ \dfrac{\hat{n}-1}{2^n} \leq t < \dfrac{\hat{n}+1}{2^n}, \\ 0, \quad otherwise, \end{cases} \tag{1.2.1}$$

where $P_m(t)$ are the well-known Legendre polynomials of order m and defined on $[-1, 1]$ by the following recurrence relation:

$$P_0(t) = 1, \quad P_1(t) = t \quad and \quad P_{m+1}(t) = \frac{2m+1}{m+1} t\, P_m(t) - \frac{m}{m+1} P_{m-1}(t), \quad m = 1, 2, 3, \ldots .$$

The Legendre wavelets $\psi_{nm} = \psi(k, \hat{n}, m, t)$ have four arguments; $n = 1, 2, 3, \ldots, 2^{k-1}, k = 1,2,3,\ldots$, m is the order of Legendre polynomials and

t is the normalized time. The family $\{\psi_{nm}\}_{n,m}$ forms an orthonormal basis for $L^2[0,1]$.

1.3 The Legendre's Scaling Functions

Legendre scaling functions $\{\phi_i(t)\}$ are defined by

$$\phi_i(t) = \begin{cases} \sqrt{(2i+1)}P_i(2t-1), & \text{for } 0 \le t < 1 \\ 0, & \textit{otherwise}, \end{cases} \tag{1.3.1}$$

where $P_i(t)$ is Legendre polynomials of order i which are orthogonal on the interval $[-1,1]$ with respect to the weight function $w(t)=1$. Legendre polynomials satisfy the following relation

$$P_0(t)=1, \qquad P_1(t)=t,$$

$$P_{i+1}(t) = \left(\frac{2i+1}{i+1}\right)tP_i(t) - \left(\frac{i}{i+1}\right)P_{i-1}(t) \qquad i = 1,2,3,.........$$

The Legendre scaling function of degree i is given by

$$\phi_i(t) = (2i+1)^{\frac{1}{2}} \sum_{k=0}^{i} (-1)^{i+k} \frac{(i+k)!}{(i-k)!} \frac{t^k}{(k!)^2} \tag{1.3.2}$$

1.4 The Bernstein Polynomials

The Bernstein polynomial, named after Sergei Natanovich Bernstein, is a polynomial in the Bernstein form that is a linear combination of Bernstein basis polynomials.

The Bernstein basis polynomials of degree n are defined by

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i} , \qquad \textit{for } i = 0,1,2, \cdots, n . \tag{1.4.1}$$

There are $(n+1)$ n^{th} degree Bernstein basis polynomials forming a basis for the linear space V_n - consisting of all polynomials of degree less than or equal to n in R[x]-the ring of polynomials over the field R. For mathematical convenience, we usually set $B_{i,n} = 0$ if $i < 0$ or $i > n$.

Any polynomial $B(x)$ in R[x] may be written as

$$B(x) = \sum_{i=0}^{n} \beta_i B_{i,n}(x), \text{ for some } n. \tag{1.4.2}$$

Then $B(x)$ is called a polynomial in Bernstein form or Bernstein polynomial of degree n. The coefficients β_i are called Bernstein or Bezier coefficients. But several mathematicians call Bernstein basis polynomials $B_{i,n}(x)$ as the Bernstein polynomials. We will follow this convention as well. These polynomials have the following properties:

(i) $B_{i,n}(0) = \delta_{i0}$ and $B_{i,n}(1) = \delta_{in}$, where δ is the Kronecker delta function.

(ii) $B_{i,n}(t)$ has one root, each of multiplicity i and $n-i$, at $t = 0$ and $t = 1$, respectively.

(iii) $B_{i,n}(t) \geq 0$ for $t \in [0,1]$ and $B_{i,n}(1-t) = B_{n-i,n}(t)$.

(iv) For $i \neq 0$, $B_{i,n}$ has a unique local maximum in $[0,1]$ at $t = i/n$ and the

maximum value $i^i n^{-n} (n-i)^{n-i} \binom{n}{i}$.

(v) The Bernstein polynomials form a partition of unity i.e. $\sum_{i=0}^{n} B_{i,n}(t) = 1$.

(vi) It has a degree raising property in the sense that any of the lower-degree polynomials (degree $< n$) can be expressed as linear combinations of polynomials of degree n. We have,

$$B_{i,n-1}(t) = \left(\frac{n-i}{n}\right) B_{i,n}(t) + \left(\frac{i+1}{n}\right) B_{i+1,n}(t).$$

(vii) Let $f(x) \in C\,[0,1]-$ the class of continuous functions on $[0,1]$, then

$$B_n(f)(x) \;=\; \sum_{i=0}^{n} f\left(\frac{i}{n}\right) B_{i,n}(x)\,, \tag{1.4.3}$$

converges to $f(x)$ uniformly on [0,1] as $n \to \infty$.

$(viii)$ Let $f(x) \in C^{(k)}[0,1]-$ the class of $k-$ times differentiable function with $f^{(k)}$ continuous, then $\left\| B_n(f)^{(k)} \right\|_\infty \le \frac{(n)_k}{n^k} \left\| f^{(k)} \right\|_\infty$ and $\left\| f^{(k)} - B_n(f)^{(k)} \right\|_\infty \to 0$ as $k \to \infty$, where $\|.\|_\infty$ is the sup. norm and $\frac{(n)_k}{n^k} = \left(1-\frac{0}{n}\right)\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)$ is an eigen value of B_n; the corresponding eigen function is a polynomial of degree k.

1.5 Abel Integral Equations

Although courses in differential equations are standard components of most scientific and engineering curricula, less attention has generally been given to the subject of integral equations. Scientific theories have generally been formulated in differential form, even though such theories are often derived from variational principles, which require the optimization of some integral functional. For the numerical solution of the resulting differential equations this process is frequently reversed, leading to the integral equation reformulation of the given differential equation. Because of this, integral equations have become increasingly important in formulating, analyzing and obtaining numerical solutions to the scientific problems.

Historically, the first integral equation to be studied was analyzed by Abel (1826), which now takes his name, had the form

$$\int_0^x \frac{\phi(t)}{(x-t)^{1/2}}\,dt = f(x), \qquad 0 \le x \le \infty, \tag{1.5.1}$$

where the function $f(x)$ in the right hand side of Eq. (1.5.1), is a known function and Eq. (1.5.1) must be solved for $\phi(t)$. Even though they have a rather specialized structure, Abel equations form an important class of integral equations in applications. This happens because completely independent problems lead to the solution of such equations.

Its importance is then related to how Abel used the relationship between the kinetic and potential energies of falling bodies instead of Newton's law, to determine the path along which a particle must be constrained to fall, under constant vertical acceleration, in order that its time of fall be a prescribed function $f(x)$ of the distance fallen (Lonseth (1977)). Abel not only formulated the integral equation (1.5.1) but also derived (Abel (1826)) the following two inversion formulae for Eq. (1.5.1):

$$\phi(t) = \int_0^t (t-x)^{-1/2} f'(x)dx, \qquad f'(x) = df(x)/dx, \tag{1.5.2}$$

$$\phi(t) = \frac{1}{\pi}\frac{d}{dt}\left\{\int_0^t (t-x)^{-1/2} f(x)dx\right\}. \tag{1.5.3}$$

Since Abel formulated Eq. (1.5.1), many authors have generalized it (Linz (1985)). Because explicit inversion formulae (1.5.2 & 1.5.3) exist for Eq. (1.5.1), it might be thought that obtaining a numerical approximation to ϕ is straight forward, as a cursory glance at Eq. (1.5.2) suggests that a differentiation followed by integration should be stable. The fallacy of this argument is due to the singularity in the integral and it turns out that the inversion is actually (weakly) unstable.

It may therefore be expected that difficulties will arise when seeking the solution of Eq. (1.5.1) and generalizations of it, which are similar to those associated with obtaining estimates of derivatives. As is well known (Anderssen and Bloomfield (1974)), difficulties associated with estimating derivatives can become quite severe when the data is in error. It turns out that similar difficulties arise in the numerical solution of Eq. (1.5.1). Consequently, the problems which Abel integral equations model are improperly posed in the sense that certain small perturbations in the data $f(x)$ can lead to arbitrarily large perturbations in the solution $\phi(t)$.

A further complication arises due to the fact that, in general, f and ϕ are not both smooth. For example, if $\phi(x)=1$ then $f(x)=2\sqrt{x}$ while, on the other hand, if $\phi(x)=\sqrt{x}$ then $f(x)=\pi x/2$. This means that care needs to be exercised when constructing numerical schemes as some of the obvious candidates may have a slow rate of convergence.

Abel integral equations arise in a large number of physically unrelated ways, and that the available information about $f(x)$ varies greatly, depending on the context of the application. Since the analysis of such equations will vary greatly depending on the structure of $f(x)$, the subsequent examination is categorized in terms of the properties of $f(x)$ in the following ways:

(i) Exact Analytic: When $f(x)$ corresponds to a known function of x, then formally in terms of Eqs. (1.5.2) and (1.5.3), we also know $\phi(t)$. This knowledge is explicit when either Eq. (1.5.2) or Eq. (1.5.3) can be evaluated exactly. Consequently, exact solutions of Abel-type integral equations are known for a wide variety of situations and are tabulated as Riemann-Liouville fractional integrals (Erdelyi (1954)).

(ii) Discrete (Numerical): When the inversion formulae (1.5.2)-(1.5.3) cannot be evaluated exactly to determine $\phi(t)$, it will be necessary to work with discretized (numerical) values of $f(x)$ and to resort to numerical (computational) procedures in order to determine an approximation to $\phi(t)$. Because of the inherent improperly posedness of Eq. (1.5.1) (mentioned above), the construction of stable numerical processes for the solution of Eq. (1.5.1) poses additional challenges. Even though some standard numerical methods yield good approximations to $\phi(t)$ when the discretized values of $f(x)$ are evaluated with sufficient accuracy (Brunner (1982)), it is much more difficult, from the point of view of the underlying numerical analysis, to characterize when such numerical methods will work successfully. But discretized (numerical) data are like very accurate observational data, and thus only a mild form of stabilization is necessary to control the inherent improperly posedness and thereby yield good approximations to $\phi(t)$ (Anderssen (1976), Lukas (1980)).

(iii) Nonexact Observational: Here, the only available information about the exact data $f(x)$ is the discrete observational data

$$d_j = f(x_j) + \varepsilon_j, \quad j = 1, 2, \ldots n, \quad a < x_1 < x_2 < \cdots < x_n < b, \tag{1.5.4}$$

where ε_j denote observational (measurement) errors. This typifies the circumstances of a wide variety of applications including the situation where $f(x)$ is a probability distribution function and the only available information is a histogram approximation. Now, the problem is to recover as much information about $f(x)$ and $\phi(t)$ consistent with the information in the available data $d_j, j = 1, 2, \cdots n$. Because of the underlying numerical differentiation which must be performed implicitly or explicitly on the data $d_j, j = 1, 2, \cdots n$ in order to recover information about the solution $\phi(t)$, some form of stabilization must be used. There are different ways in which this can be done including regularization (Lukas (1980), Baev and Glasko (1976)), constrained optimization, linear programming inversion (Anderssen and Gustafson (1984)), and low-dimensional parameterization in the characterization of the approximations. An alternative strategy is to simply limit the information determined and used for inference purposes about the solution $\phi(t)$ to bounded linear functionals

$$L_\theta(\phi) = \int_a^b \theta(t)\phi(t)\,dt, \qquad \theta(t) \equiv \text{known}, \tag{1.5.6}$$

defined on $\phi(t)$ such as the statistical moments of mean, variance, and kurtosis which are defined in terms of linear functionals characterized by $\theta(t) = t,\ t^2,\ t^3$.

Though much of the literature devoted to the numerical solution of Abel-type integral equation is concerned with the discrete (numerical) situation, it is the nonexact observational situation which is most important, from the point of view of applications. Consequently, attention is focused on the both (discrete, nonexact observational) in this thesis.

1.6 Generalized Abel Equation

Standard Abel integral equations refer to

$$\int_0^x \frac{\phi(t)}{(x-t)^\nu}dt = f(x),\ x>0,\ 0<\nu<1, \tag{1.6.1}$$

and

$$\int_x^a \frac{\phi(t)}{(t-x)^\nu}dt = f(x),\ x<a,\ 0<\nu<1, \tag{1.6.2}$$

where $f(x)$ is a known function, and $\phi(t)$ is an unknown function to be determined. For the above two cases, the corresponding exact solutions are (Tricomi (1957))

$$\phi(x) = \frac{\sin(\nu\pi)}{\pi}\frac{d}{dx}\int_0^x \frac{f(t)}{(x-t)^{1-\nu}}dt\,, \tag{1.6.3}$$

and

$$\phi(x) = -\frac{\sin(\nu\pi)}{\pi}\frac{d}{dx}\int_x^a \frac{f(t)}{(t-x)^{1-\nu}}dt\,, \tag{1.6.4}$$

respectively. Nevertheless, both exact solutions fail in practical applications since input function $f(x)$ is given with small random error and the differential operator involved is an ill-posed or unbounded operator. In other words, small noise in the data function $f(x)$ might cause large error in the computed solution, and however an instable solution is undesired. Consequently, to avoid any instability of desired solution, other methods are needed.

A more general form of Abel equation is faced in the study of many branches of science and technology, such as plasma diagnostics and flame studies, where the most common problem of deduction of radial distributions of some important physical quantity from measurement of line-of-sight projected values is encountered

For a cylindrically symmetric, optically thin plasma source, the relation between radial distribution of the emission coefficient and the intensity measured from outside of the radial source is described by Abel transform. The challenging task of reconstruction of emission coefficient from its projection is known as Abel's inversion. The form of Abel integral equation encountered in reconstruction of emission coefficient in the study of cylindrically symmetric, optically thin plasma source is given by:

$$I(y) = 2\int_y^1 \frac{\varepsilon(r)r}{\sqrt{r^2-y^2}}dr, \qquad 0 \le y \le 1\,, \tag{1.6.5}$$

where $\varepsilon(r)$ and $I(y)$ represent, respectively the emissivity and measured intensity, as measured from outside the source (Griem (1963)).

The analytical inversion formula for Eq. (1.6.5) is given as (Tricomi (1957))

$$\varepsilon(r) = -\frac{1}{\pi}\int_r^1 \frac{1}{\sqrt{y^2 - r^2}} \frac{dI(y)}{d(y)} dy, \quad 0 \le r \le 1. \tag{1.6.6}$$

Singh et al. (2009), constructed an operational matrix of integration based on orthonormal Bernstein polynomials and used it to propose a stable algorithm to invert the following form of Abel integral equation

$$I(y) = 2\int_0^y \frac{\varepsilon(r) r}{\sqrt{y^2 - r^2}} dr, \quad 0 \le y \le 1. \tag{1.6.7}$$

In 2010, Singh et al. (2010) constructed yet another operational matrix of integration based on orthonormal Bernstein polynomials and used it to propose an algorithm to invert the Abel integral equation (1.6.5).

In 2008, Chakrabarti (2008) employed a direct function theoretic method to determine the closed form solution of the following generalized Abel integral equation

$$I(y) = a(y)\int_\alpha^y \frac{r^{\mu-1}\varepsilon(r)}{(y^\mu - r^\mu)^\gamma} dr + b(y)\int_y^\beta \frac{r^{\mu-1}\varepsilon(r)}{(r^\mu - y^\mu)^\gamma} dr, \tag{1.6.8}$$

$$\alpha \le y \le \beta, \; 0 < \gamma < 1,$$

where the coefficients $a(y)$ and $b(y)$ do not vanish simultaneously. But the numerical inversion is still needed for its application in physical models since the experimental data for the intensity $I(y)$ is available only at a discrete set of points and it may also be distorted by the noise.

1.7 Review of Operational Matrices

Approximations by orthonormal family of functions have played a vital role in the development of physical sciences, engineering and technology in general and mathematical analysis in particular since long. In the last three decades, they have been playing an important role in the evaluation of new techniques to solve problems such as identification, analysis and optimal control. The aim of these techniques has been to obtain effective algorithms that are suitable for the digital computers. Their major effort has been concentrated on the methods of the orthogonal functions.

The motivation and philosophy behind the operational matrix approach is that it transforms the underlying partial differential equation of the problem to an algebraic equation, thus simplifying the solution process of the problem to a great extent. The basic idea of this technique is as follows:

- The unknown function u(x, t) or its derivatives with respect to time in the given partial differential equation are approximated by linear combinations of the orthonormal basis functions and truncating them at optimal levels.

- Finally, substituting operational matrices in the equation leads linear algebraic equations whose solutions can be obtained using Sylvester's approach; this in turn gives approximate solutions for partial differential equation.

The key idea of the technique depends on the following properties of the basis vector $\phi(t)$:

- $$\underbrace{\int_a^t \cdots \int_a^t}_{k-times} \phi(\sigma)(d\sigma)^k \approx I_{m+1}^{(k)}\phi(t),$$

- $$\frac{d^k\phi(\sigma)}{d\sigma^k} \approx D_{m+1}^{(k)}\phi(t),$$

where $\phi(t)=[\phi_0(t),\phi_1(t),...,\phi_m(t)]^T$ in which the elements $\phi_0(t),\phi_1(t),...,\phi_m(t)$ are the discrete representation of the basis functions, orthogonal on a certain interval $[a,b]$, P_{m+1} and D_{m+1} are the operational matrices for the integration and differentiation of $\phi(t)$ respectively. Note that P_{m+1} and D_{m+1} are constant matrices of order $(m+1)\times(m+1)$.

Using the operational matrix of an orthonormal system of functions to perform integration for solving, identifying and optimizing a linear dynamic system has several advantages:

- The method is computer oriented, thus solving higher order partial differential equation becomes a matter of dimension increasing.
- The solution is convergent, even though the size of increment may be large.

Until now, the operational matrix of integration has been determined for several types of orthogonal basis functions, such as the Walsh function (Chen and Hsiao (1975a, 1975b)), block-pulse function (Chen *et al.* (1977), Sannuti (1977)), Laguerre series (King and Paraskevopoulos (1979), Hwang and Shih (1982a, 1982b), Chebyshev polynomials (Paraskevopoulos (1983), Horng and Chou (1985)), Legendre polynomials (Chang and Wang (1982)), Fourier series (Paraskevopoulos (1985), Razzaghi (1988)) and Bessel series (Paraskevopoulos *et al.* (1990)). Later Gu and Jiang (1996) derived the Haar wavelets operational matrix of integration followed by Razzaghi and Yousefi (2001) who gave the Legendre wavelets operational matrix of integration.

Similarly the operational matrix of differentiations has been determined for several types of orthogonal basis functions, such as Legendre polynomial (Saadatmandi and M. Dehghan (2010)), Jacobi operational matrix (Doha et al. (2012)), Legendre wavelet (Mohammadi and Hosseini (2011)), Chebyshew wavelet (Hosseini and Mohammadi (2011)).

The conventional method of deriving the operational matrix is not only complicated but also time consuming. This is the case with Legendre scaling basis functions which are used as basis functions for approximations in almost chapters of the thesis. We have constructed two operational matrices based on the Legendre scaling basis functions for the numerical solution of fractional singular volterra integro-differentail equation in chapter 2, 3 and 4.

Specially in chapter 2, we have constructed operational matrices for the fractional integration and fractional differentation and used them in proposed alogoritm for numerical evaluations of the singular volterra integro-differential equations of both fraction and integral order.

Chapter 3 of this thesis presents a study of linear and nonlinear generalized Abel integral equations and few operational matrices have been constructed to solve generalized Abel integral equations.

Extending the ambience of applications of operational matrices, we have taken up the class of system of singular volterra integro-differential equations. In Chapter 4, we developed numerical alogorithm to find solution of such system by constructing operational and almost matrices of integration based on wavelets and orthogonal polynomials.

In Chapter 5, again we have constructed some new operational matrices and used them to evolved a method to find solution of intial and boundary value problem for non-homogeneous fractional partial differential equations.

Special kind of operational matrices namely Bernstein's operational matrix of fractional differentiation (BOMFD) is applied to solve nonlinear system of fractional differential equations and fractional stiff system nonlinear differential equations, in chapter 6.

1.8 Fractional Calculus

Many definitions and studies of fractional calculus have been proposed in the last two centuries. These definitions include, Riemann-Liouville, Weyl, Reize, Compos, Caputo and Nashimoto fractional operators. A brief description of the fractional calculus needed for the further development is given below.

Definition 1.8.1. The Riemann-Liouville Fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$, is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt \ , \qquad \alpha > 0, x > 0,$$

$$J^0 f(x) = f(x) . \tag{1.8.1}$$

Properties of the operator J^α can be found in (Gorenflo and Mainardi (1997)), we mention the following:

For $f \in C_\mu^m$, $\alpha, \ \beta > 0$, $\mu > -1$ and $\gamma > -1$,

1. $J^\alpha (x-a)^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(1+\gamma+\alpha)} (x-a)^{\gamma+\alpha}$

2. $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$

3. $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$.

Definition 1.8.2. The fractional derivative of $f(x)$ in Caputo sense is defined as

$$D^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^m(t) \, d\, t \, , \tag{1.8.2}$$

for $m-1 < \alpha \leq m, m \in N, x > 0, f \in C_{-1}^m$.

Also, we need here two basic properties of the Caputo's fractional derivative (Gorenflo and Mainardi (1997)).

Definition 1.8.3. If $m-1<\alpha\leq m,\ m\in N$ and $f\in C_{-1}^{m}$ and $\mu\geq -1$, then

$$D^{\alpha}J^{\alpha}f(x)=f(x),$$

and

$$J^{\alpha}D^{\alpha}f(x)=f(x)-\sum_{k=0}^{m-1}f^{(k)}(0^{+})\frac{(x-a)^{k}}{k!},\ x>0. \qquad (1.8.3)$$

The Caputo fractional derivative is considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem. For mathematical properties of fractional derivatives and integrals one can consult (Gorenflo and Mainardi (1997), Luchko and Gorenflo (1998), Podlubny (1999), Carpinteri and Mainardi (1997)).