Chapter 3

Generalized Hukuhara Weak Subdifferential and Its Application on Identifying Optimality Conditions for Nonsmooth Interval-Valued Functions

3.1 Introduction

Introduced by Azimov and Gasimov [9], the notion of weak subdifferential is a generalization of the classical subdifferential. It is especially useful when subdifferential is incapable of characterizing the optimality conditions for nonconvex and nonsmooth optimization problems. Its main advantage is that it uses a conic surface, instead of hyperplane, to support the graph of the nonconvex function at a given point [106–108,110], and, thus, does not need any convexity assumption. In terms of application of weak subdifferential, a strong duality theorem for nonconvex inequality-constrained problems have been shown by defining a weak conjugate function [189]. [164] has applied this notion in duality theory in the form of a weak subdifferentiable perturbation function.

3.2 Motivation

As is the case with conventional optimization problems, gH-subgradients might be inadequate for capturing the efficient solution of nonsmooth and nonconvex interval optimization problems as well. At this point, we need the concept of weak subgradients which can relax the convexity restriction by employing a conic surface to support the graph of the nonconvex interval-valued function. Further, from implementation point of view, weak subgradients might be more easy to implement and computational efficient compared to other numerical methods in the literature for IOPs, such as bundle-type methods, gradient sampling algorithm, variable metric method etc. These methods are based on iterative solving of subproblems which will naturally take more computational resources [41].

3.3 Contributions

In this chapter, we define the concept of gH-weak subdifferential for interval-valued functions (IVFs). Using an example, we provide a geometrical interpretation of the notion to further clear the understanding. We show that a nonempty gH-weak subdifferential set is closed and convex, and the class of IVFs for which the gH-weak subdifferential set is nonempty gH-lower Lipschitz. Next, using a counter example, we show that the sum rule of gH-weak subdifferential for a pair of IVFs does not hold in general. However, one-sided inclusion for the sum rule can be shown to hold if we put a mild restriction on one of the IVFs. Finally, we show the application of gH-weak subdifferential by providing some optimality conditions for nonsmooth IVFs.

The novel contributions of this chapter are as follows:

(i) The use of gH-Fréchet differentiability in obtaining the weak efficiency condition for gH-weak subdifferentiable IVFs is shown.

(ii) A necessary optimality condition for IOPs where the objective is the difference of two nonsmooth IVFs is presented under mild assumptions. This necessary but not sufficient condition is shown using an example.

(iii) For finding weak efficient points, a necessary and sufficient condition using augmented normal cone and gH-weak subdifferential of IVFs is shown.

(iv) For studying a 'sup-relation' between gH-direction derivative and gH-weak subgradients, we compute an approximation of gH-weak subgradient at each iteration. We then present $\mathcal{W}$-gH-weak subgradient technique to find a weak efficient solution of an unconstrained nonconvex and nonsmooth IOP. The method is explained using an illustrative example.

(v) Lastly, convergence analysis of the proposed method for the two cases of constant and diminishing stepsize are provided.

3.4 gH-weak subdifferential calculus for IVFs

In this section, we present the concepts of gH-weak subgradient and gH-weak subdifferential for IVFs. We also discuss a few properties of gH-weak subdifferential and illustrate the inclusion for sum rule. We explore its connection with gH-Fréchet lower subdifferential.

Definition 3.1 (gH-weak subdifferential). *Let $\emptyset \neq \mathcal{Y} \subseteq \mathbb{R}^n$ and $\mathbf{\Phi}$ be an IVF defined on $\mathcal{Y}$. The gH-weak subgradient of $\mathbf{\Phi}$ at $u \in \mathcal{Y}$ is defined as a pair $(\widehat{\boldsymbol{G}^w}, c) \in I(\mathbb{R})^n \times \mathbb{R}_+$ satisfying the inequality*

$$\widehat{\boldsymbol{G}^w}^\top \odot (y-u) \ominus_{gH} c\|y-u\| \preceq \mathbf{\Phi}(y) \ominus_{gH} \mathbf{\Phi}(u) \tag{3.1}$$

for every $y \in \mathcal{Y}$. The gH-weak subdifferential of $\mathbf{\Phi}$ at $u \in \mathcal{Y}$ is defined as the following set:

$$\partial^w \mathbf{\Phi}(u) = \left\{ (\widehat{\boldsymbol{G}^w}, c) \in I(\mathbb{R})^n \times \mathbb{R}_+ : \widehat{\boldsymbol{G}^w}^\top \odot (y-u) \ominus_{gH} c\|y-u\| \preceq \mathbf{\Phi}(y) \ominus_{gH} \mathbf{\Phi}(u) \ \forall\, y \in \mathcal{Y} \right\}.$$

Example 3.4.1 *Consider an IVF $\mathbf{\Phi} : [-1,1] \to I(\mathbb{R})$ which is given by*

$$\mathbf{\Phi}(y) = \left[y^2, |y|\right], \textit{ where } y \in [-1,1].$$

Let us estimate the gH-weak subdifferential of $\mathbf{\Phi}$ at 0 and 1, i.e., $\partial^w \mathbf{\Phi}(0)$ and $\partial^w \mathbf{\Phi}(1)$, respectively. Here, we make a note that

$$\begin{aligned} \partial^w \mathbf{\Phi}(0) &= \left\{ (\boldsymbol{G}_1^w, c) \in I(\mathbb{R}) \times \mathbb{R}_+ : \boldsymbol{G}_1^w \odot y \ominus_{gH} c|y| \preceq \left[y^2, |y|\right] \ \ \forall\, y \in [-1,1] \right\} \\ &= \left\{ \left([\underline{g_1^w}, \overline{g_1^w}], c \right) \in I(\mathbb{R}) \times \mathbb{R}_+ : [\underline{g_1^w}, \overline{g_1^w}] \odot y \ominus_{gH} c|y| \preceq \left[y^2, |y|\right] \ \ \forall\, y \in [-1,1] \right\}, \end{aligned}$$

which splits into the following two cases corresponding to $y \in [0,1]$ and $y \in [-1,0]$.

(i)

$$\begin{aligned} \partial^w \mathbf{\Phi}(0) &= \left\{ \left([\underline{g_1^w}, \overline{g_1^w}], c \right) \in I(\mathbb{R}) \times \mathbb{R}_+ : [\underline{g_1^w}, \overline{g_1^w}] \odot y \ominus_{gH} c|y| \preceq \left[y^2, |y|\right] \ \ \forall\, y \in [0,1] \right\} \\ &= \left\{ \left([\underline{g_1^w}, \overline{g_1^w}], c \right) \in I(\mathbb{R}) \times \mathbb{R}_+ : \underline{g_1^w} y - cy \leq y^2 \textit{ and } \overline{g_1^w} y - cy \leq y \ \forall\, y \in [0,1] \right\} \\ &= \left\{ \left([\underline{g_1^w}, \overline{g_1^w}], c \right) \in I(\mathbb{R}) \times \mathbb{R}_+ : \underline{g_1^w} - c \leq 0 \textit{ and } \overline{g_1^w} - c \leq 1 \right\}. \end{aligned}$$

(ii) Likewise,

$$\partial^w \mathbf{\Phi}(0) = \left\{ \left([\underline{g_1^w}, \overline{g_1^w}], c \right) \in I(\mathbb{R}) \times \mathbb{R}_+ : -1 \leq \underline{g_1^w} + c \textit{ and } \ 0 \leq \overline{g_1^w} + c \right\}.$$

Hence, by assembling Case (i) and Case (ii), we obtain

$$\partial^w \boldsymbol{\Phi}(0) = \left\{(\boldsymbol{G}_1^w, c) \in I(\mathbb{R}) \times \mathbb{R}_+ : [-1-c, -c] \preceq \boldsymbol{G}_1^w \preceq [c, 1+c]\right\}.$$

In the same manner,

$$\partial^w \boldsymbol{\Phi}(1) = \left\{(\boldsymbol{G}_2^w, c) \in I(\mathbb{R}) \times \mathbb{R}_+ : [1-c, 2-c] \preceq \boldsymbol{G}_2^w\right\}.$$

Remark 3.4.1 *To get insight into the geometric meaning of the gH-weak subdifferential of an IVF $\boldsymbol{\Phi}$, let $(\widehat{\boldsymbol{G}^w}, c) \in \partial^w \boldsymbol{\Phi}(u)$. This indicates that $(\widehat{\boldsymbol{G}^w}, c) \in I(\mathbb{R})^n \times \mathbb{R}_+$, for every $c \geq 0$, serves as a gH-weak subgradient of $\boldsymbol{\Phi}$ at $u \in \mathcal{Y}$ if and only if there exists a concave and gH-continuous IVF $\boldsymbol{H} : \mathcal{Y} \to I(\mathbb{R})$, which is defined by*

$$\boldsymbol{H}(y) = \boldsymbol{\Phi}(u) \oplus \widehat{\boldsymbol{G}^w}^\top \odot (y-u) \ominus_{gH} c\|y-u\| \ \forall\ y \in \mathcal{Y},$$

that satisfies

$$(\forall\ y \in \mathcal{Y})\ \boldsymbol{H}(y) \preceq \boldsymbol{\Phi}(y) \text{ and } \boldsymbol{H}(u) = \boldsymbol{\Phi}(u).$$

As per the above condition, $\boldsymbol{H}$ should intersect $\boldsymbol{\Phi}$ at minimum one point $(u, \boldsymbol{\Phi}(u))$ from below. Hence, it concludes that if $\boldsymbol{\Phi}$ is gH-weak subdifferentiable at u and $(\widehat{\boldsymbol{G}^w}, c) \in \partial^w \boldsymbol{\Phi}(u)$, then the graph of IVF $\boldsymbol{H}$, that is,

$$Gr(\boldsymbol{H}) = \{(y, \boldsymbol{Y}) \in \mathcal{Y} \times I(\mathbb{R}) : \boldsymbol{Y} = \boldsymbol{H}(y)\}$$

always lie below the epigraph of $\boldsymbol{\Phi}$, i.e.,

$$Epi(\boldsymbol{\Phi}) = \{(y, \boldsymbol{Y}) \in \mathcal{Y} \times I(\mathbb{R}) : \boldsymbol{\Phi}(y) \preceq \boldsymbol{Y}\},$$

such that

$$Epi(\boldsymbol{\Phi}) \subset Epi(\boldsymbol{H}) \text{ and } cl(Epi(\boldsymbol{\Phi})) \bigcap Gr(\boldsymbol{H}) \text{ is nonempty.}$$

For example, Let $\mathcal{Y} = [-1, 2]$. Consider an IVF $\boldsymbol{\Phi} : \mathcal{Y} \to I(\mathbb{R})$ which is given by

$$\boldsymbol{\Phi}(y) = \begin{cases} [y^2 - 1, (y-1)^2], & \text{if } y \in [-1, 1] \\ [(y-1)^2, y^2 - 1], & \text{if } y \in (1, 2]. \end{cases}$$

The gH-weak subdifferential of $\boldsymbol{\Phi}$ at $u = 1$ is

$$\partial^w \boldsymbol{\Phi}(1) = \{(\mathbf{G}^w, c) \in I(\mathbb{R}) \times \mathbb{R}_+ : [-c, 2-c] \preceq \mathbf{G}^w \preceq [c, 2+c]\}.$$

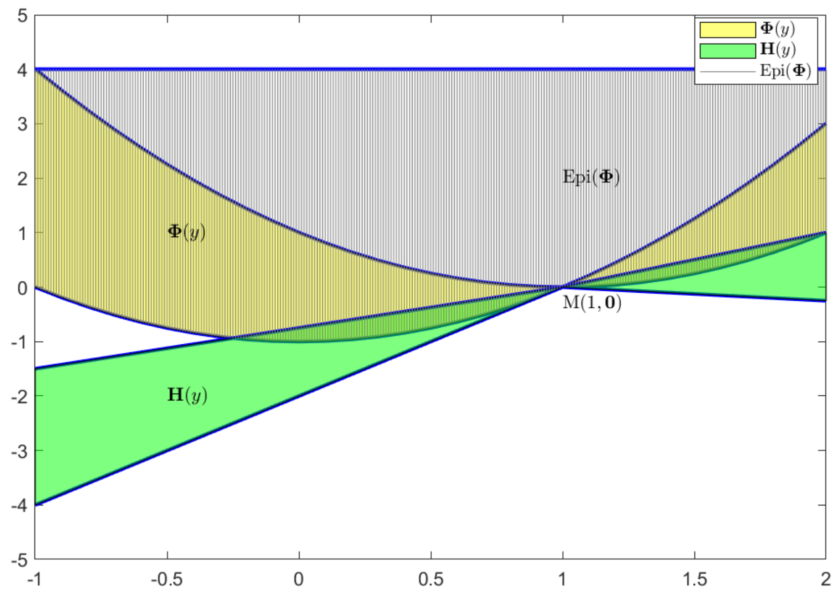


Figure 3.1: Geometrical representation of gH-weak subgradients of $\mathbf{\Phi}$ of Example 3.4.1

For instance, $(\mathbf{G}^w, c) = ([0.25, 1.5], 0.5) \in \partial^w \mathbf{\Phi}(1)$, geometrically indicates that the IVF

$$\mathbf{H}(y) = \mathbf{\Phi}(1) \oplus [0.25, 1.5] \odot (y-1) \ominus_{gH} 0.5|y-1|$$

intersects

$$\text{Epi}(\mathbf{\Phi}) = \{(y, \mathbf{4}) \in \mathcal{Y} \times \mathbb{R} : \mathbf{\Phi}(y) \preceq \mathbf{4}\}$$

at the point M$(1, \mathbf{0})$ from below as shown in Figure 3.1. We also observe from the figure that

$$\text{Epi}(\mathbf{\Phi}) \subset \text{Epi}(\mathbf{H}), \text{ and } cl(\text{Epi}(\mathbf{\Phi})) \bigcap Gr(\mathbf{H}) \text{ is nonempty.}$$

Theorem 3.1 (Convexity of gH-weak subdifferential). *Let $\mathcal{Y} \subset \mathbb{R}^n$. Let the gH-weak subdifferential of $\mathbf{\Phi} : \mathcal{Y} \to I(\mathbb{R})$ at u be nonempty. Then, the set $\partial^w \mathbf{\Phi}(u)$ is convex.*

Proof: Let $(\widehat{\mathbf{G}_1^w}, c_1)$ and $(\widehat{\mathbf{G}_2^w}, c_2) \in \partial^w \mathbf{\Phi}(u)$, where $\widehat{\mathbf{G}_1^w} = (\mathbf{G}_{11}^w, \mathbf{G}_{12}^w, \ldots, \mathbf{G}_{1n}^w)^\top, \widehat{\mathbf{G}_2^w} = (\mathbf{G}_{21}^w, \mathbf{G}_{22}^w, \ldots, \mathbf{G}_{2n}^w)^\top$. Let $\beta \in [0, 1]$. From the definition of $\partial^w \mathbf{\Phi}(u)$, we have

$$\widehat{\mathbf{G}_1^w}^\top \odot (y-u) \ominus_{gH} c_1\|y-u\| \preceq \mathbf{\Phi}(y) \ominus_{gH} \mathbf{\Phi}(u) \text{ and} \tag{3.2}$$

$$\widehat{\mathbf{G}_2^w}^\top \odot (y-u) \ominus_{gH} c_2\|y-u\| \preceq \mathbf{\Phi}(y) \ominus_{gH} \mathbf{\Phi}(u), \tag{3.3}$$

for all $y \in \mathcal{Y}$. Up to a rearrangement of terms, let the first m components of $(y-u)$ be non-negative, and the rest be negative. Then, from the inequalities (3.2) and (3.3), we get

$$\bigoplus_{i=1}^{m}(y_i-u_i)\odot\mathbf{G}_{1i}^w\bigoplus_{j=m+1}^{n}(y_j-u_j)\odot\mathbf{G}_{1j}^w\ominus_{gH}c_1\|y-u\|\preceq\mathbf{\Phi}(y)\ominus_{gH}\mathbf{\Phi}(u)$$

and

$$\bigoplus_{i=1}^{m}(y_i-u_i)\odot\mathbf{G}_{2i}^w\bigoplus_{j=m+1}^{n}(y_j-u_j)\odot\mathbf{G}_{2j}^w\ominus_{gH}c_2\|y-u\|\preceq\mathbf{\Phi}(y)\ominus_{gH}\mathbf{\Phi}(u).$$

Thus,

$$\bigoplus_{i=1}^{m}\beta\odot((y_i-u_i)\odot\mathbf{G}_{1i}^w)\bigoplus_{j=m+1}^{n}\beta\odot((y_j-u_j)\odot\mathbf{G}_{1j}^w)\ominus_{gH}\beta c_1\|y-u\|$$
$$\preceq\beta\odot(\mathbf{\Phi}(y)\ominus\mathbf{\Phi}(u))\tag{3.4}$$

and

$$\bigoplus_{i=1}^{m}(1-\beta)\odot((y_i-u_i)\odot\mathbf{G}_{2i}^w)\bigoplus_{j=m+1}^{n}(1-\beta)\odot((y_j-u_j)\odot\mathbf{G}_{2j}^w)\ominus_{gH}(1-\beta)c_2\|y-u\|$$
$$\preceq(1-\beta)\odot(\mathbf{\Phi}(y)\ominus\mathbf{\Phi}(u)).\tag{3.5}$$

By adding (3.4) and (3.5), we obtain

$$\bigoplus_{i=1}^{m}(y_i-u_i)\odot\{\beta\odot\mathbf{G}_{1i}^w\oplus(1-\beta)\odot\mathbf{G}_{2i}^w\}\bigoplus_{j=m+1}^{n}(y_j-u_j)\odot\{\beta\odot\mathbf{G}_{1j}^w\oplus(1-\beta)\odot\mathbf{G}_{2j}^w\}$$
$$\ominus_{gH}(\beta c_1\oplus(1-\beta)c_2)\|y-u\|\preceq\mathbf{\Phi}(y)\ominus_{gH}\mathbf{\Phi}(u).\tag{3.6}$$

Therefore, we have

$$\{\beta\odot\widehat{\mathbf{G}_1^w}\oplus(1-\beta)\odot\widehat{\mathbf{G}_2^w}\}^\top\odot(y-u)\ominus_{gH}(\beta c_1\oplus(1-\beta)c_2)\|y-u\|\preceq\mathbf{\Phi}(y)\ominus_{gH}\mathbf{\Phi}(u),$$

i.e., $(\beta\odot\widehat{\mathbf{G}_1^w}\oplus(1-\beta)\odot\widehat{\mathbf{G}_2^w},\ \beta c_1\oplus(1-\beta)c_2)\in\partial^w\mathbf{\Phi}(u)$, which proves the convexity of $\partial^w\mathbf{\Phi}(u)$. □

Theorem 3.2 (Closedness of gH-weak subdifferential). *Let $\emptyset\neq\mathcal{Y}\subseteq I(\mathbb{R})^n$. If for an IVF $\mathbf{\Psi}:\mathcal{Y}\to I(\mathbb{R})$, the set $\partial^w\mathbf{\Psi}(u)$ is nonempty at $u\in\mathcal{Y}$, then $\partial^w\mathbf{\Psi}(u)$ is closed.*

Proof: Let $\{(\widehat{\mathbf{G}_k^w},c_k)\}$ be an arbitrary sequence in $\partial^w\mathbf{\Psi}(y)$ converging to $(\widehat{\mathbf{G}^w},c)\in I(\mathbb{R})^n\times\mathbb{R}_+$, where $\widehat{\mathbf{G}_k^w}=(\mathbf{G}_{k1}^w,\mathbf{G}_{k2}^w,\ldots,\mathbf{G}_{kn}^w)^\top$ and $\widehat{\mathbf{G}^w}=(\mathbf{G}_1^w,\mathbf{G}_2^w,\ldots,\mathbf{G}_n^w)^\top$. Since

$(\widehat{\mathbf{G}^w_k}, c) \in \partial^w \mathbf{\Psi}(y)$ for all $d \in \mathcal{Y}$, we obtain

$$\widehat{\mathbf{G}^w_k}^\top \odot d \ominus_{gH} c_k\|d\| \preceq \mathbf{\Psi}(u+d) \ominus_{gH} \mathbf{\Psi}(u),$$

which implies

$$\bigoplus_{i=1}^{n} d_i \odot \mathbf{G}^w_{ki} \ominus_{gH} c_k\|d\| \preceq \mathbf{\Psi}(u+d) \ominus_{gH} \mathbf{\Psi}(u). \tag{3.7}$$

Up to a rearrangement of terms, let the first p components of d be non-negative, and the rest be negative. Then, from (3.7), we get

$$\bigoplus_{i=1}^{p} d_i \odot \mathbf{G}^w_{ki} \bigoplus_{j=p+1}^{n} d_j \odot \mathbf{G}^w_{kj} \ominus_{gH} c_k\|d\| \preceq \mathbf{\Psi}(u+d) \ominus_{gH} \mathbf{\Psi}(u)$$

$$\implies \bigoplus_{i=1}^{p} d_i \odot [\underline{g^w_{ki}}, \overline{g^w_{ki}}] \bigoplus_{j=p+1}^{n} d_j \odot [\underline{g^w_{kj}}, \overline{g^w_{kj}}] \ominus_{gH} c_k\|d\| \preceq \mathbf{\Psi}(u+d) \ominus_{gH} \mathbf{\Psi}(u).$$

Therefore,

$$\sum_{i=1}^{p} \underline{g^w_{ki}} d_i + \sum_{j=p+1}^{n} \overline{g^w_{kj}} d_j - c_k\|d\| \preceq \min\left\{\underline{\Psi}(u+d) - \underline{\Psi}(u), \overline{\Psi}(u+d) - \overline{\Psi}(u)\right\} \tag{3.8}$$

and

$$\sum_{i=1}^{p} \overline{g^w_{ki}} d_i + \sum_{j=p+1}^{n} \underline{g^w_{kj}} d_j - c_k\|d\| \preceq \max\left\{\underline{\Psi}(u+d) - \underline{\Psi}(u), \overline{\Psi}(u+d) - \overline{\Psi}(u)\right\}. \tag{3.9}$$

Since the sequence $\widehat{\mathbf{G}^w_k}$ converges to $\widehat{\mathbf{G}^w}$, the sequences $\{\underline{g^w_{ki}}\}$ and$\{\overline{g^w_{ki}}\}$ converge to $\{\underline{g^w_i}\}$ and $\{\overline{g^w_i}\}$, respectively for all i. Thus, by (3.8) and (3.9), we have

$$\begin{aligned}\sum_{i=1}^{p} \underline{g^w_{ki}} d_i + \sum_{j=p+1}^{n} \overline{g^w_{kj}} d_j - c_k\|d\| &\to \sum_{i=1}^{p} \underline{g^w_i} d_i + \sum_{j=p+1}^{n} \overline{g^w_j} d_j - c\|d\| \\ &\preceq \min\left\{\underline{\Psi}(u+d) - \underline{\Psi}(u), \overline{\Psi}(u+d) - \overline{\Psi}(u)\right\}\end{aligned}$$

and

$$\sum_{i=1}^{p} \overline{g^w_{ki}} d_i + \sum_{j=p+1}^{n} \underline{g^w_{kj}} d_j - c_k\|d\| \quad \to \sum_{i=1}^{p} \overline{g^w_i} d_i + \sum_{j=p+1}^{n} \underline{g^w_j} d_j - c\|d\|$$

$$\preceq \max\left\{\underline{\Psi}(u+d)-\underline{\Psi}(u), \overline{\Psi}(u+d)-\overline{\Psi}(u)\right\}.$$

Hence, for any $u \in \mathcal{Y}$,

$$\left[\sum_{i=1}^{p}\underline{g_i^w}d_i + \sum_{j=p+1}^{n}\overline{g_j^w}d_j - c\|d\|, \sum_{i=1}^{p}\overline{g_i^w}d_i + \sum_{j=p+1}^{n}\underline{g_j^w}d_j - c\|d\|\right] \preceq \mathbf{\Psi}(u+d) \ominus_{gH} \mathbf{\Psi}(u)$$

$$\implies \bigoplus_{i=1}^{p} [\underline{g_i^w}d_i, \overline{g_i^w}d_i] \bigoplus_{j=p+1}^{n} [\overline{g_j^w}d_j, \underline{g_j^w}d_j] \ominus_{gH} c\|d\| \preceq \mathbf{\Psi}(u+d) \ominus_{gH} \mathbf{\Psi}(u)$$

$$\implies \bigoplus_{i=1}^{p} d_i \odot \mathbf{G}_i^w \bigoplus_{j=p+1}^{n} d_j \odot \mathbf{G}_j^w \ominus_{gH} c\|d\| \preceq \mathbf{\Psi}(u+d) \ominus_{gH} \mathbf{\Psi}(u)$$

$$\implies \widehat{\mathbf{G}^w}^\top \odot d \ominus_{gH} c\|d\| \preceq \mathbf{\Psi}(u+d) \ominus_{gH} \mathbf{\Psi}(u).$$

Therefore, $\widehat{\mathbf{G}^w} \in \partial^w \mathbf{\Psi}(u)$, and hence $\partial^w \mathbf{\Psi}(u)$ is closed. □

Definition 3.2 (gH-Fréchet lower subdifferential). *Let $\mathbf{\Phi} : \mathcal{Y} \to I(\mathbb{R}) \cup \{-\infty, +\infty\}$ be an IVF that is finite at an $u \in \mathcal{Y}$. Then, the gH-Fréchet lower subdifferential of $\mathbf{\Phi}$ at u is defined by*

$$\partial_{\mathscr{F}}^{-}\mathbf{\Phi}(u) = \left\{\widehat{\boldsymbol{G}} : \boldsymbol{0} \preceq \liminf_{\substack{y\to u\\ y\neq u}} \frac{1}{\|y-u\|} \odot \{\mathbf{\Phi}(y) \ominus_{gH} \mathbf{\Phi}(u) \ominus_{gH} \widehat{\boldsymbol{G}}^\top \odot (y-u)\},\right.$$
$$\left. \text{where } \widehat{\boldsymbol{G}} : \mathcal{Y} \to I(\mathbb{R}) \text{ is } gH\text{-continuous and linear IVF}\right\}.$$

One important fact is that gH-weak subdifferential is an immediate consequence of gH-Fréchet lower subdifferential.

Theorem 3.3 *Let $\emptyset \neq \mathcal{Y} \subseteq \mathbb{R}^n$. If $\mathbf{\Phi} : \mathcal{Y} \to I(\mathbb{R})$ has gH-Fréchet lower subdifferential $\widehat{\boldsymbol{G}}$ at the point u, then $(\widehat{\boldsymbol{G}}, \epsilon)$ is a gH-weak subgradient of $\mathbf{\Phi}$ at u for any $\epsilon \in \mathbb{R}_+$.*

Proof: Let $\widehat{\mathbf{G}} \in \partial_{\mathscr{F}}^{-}\mathbf{\Phi}(u)$. Due to Definition 3.2, we can write

$$\mathbf{0} \preceq \liminf_{\substack{y\to u\\ y\neq u}} \frac{1}{\|y-u\|} \odot \{\mathbf{\Phi}(y) \ominus_{gH} \mathbf{\Phi}(u) \ominus_{gH} \widehat{\mathbf{G}}^\top \odot (y-u)\}.$$

Then, for the $\epsilon > 0$ in the hypothesis there exists $\delta > 0$ such that

$$-\epsilon\|y-u\| \preceq \mathbf{\Phi}(y) \ominus_{gH} \mathbf{\Phi}(u) \ominus_{gH} \widehat{\mathbf{G}}^\top \odot (y-u) \;\forall\; y \in B_\delta(u),$$

Then, from lemma 1.2, we have

$$\widehat{\mathbf{G}}^\top \odot (y-u) \ominus_{gH} \epsilon\|y-u\| \preceq \mathbf{\Phi}(y) \ominus_{gH} \mathbf{\Phi}(u).$$

By Definition 3.1, $(\widehat{\mathbf{G}}, \epsilon)$ is a gH-weak subdifferential of $\mathbf{\Phi}$ at u. □

Lemma 3.1 *For any $y \in \mathbb{R}^n$ and $\widehat{\boldsymbol{C}} = (\boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{C}_3, \ldots, \boldsymbol{C}_n) \in I(\mathbb{R})^n$,*

$$-\|y\|\|\widehat{\boldsymbol{C}}\|_{I(\mathbb{R})^n} \preceq \|y^\top \odot \widehat{\boldsymbol{C}}\|_{I(\mathbb{R})}.$$

Proof: See Appendix 10.5. □

To investigate the class of interval-valued functions for which weak subgradients always exist, we need the following definition.

Definition 3.3 (gH-lower Lipschitz IVF). *Let $\emptyset \neq \mathcal{Y} \subseteq \mathbb{R}^n$. An IVF $\mathbf{\Phi} : \mathcal{Y} \to \overline{I(\mathbb{R})}$ is called gH-lower locally Lipschitz at $u \in \mathcal{Y}$ if $\exists\ L \geq 0$ and a neighbourhood $\mathcal{N}(u)$ of u such that*

$$-L\|y - u\| \preceq \mathbf{\Phi}(y) \ominus_{gH} \mathbf{\Phi}(u)\ \forall\ y \in \mathcal{N}(u). \tag{3.10}$$

If the inequality (3.10) satisfies for all $y \in \mathcal{Y}$, then $\mathbf{\Phi}$ is called gH-lower Lipschitz at $u \in \mathcal{Y}$ with Lipschitz constant L.

Example 3.4.2 *Let $\mathbf{\Phi} : [1, \infty) \to I(\mathbb{R})$ be an IVF, defined by $\mathbf{\Phi}(y) = \ln y \odot \boldsymbol{C}$ for all $y \in [1, \infty)$, where $\boldsymbol{0} \preceq \boldsymbol{C} = [\underline{c}, \overline{c}]$. Let $\delta > 0$. We choose the neighbourhood of u, $\mathcal{N}_\delta(u) = \{y : |y - u| < \delta\}$.*
If $0 < y - u < \delta$, then $u < y$ and also then $\frac{y}{u} > 1$ and then

$$\begin{aligned} 0 < \ln\frac{y}{u} &< \frac{y}{u} - 1,\ \textit{since}\ \ln(1+p) < p\ \textit{if}\ p > 0 \\ &\leq y - u. \end{aligned} \tag{3.11}$$

Since $\underline{c}, \overline{c} \geq 0$, we have

$$(\ln y - \ln u)\underline{c} \leq (y - u)\underline{c}\ \textit{and}\ (\ln y - \ln u)\overline{c} \leq (y - u)\overline{c}.$$

Then,

$$(\ln y - \ln u) \odot \boldsymbol{C} \preceq (y - u) \odot \boldsymbol{C}. \tag{3.12}$$

If $-\delta < y - u < 0$, then $y < u$ and also then $\frac{u}{y} > 1$ and then

$$\begin{aligned} 0 < \ln\frac{u}{y} &< \frac{u}{y} - 1,\ \textit{since}\ \ln(1+p) < p\ \textit{if}\ p > 0 \\ &\leq u - y. \end{aligned} \tag{3.13}$$

Then, similarly, as seen in (3.12),

$$(\ln u - \ln y) \odot \boldsymbol{C} \preceq (u - y) \odot \boldsymbol{C}. \tag{3.14}$$

Combining (3.12) and (3.14), we have

$$\begin{aligned} & |\ln y - \ln u| \odot \boldsymbol{C} \preceq |y - u| \odot \boldsymbol{C} \\ \Longrightarrow \ & \ln u \odot \boldsymbol{C} \ominus_{gH} \ln y \odot \boldsymbol{C} \preceq |y - u| \odot \boldsymbol{C} \\ \Longrightarrow \ & -|y - u| \odot \boldsymbol{C} \preceq \ln y \odot \boldsymbol{C} \ominus_{gH} \ln u \odot \boldsymbol{C} \\ \Longrightarrow \ & -\bar{c}|y - u| \preceq \boldsymbol{\Phi}(y) \ominus_{gH} \boldsymbol{\Phi}(u). \end{aligned}$$

This shows that $\boldsymbol{\Phi}$ is gH-lower locally Lipschitz on $\mathcal{N}_\delta(u)$ with $L = \bar{c}$. From arbitrariness of y, u in $[1, \infty)$, we conclude that $\boldsymbol{\Phi}$ is gH-lower Lipschitz on $[1, \infty)$.

Theorem 3.4 *Let $\emptyset \neq \mathcal{Y} \subseteq \mathbb{R}^n$. Let $\boldsymbol{\Phi} : \mathcal{Y} \to \overline{I(\mathbb{R})}$ be an IVF, where $\boldsymbol{\Phi}(u)$ is finite for some $u \in \mathcal{Y}$. Then, the following three statements are equivalent:*

(a) $\boldsymbol{\Phi}$ is gH-weak subdifferentiable at u.

(b) $\boldsymbol{\Phi}$ is gH-lower Lipschitz at u.

(c) $\boldsymbol{\Phi}$ is gH-lower locally Lipschitz at u, and there exists a number $p \geq 0$ and an interval $\boldsymbol{Q}$ such that

$$-p\|y\| \oplus \boldsymbol{Q} \preceq \boldsymbol{\Phi}(y) \ \forall \ y \in \mathcal{Y}. \tag{3.15}$$

Proof: (a) implies (b) : Suppose $\boldsymbol{\Phi}$ is gH-weak subdifferentiable at u. Then, there exists $(\widehat{\mathbf{G}^w}, c) \in I(\mathbb{R})^n \times \mathbb{R}_+$ such that for any $y \in \mathcal{Y}$, we have

$$\widehat{\mathbf{G}^w}^\top \odot (y - u) \ominus_{gH} c\|y - u\| \preceq \boldsymbol{\Phi}(y) \ominus_{gH} \boldsymbol{\Phi}(u). \tag{3.16}$$

From lemma 3.1, we have $-\|\widehat{\mathbf{G}^w}\|_{I(\mathbb{R})^n}\|y - u\| - c\|y - u\| \preceq \widehat{\mathbf{G}^w} \odot (y - u) \ominus_{gH} c\|y - u\|$. Hence, the inequality (3.16) yields

$$-(\|\widehat{\mathbf{G}^w}\| + c)\|y - u\| \preceq \boldsymbol{\Phi}(y) \ominus_{gH} \boldsymbol{\Phi}(u) \text{ by Lemma 2.3 (ii) of [66].}$$

By choosing $L = (\|\widehat{\mathbf{G}^w}\| + c)$, we obtain

$$-L\|y - u\| \preceq \boldsymbol{\Phi}(y) \ominus_{gH} \boldsymbol{\Phi}(u) \ \forall \ y \in \mathcal{Y}. \tag{3.17}$$

So, $\mathbf{\Phi}$ is gH-lower Lipschitz at u.

(b) implies (c) : Suppose that (b) is satisfied. It needs to prove that the inequality (3.15) holds. Then, there exists an $L \geq 0$ such that

$$-L\|y-u\| \preceq \mathbf{\Phi}(y) \ominus_{gH} \mathbf{\Phi}(u). \tag{3.18}$$

Note that $-L\|y\| - L\|u\| \leq -L\|y-u\|$. So, the inequality (3.18) gives

$$-L\|y\| - L\|u\| \preceq \mathbf{\Phi}(y) \ominus_{gH} \mathbf{\Phi}(u),$$

which gives $\mathbf{\Phi}(u) \ominus_{gH} L\|u\| - L\|y\| \preceq \mathbf{\Phi}(y)$ by (iv) of Lemma 1.5. Taking $\mathbf{Q} = \mathbf{\Phi}(u) \ominus_{gH} L\|u\|$ and $p = L$, we obtain $-p\|y\| \oplus \mathbf{Q} \preceq \mathbf{\Phi}(y)$ for all $y \in \mathcal{Y}$.

(c) implies (a) : Let $\mathcal{N}(u)$ be an ϵ-neighbourhood of u such that (3.10) holds. Then, we get

$$-L\|y-u\| \preceq \mathbf{\Phi}(y) \ominus_{gH} \mathbf{\Phi}(u) \ \forall\ y \in \mathcal{N}(u) \tag{3.19}$$

and

$$-p\|y\| \oplus \mathbf{Q} \preceq \mathbf{\Phi}(y) \ \forall\ y \in \mathbb{R}^n. \tag{3.20}$$

Assume to the contrary that $\mathbf{\Phi}$ is not gH-weak subdifferentiable at u. Then, for any $(\widehat{\mathbf{G}_n^w}, c_n) \in I(\mathbb{R})^n \times \mathbb{R}_+$, there exists y_n such that

$$\mathbf{\Phi}(y_n) \ominus_{gH} \mathbf{\Phi}(u) \prec \widehat{\mathbf{G}_n^w}^\top \odot (y_n - u) \ominus_{gH} c_n\|y_n - y\|.$$

If the sequence $\{\widehat{\mathbf{G}_n^w}\}$ is assumed to be converging to $\widehat{\mathbf{G}^w}$, then we get

$$\begin{aligned}\mathbf{\Phi}(y_n) \ominus_{gH} \mathbf{\Phi}(u) &\preceq \widehat{\mathbf{G}^w}^\top \odot (y_n - u) \ominus_{gH} c_n\|y_n - y\| \\ &\preceq \|\widehat{\mathbf{G}^w}\|\|y_n - u\| - c_n\|y_n - u\|, \text{ by Theorem 3.1 of [66].}\end{aligned} \tag{3.21}$$

By putting $y = y_n$ in (3.20), we get

$$-p\|y_n - u\| - p\|y\| \oplus \mathbf{Q} \preceq -p\|y_n\| \oplus \mathbf{Q} \preceq \mathbf{\Phi}(y_n),$$

which implies

$$(-p\|y_n - u\| - p\|u\| \oplus \mathbf{Q}) \ominus_{gH} \mathbf{\Phi}(u) \preceq \mathbf{\Phi}(y_n) \ominus_{gH} \mathbf{\Phi}(u) \text{ by Note 2 of [66].} \tag{3.22}$$

From (3.21) and (3.22), by Lemma 2.3 (ii) of [68], we deduce that

$$(-p\|y_n - u\| - p\|u\| \oplus \mathbf{Q}) \ominus_{gH} \mathbf{\Phi}(u) \preceq \|\widehat{\mathbf{G}^w}\|\|y_n - u\| - c_n\|y_n - u\|,$$
$$\text{or, } (c_n - p - \|\widehat{\mathbf{G}^w}\|)\|y_n - u\| \preceq \mathbf{\Phi}(u) \oplus p\|u\| \ominus_{gH} \mathbf{Q} \text{ by (iii) of Lemma 1.5.} \tag{3.23}$$

Assume, without loss of generality, that $c_n - p - \|\widehat{\mathbf{G}^w}\| \neq 0$. Then, from (1.5), we obtain

$$\|y_n - u\| \preceq \frac{1}{c_n - p - \|\widehat{\mathbf{G}^w}\|} \odot \{\mathbf{\Phi}(u) \oplus p\|u\| \ominus_{gH} \mathbf{Q}\}.$$

As $(\mathbf{\Phi}(u) \oplus p\|u\| \ominus_{gH} \mathbf{Q})$ is bounded below on $\mathcal{N}(u)$, we get $y_n \to u$ as $c_n \to \infty$. Thus, $y_n \in \mathcal{N}(u)$ for large n. Then, from (3.19) it follows that

$$-L\|y_n - u\| \preceq \mathbf{\Phi}(y_n) \ominus_{gH} \mathbf{\Phi}(u). \tag{3.24}$$

In view of (3.21), we obtain

$$\mathbf{\Phi}(y_n) \ominus_{gH} \mathbf{\Phi}(u) \preceq \|\widehat{\mathbf{G}^w}\|\|y_n - u\| - c_n\|y_n - u\| = -(c_n - \|\widehat{\mathbf{G}^w}\|)\|y_n - u\|.$$

Since $c_n \to +\infty$ and $L \geq 0$, we can pick c_n sufficiently large so that $c_n - \|\widehat{\mathbf{G}^w}\| \geq L$. So,

$$\mathbf{\Phi}(y_n) \ominus_{gH} \mathbf{\Phi}(u) \preceq -L\|y_n - u\|.$$

This inequality leads to a contradiction. So, the result follows. □

Theorem 3.5 *Let $\emptyset \neq \mathcal{Y} \subseteq \mathbb{R}^n$. Let $\mathbf{\Psi} : \mathcal{Y} \to I(\mathbb{R})$ be gH-Fréchet differentiable at u with gH-Fréchet derivative $\mathbf{\Psi}_{\mathscr{F}}(u)$. Then,*

$$\{(\mathbf{\Psi}_{\mathscr{F}}(u), c) : c \geq 0\} \subset \partial^w \mathbf{\Psi}(u).$$

Proof: Since $\mathbf{\Psi}$ admits gH-Fréchet derivative $\mathbf{\Psi}_{\mathscr{F}}(u)$ at u, we get

$$\lim_{y \to u} \frac{1}{\|y - u\|} \odot \{\mathbf{\Psi}(y) \ominus_{gH} \mathbf{\Psi}(u) \ominus_{gH} \mathbf{\Psi}_{\mathscr{F}}(u)^\top \odot (y - u)\} = \mathbf{0}$$
$$\implies \liminf_{\substack{y \to u \\ y \neq u}} \frac{1}{\|y - u\|} \odot \{\mathbf{\Psi}(y) \ominus_{gH} \mathbf{\Psi}(u) \ominus_{gH} \mathbf{\Psi}_{\mathscr{F}}(u)^\top \odot (y - u)\} = \mathbf{0}.$$

Therefore, according to Definition 3.2, $\mathbf{\Psi}_{\mathscr{F}}(u) \in \partial^-_{\mathscr{F}} \mathbf{\Psi}(u)$. So,

$$\mathbf{\Psi}_{\mathscr{F}}(u)^\top \odot (y - u) \preceq \mathbf{\Psi}(y) \ominus_{gH} \mathbf{\Psi}(u) \ \forall\ y \in \mathcal{Y}$$
$$\implies \mathbf{\Psi}_{\mathscr{F}}(u)^\top \odot (y - u) \ominus_{gH} c\|y - u\| \preceq \mathbf{\Psi}(y) \ominus_{gH} \mathbf{\Psi}(u), \text{ for any } c \geq 0.$$

Hence, $(\mathbf{\Psi}_{\mathscr{F}}(u), c) \in \partial^w \mathbf{\Psi}(u)$. □

Lemma 3.2 *Let $\emptyset \neq \mathcal{Y} \subseteq \mathbb{R}^n$. Let $\mathbf{\Phi} : \mathcal{Y} \to I(\mathbb{R})$ be gH-Fréchet differentiable at u with gH-Fréchet derivative $\mathbf{\Phi}_{\mathscr{F}}(u)$. Then, $-1 \odot \mathbf{\Phi}_{\mathscr{F}}(u) \in \partial^-_{\mathscr{F}}(-1 \odot \mathbf{\Phi})(u)$.*

Proof: Since $\mathbf{\Phi}$ admits gH-Fréchet derivative $\mathbf{\Phi}_{\mathscr{F}}(u)$ at u, one gets

$$\lim_{y \to u} \frac{1}{\|y-u\|} \odot \{\mathbf{\Phi}(y) \ominus_{gH} \mathbf{\Phi}(u) \ominus_{gH} \mathbf{\Phi}_{\mathscr{F}}(u)^\top \odot (y-u)\} = \mathbf{0}.$$

By applying Lemma 1.4, we have

$$\lim_{\substack{y \to u \\ y \neq u}} \frac{1}{\|y-u\|} \odot \left\{\mathbf{0} \ominus_{gH} \{(-1 \odot \mathbf{\Phi})(y) \ominus_{gH} (-1 \odot \mathbf{\Phi})(u) \ominus_{gH} (-1 \odot \mathbf{\Phi}_{\mathscr{F}}(u)^\top) \odot (y-u)\}\}\right\} = \mathbf{0}$$

$$\implies \lim_{\substack{y \to u \\ y \neq u}} \frac{1}{\|y-u\|} \odot \left\{(-1 \odot \mathbf{\Phi})(y) \ominus_{gH} (-1 \odot \mathbf{\Phi})(u) \ominus_{gH} (-1 \odot \mathbf{\Phi}_{\mathscr{F}}(u))^\top \odot (y-u)\right\} = \mathbf{0}$$

$$\implies \liminf_{\substack{y \to u \\ y \neq u}} \frac{1}{\|y-u\|} \odot \{(-1 \odot \mathbf{\Phi})(y) \ominus_{gH} (-1 \odot \mathbf{\Phi})(u) \ominus_{gH} (-1 \odot \mathbf{\Phi}_{\mathscr{F}}(u))^\top \odot (y-u)\} = \mathbf{0}.$$

Hence, $-1 \odot \mathbf{\Phi}_{\mathscr{F}}(u) \in \partial^-_{\mathscr{F}}(-1 \odot \mathbf{\Phi})(u)$. □

Next, we aim to investigate the sum rule [109] stating that for two real-valued functions f_1 and f_2, their weak subdifferential is $\partial^w(f_1 + f_2)(x) = \partial^w f_1(x) + \partial^w f_2(x)$, does not apply to interval-valued functions, as illustrated in the following example.

Consider the interval-valued functions $\mathbf{\Phi}_1 : [-1, 1] \to I(\mathbb{R})$ and $\mathbf{\Phi}_2 : [-1, 1] \to I(\mathbb{R})$, defined by

$$\mathbf{\Phi}_1(y) = \begin{cases} \left[-y, \frac{1}{2}y\right], & \text{if } y \in [0,1] \\ \left[-\frac{1}{2}y - y\right], & \text{if } y \in [-1,0] \end{cases} \quad \text{and} \quad \mathbf{\Phi}_2(y) = [y^2, -y+3],$$

respectively. For these two functions, the gH-weak subdifferential at $u = 0$ are given by

$$\partial^w \mathbf{\Phi}_1(0) = \{(\mathbf{G}_1^w, c_1) \in I(\mathbb{R}) \times \mathbb{R}_+ : \left[-1, -\tfrac{1}{2}\right] \preceq \mathbf{G}_1^w \oplus c_1,\ \mathbf{G}_1^w \ominus_{gH} c_1 \preceq \left[-1, \tfrac{1}{2}\right]\ \forall\, y \in [-1,1]\}$$

and

$$\partial^w \mathbf{\Phi}_2(0) = \{(\mathbf{G}_2^w, c_2) \in I(\mathbb{R}) \times \mathbb{R}_+ : [-1, 0] \preceq \mathbf{G}_2^w \oplus c_2,\ \mathbf{G}_2^w \ominus_{gH} c_2 \preceq [-1, 0]\ \forall\, y \in [-1,1]\}.$$

Thus, we have

$$\partial^w \mathbf{\Phi}_1(0) \oplus \partial^w \mathbf{\Phi}_2(0)$$

$$= \{(\mathbf{H}^w, c) \in I(\mathbb{R}) \times \mathbb{R}_+ : \left[-2, -\tfrac{1}{2}\right] \preceq \mathbf{H}^w \oplus c,\ \mathbf{H}^w \ominus_{gH} c \preceq \left[-2, \tfrac{1}{2}\right]\ \forall\ y \in [-1, 1]\}. \tag{3.25}$$

Now, let $(\mathbf{H}^w, c) \in \partial^w(\mathbf{\Phi}_1 \oplus \mathbf{\Phi}_2\}(0)$, where

$$(\mathbf{\Phi}_1 \oplus \mathbf{\Phi}_2)(y) = \begin{cases} \left[y^2 - y, -\frac{1}{2}y + 3\right] & \text{if } y \in [0, 1] \\ \left[y^2 - \frac{1}{2}y, -2y + 3\right] & \text{if } y \in [-1, 0]. \end{cases}$$

There are the following two cases corresponding to $y \in [0, 1]$ and $y \in [-1, 0]$.

(i) As $y \geq 0$, we have

$$\begin{aligned} & \mathbf{H}^w \odot y \ominus_{gH} c \odot y \preceq (\mathbf{\Phi}_1 \oplus \mathbf{\Phi}_2)(y) \ominus_{gH} (\mathbf{\Phi}_1 \oplus \mathbf{\Phi}_2)(0) \\ \Longrightarrow\ & [\underline{h^w} - c, \overline{h^w} - c] \odot y \preceq \left[y^2 - y, -\tfrac{1}{2}y\right] \\ \Longrightarrow\ & \underline{h^w} - c \leq -1 \text{ and } \overline{h^w} - c \leq -\tfrac{1}{2}. \end{aligned}$$

(ii) As $-1 \leq y \leq 0$, we have

$$\begin{aligned} & [(\overline{h^w} + c)y, (\underline{h^w} + c)y] \preceq \left[y^2 - \tfrac{1}{2}y, -2y + 3\right] \ominus_{gH} [0, 3] \\ \Longrightarrow\ & [(\overline{h^w} + c)y, (\underline{h^w} + c)y] \preceq \left[y^2 - \tfrac{1}{2}y, -2y\right] \\ \Longrightarrow\ & -2 - c \leq \underline{h^w} \text{ and} - \tfrac{1}{2} - c \leq \overline{h^w}. \end{aligned}$$

Therefore, from Case (i) and Case (ii), we have

$$\begin{aligned} & \partial^w(\mathbf{\Phi}_1 \oplus \mathbf{\Phi}_2)(0) \\ = & \{(\mathbf{H}^w, c) \in I(\mathbb{R}) \times \mathbb{R}_+ : \left[-2, -\tfrac{1}{2}\right] \preceq (\mathbf{H}^w \oplus c), (\mathbf{H}^w \ominus_{gH} c) \preceq \left[-1, -\tfrac{1}{2}\right] \forall\ y \in [0, 1]\}. \end{aligned} \tag{3.26}$$

Thus, (3.25) and (3.26) are not equal.

In the following Theorem 3.6, we show that under certain conditions on $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ one-sided inclusion holds for the sum rule.

Theorem 3.6 *Let $\emptyset \neq \mathcal{Y} \subseteq \mathbb{R}^n$. Let $\mathbf{\Phi}_1 : \mathcal{Y} \to I(\mathbb{R})$ be gH-weak subdifferential at u and $\mathbf{\Phi}_2 : \mathcal{Y} \to \mathbb{R}$ be gH-Fréchet differentiable at u. Then,*

$$\partial^w(\mathbf{\Phi}_1 \oplus \mathbf{\Phi}_2)(u) \subset \partial^w\mathbf{\Phi}_1(u) \oplus \partial^w\mathbf{\Phi}_2(u),$$

provided that $w(\widehat{\boldsymbol{G}_1^w}) \leq w(\widehat{\boldsymbol{G}_2^w})$ for all $\widehat{\boldsymbol{G}_1^w} \in \partial\mathbf{\Phi}_2(y)$ and $\widehat{\boldsymbol{G}_2^w} \in \partial(\mathbf{\Phi}_1 \oplus \mathbf{\Phi}_2)(y)$, where $w(\boldsymbol{A}) = \overline{a} - \underline{a}$, is the width of the interval $\boldsymbol{A} = [\underline{a}, \overline{a}] \in I(\mathbb{R})$.

Proof: If $(\widehat{\mathbf{G}^w}, c) \in \partial^w(\mathbf{\Phi}_1 \oplus \mathbf{\Phi}_2)(u)$, then

$$\widehat{\mathbf{G}^w}^\top \odot (y-u) \ominus_{gH} c\|y-u\| \preceq (\mathbf{\Phi}_1 \oplus \mathbf{\Phi}_2)(y) \ominus_{gH} (\mathbf{\Phi}_1 \oplus \mathbf{\Phi}_2)(u). \tag{3.27}$$

We know that $\mathbf{\Phi}_2 : \mathcal{Y} \to I(\mathbb{R})$ is gH-Fréchet differentiable at u with the gH-Fréchet derivative $\mathbf{\Phi}_{2\mathscr{F}}(u)$. Hence, $\mathbf{\Phi}_{2\mathscr{F}}(u) \in \partial^-_{\mathscr{F}}\mathbf{\Phi}_2(u)$ implies $-1 \odot \mathbf{\Phi}_{2\mathscr{F}}(u) \in \partial^-_{\mathscr{F}}(-1 \odot \mathbf{\Phi}_2)(u)$. We can then express

$$\begin{aligned} & -1 \odot \mathbf{\Phi}_{2\mathscr{F}}(u) \odot (y-u) \preceq (-1 \odot \mathbf{\Phi}_2)(u) \ominus_{gH} (-1 \odot \mathbf{\Phi}_2)(u) \\ \Longrightarrow\ & -1 \odot \mathbf{\Phi}_{2\mathscr{F}}(u) \odot (y-u) \preceq -1 \odot (\mathbf{\Phi}_2(y) \ominus_{gH} \mathbf{\Phi}_2(u)) \\ & \text{by properties of } gH\text{-difference (iv) of [174].} \end{aligned} \tag{3.28}$$

In view of Lemma 1.3, (3.27) becomes

$$\widehat{\mathbf{G}^w}^\top \odot (y-u) \ominus_{gH} c\|y-u\| \preceq (\mathbf{\Phi}_1(y) \ominus_{gH} \mathbf{\Phi}_1(u)) \oplus (\mathbf{\Phi}_2(y) \ominus_{gH} \mathbf{\Phi}_2(u)).$$

Using (v) of Lemma 1.5, this inequality turns into

$$\widehat{\mathbf{G}^w}^\top \odot (y-u) \ominus_{gH} (\mathbf{\Phi}_2(y) \ominus_{gH} \mathbf{\Phi}_2(u)) \ominus_{gH} c\|y-u\| \preceq \mathbf{\Phi}_1(y) \ominus_{gH} \mathbf{\Phi}_1(u).$$

Now, from the inequality (3.28), we see that

$$\widehat{\mathbf{G}^w}^\top \odot (y-u) \ominus_{gH} \mathbf{\Phi}_{2\mathscr{F}}(u) \odot (y-u) \ominus_{gH} c\|y-u\| \preceq \mathbf{\Phi}_1(y) \ominus_{gH} \mathbf{\Phi}_1(u).$$

Thus,

$$(\widehat{\mathbf{G}^w} \ominus_{gH} \mathbf{\Phi}_{2\mathscr{F}}(u))^\top \odot (y-u) \ominus_{gH} c\|y-u\| \preceq \mathbf{\Phi}_1(y) \ominus_{gH} \mathbf{\Phi}_1(u).$$

Then, $(\widehat{\mathbf{G}^w} \ominus_{gH} \mathbf{\Phi}_{2\mathscr{F}}(u), c) \in \partial^w\mathbf{\Phi}_1(u)$ and $(\mathbf{\Phi}_{2\mathscr{F}}(u), 0) \in \partial^w\mathbf{\Phi}_2(u)$. Therefore, $(\widehat{\mathbf{G}^w}, c) \in \partial^w\mathbf{\Phi}_1(u) \oplus \partial^w\mathbf{\Phi}_2(u)$. Hence, the result follows. $\square$

Theorem 3.7 *Let $\mathcal{Y}$ be a nonempty set of $\mathbb{R}^n$. Let $\mathbf{\Phi}_1 : \mathcal{Y} \to I(\mathbb{R})$ be gH-Fréchet differentiable at u. Let $\mathbf{\Phi}_2 : \mathcal{Y} \to I(\mathbb{R})$ be an IVF. If u is a weak efficient point of $\mathbf{\Phi}_1 \oplus \mathbf{\Phi}_2$, then $(-1 \odot \mathbf{\Phi}_{1\mathscr{F}}(u), 0) \in \partial^w\mathbf{\Phi}_2(u)$.*

Proof: Since u is a weak efficient point of $\mathbf{\Phi}_1 \oplus \mathbf{\Phi}_2$, for any $y \in \mathcal{Y}$,

$$\begin{aligned} & (\mathbf{\Phi}_1 \oplus \mathbf{\Phi}_2)(u) \preceq (\mathbf{\Phi}_1 \oplus \mathbf{\Phi}_2)(y) \\ \Longrightarrow\ & \mathbf{\Phi}_1(u) \oplus \mathbf{\Phi}_2(u) \preceq \mathbf{\Phi}_1(y) \oplus \mathbf{\Phi}_2(y) \\ \Longrightarrow\ & \mathbf{\Phi}_1(u) \ominus_{gH} \mathbf{\Phi}_1(y) \preceq \mathbf{\Phi}_2(y) \ominus_{gH} \mathbf{\Phi}_2(u), \text{ using Lemma 2.3 of [68]} \\ \Longrightarrow\ & (-1) \odot \{\mathbf{\Phi}_1(y) \ominus_{gH} \mathbf{\Phi}_1(u)\} \preceq \mathbf{\Phi}_2(y) \ominus_{gH} \mathbf{\Phi}_2(u), \text{ by } \ominus_{gH} \text{ property in (iv) of [174]} \end{aligned}$$

$$\implies (-1 \odot \mathbf{\Phi}_1)(y) \ominus_{gH} (-1 \ \odot \mathbf{\Phi}_1)(u) \preceq \mathbf{\Phi}_2(y) \ominus_{gH} \mathbf{\Phi}_2(u),$$
$$\text{by } \ominus_{gH} \text{ property in (iv) of [174].} \tag{3.29}$$

By the Lemma 3.2, we also obtain that

$$(-1) \odot \mathbf{\Phi}_{1\mathscr{F}}(u) \odot (y - u) \preceq (-1 \odot \mathbf{\Phi}_1)(y) \ominus_{gH} (-1 \ \odot \mathbf{\Phi}_1)(u) \ \forall \ y \in \mathcal{Y}. \tag{3.30}$$

We get, from (3.29) and (3.30) that

$$(-1) \odot \mathbf{\Phi}_{1\mathscr{F}}(u) \odot (y - u) \preceq \mathbf{\Phi}_2(y) \ominus_{gH} \mathbf{\Phi}_2(u) \text{ by Lemma 2.3 (ii) of [68].}$$

This means that $((-1) \odot \mathbf{\Phi}_{1\mathscr{F}}(u), 0) \in \partial^w \mathbf{\Phi}_2(u)$. □

Theorem 3.8 *Let $\emptyset \neq \mathcal{Y} \subseteq \mathbb{R}^n$. Let $\mathbf{\Psi}$ be gH-Fréchet differentiable at u with the gH-Fréchet derivative $\mathbf{\Psi}_{\mathscr{F}}(u)$. Then, $\mathbf{\Psi}$ has weak efficient solution at u if and only if for any $y \in \mathcal{Y}$,*

$$\mathbf{\Psi}_{\mathscr{F}}(u)^\top \odot (y - u) = \mathbf{0}.$$

Proof: If $\mathbf{\Psi}$ has a weak efficient point at u, then

$$\begin{aligned} &\mathbf{\Psi}(u) \preceq \mathbf{\Psi}(y) \\ \text{or, } &\mathbf{0} \preceq \mathbf{\Psi}(y) \ominus_{gH} \mathbf{\Psi}(u), \ \text{by Lemma 2.1 of [64].} \end{aligned}$$

By gH-Fréchet differentiability of $\mathbf{\Psi}$ at u, we get

$$\lim_{\|h\| \to 0} \frac{\|(\mathbf{\Psi}(u+h) \ominus_{gH} \mathbf{\Psi}(u)) \ominus_{gH} \mathbf{\Psi}_{\mathscr{F}}(u)^\top \odot h\|_{I(\mathbb{R})}}{\|h\|} = 0.$$

If we take $h = \lambda(y - u)$, then

$$\lim_{\lambda \to 0} \frac{\|(\mathbf{\Psi}(u + \lambda(y-u)) \ominus_{gH} \mathbf{\Psi}(u)) \ominus_{gH} \mathbf{\Psi}_{\mathscr{F}}(u)^\top \odot \{\lambda(y-u)\}\|_{I(\mathbb{R})}}{\|\lambda(y-u)\|} = 0. \tag{3.31}$$

Since u is a weak efficient point of $\mathbf{\Psi}$, from (3.31) we have

$$\begin{aligned} &\lim_{\lambda \to 0} \frac{\|\mathbf{0} \ominus_{gH} \lambda \odot \{\mathbf{\Psi}_{\mathscr{F}}(u)^\top \odot (y-u)\}\|_{I(\mathbb{R})}}{\|\lambda(y-u)\|} \leq 0 \text{ by (i) of Lemma 1.5} \\ \implies \quad &\lim_{\lambda \to 0} \frac{\|\lambda \odot \{\mathbf{\Psi}_{\mathscr{F}}(u)^\top \odot (y-u)\}\|_{I(\mathbb{R})}}{\|\lambda(y-u)\|} \leq 0 \\ \implies \quad &\lim_{\lambda \to 0} \frac{\lambda \|\mathbf{\Psi}_{\mathscr{F}}(u)^\top \odot (y-u)\|_{I(\mathbb{R})}}{\lambda \|(y-u)\|} \leq 0. \end{aligned}$$

Since the norm gives a non-negative value,

$$\frac{1}{\|y-u\|} \odot \{\mathbf{\Psi}_{\mathscr{F}}(u)^\top \odot (y-u)\} = \mathbf{0}.$$

Thus, we obtain

$$\mathbf{\Psi}_{\mathscr{F}}(u)^\top \odot (y-u) = \mathbf{0} \text{ for any } y \in \mathcal{Y}.$$

To show the reverse part, we suppose that $\mathbf{\Psi}_{\mathscr{F}}(u)^\top \odot (y-u) = \mathbf{0}$ for all y. Then, we have $\mathbf{\Psi}_{\mathscr{F}}(u) \in \partial_{\mathscr{F}}^{-}\mathbf{\Psi}(u)$ and this clearly yields

$$\begin{aligned} \mathbf{0} = \mathbf{\Psi}_{\mathscr{F}}(u)^\top \odot (y-u) \preceq \mathbf{\Psi}(y) \ominus_{gH} \mathbf{\Psi}(u) \\ \Longrightarrow \mathbf{\Psi}(u) \preceq \mathbf{\Psi}(y) \text{ by (ii) of Lemma 2.1 in [64]}, \end{aligned}$$

and this means that u is weak efficient point of $\mathbf{\Psi}$. □

Theorem 3.9 *Let $\emptyset \neq \mathcal{Y} \subseteq \mathbb{R}^n$. If $\mathbf{\Psi}$ is gH-Fréchet differentiable at u, then $\mathbf{\Psi}$ is gH-weak subdifferentiable at u if and only if $\mathbf{\Psi}_{\mathscr{F}}(u)$ is gH-weak subdifferentiable at $0 \in \mathcal{Y}$, and*

$$\partial^w(\mathbf{\Psi}(u)) = \partial^w(\mathbf{\Psi}_{\mathscr{F}}(u)(0)).$$

Proof: By the gH-Fréchet differentiability of $\mathbf{\Psi}$ at u, we have

$$\lim_{\|h\|\to 0} \frac{1}{\|h\|} \odot \{(\mathbf{\Psi}(u+h) \ominus_{gH} \mathbf{\Psi}(u)) \ominus_{gH} \mathbf{\Psi}_{\mathscr{F}}(u)^\top \odot h\} = \mathbf{0}.$$

Inserting $h = \lambda \odot (y-u)$, by gH-weak subdifferentiability of $\mathbf{\Psi}$ at u, there exists $(\widehat{\mathbf{G}^w}, c) \in \partial^w \mathbf{\Psi}(u)$ such that for any $y \in \mathcal{Y}$,

$$\widehat{\mathbf{G}^w}^\top \odot (y-u) \ominus_{gH} c\|y-u\| \preceq \mathbf{\Psi}(y) \ominus_{gH} \mathbf{T}(u).$$

Hence,

$$\lim_{\lambda\to 0} \frac{1}{\|\lambda(y-u)\|} \odot \{(\mathbf{\Psi}(u+\lambda(y-u)) \ominus_{gH} \mathbf{\Psi}(u)) \ominus_{gH} \mathbf{\Psi}_{\mathscr{F}}(u)^\top \odot \lambda(y-u)\} = \mathbf{0}$$

and by gH-weak subdifferentiability of $\mathbf{\Psi}$ at u, we get, for any $y \in \mathcal{Y}$ that

$$\lim_{\lambda\to 0} \frac{1}{\|\lambda(y-u)\|} \odot \left\{(\widehat{\mathbf{G}^w}^\top \odot \lambda(y-u) \ominus_{gH} \lambda c\|y-u\|) \ominus_{gH} \mathbf{\Psi}_{\mathscr{F}}(u)^\top \odot \lambda(y-u)\right\} \preceq \mathbf{0},$$

(by (ii) of Lemma 1.5)

$$\implies \frac{1}{\|(y-u)\|} \odot \{(\widehat{\mathbf{G}^w}^\top \odot (y-u) \ominus_{gH} c\|y-u\|) \ominus_{gH} \mathbf{\Psi}_{\mathscr{F}}(u)^\top \odot (y-u)\} \preceq \mathbf{0}.$$

Therefore,

$$\widehat{\mathbf{G}^w}^\top \odot (y-u) \ominus_{gH} c\|y-u\| \ominus_{gH} \mathbf{\Psi}_{\mathscr{F}}(u)^\top \odot (y-u) \preceq \mathbf{0} \quad \forall\, y \in \mathcal{Y}$$

and so by letting $z = y - u$, we obtain

$$\widehat{\mathbf{G}^w}^\top \odot z \ominus_{gH} c\|z\| \preceq \mathbf{\Psi}_{\mathscr{F}}(u)^\top \odot z \;\forall\; z \in \mathcal{Y}. \tag{3.32}$$

Note that the gH-Fréchet derivative $\mathbf{\Psi}_{\mathscr{F}}(u)$ is also gH-Gáteaux derivative as in (see Theorem 5.2 of [64]). Hence, it is a linear IVF as in Definition 4.1 of [64]. By this fact, we have $\mathbf{\Psi}_{\mathscr{F}}(u)^\top \odot (0) = \mathbf{0}$. Then, the inequality (3.32) implies that $(\widehat{\mathbf{G}^w}, c) \in \partial^w(\mathbf{\Psi}_{\mathscr{F}}(u)(0))$.

Conversely, let $(\widehat{\mathbf{G}^w}, c) \in \partial^w(\mathbf{\Psi}_{\mathscr{F}}(u)(0))$. Then, we can write

$$\begin{aligned} &\widehat{\mathbf{G}^w}^\top \odot y \ominus_{gH} c\|y\| \preceq \mathbf{\Psi}_{\mathscr{F}}(u)^\top \odot y \;\forall\; y \in \mathcal{Y} \\ \implies &\widehat{\mathbf{G}^w}^\top \odot (y-u) \ominus_{gH} c\|y-u\| \preceq \mathbf{\Psi}_{\mathscr{F}}(u)^\top \odot (y-u) \;\forall\; y \in \mathcal{Y}. \end{aligned}$$

Since $\mathbf{\Psi}$ has gH-Fréchet derivative $\mathbf{\Psi}_{\mathscr{F}}(u)$ and it is also a gH-subgradient, it follows that

$$\mathbf{\Psi}_{\mathscr{F}}(y)^\top \odot (y-u) \preceq \mathbf{\Psi}(y) \ominus_{gH} \mathbf{\Psi}(u) \;\forall\; y \in \mathcal{Y}.$$

Then, $\widehat{\mathbf{G}^w}^\top \odot (y-u) \ominus_{gH} c\|y-u\| \preceq \mathbf{\Psi}(y) \ominus_{gH} \mathbf{\Psi}(u)$. Hence the proof is complete. □

Theorem 3.10 *Let $\emptyset \neq \mathcal{Y} \subseteq \mathbb{R}^n$. Let $\mathbf{\Phi}$ is gH-Fréchet differentiable at u. If u is a weak efficient point of $\mathbf{\Phi}$, then*

$$sup\left\{\widehat{\boldsymbol{G}^w}^\top \odot (y-u) \ominus_{gH} c\|y-u\| : (\widehat{\boldsymbol{G}^w}, c) \in \partial^w \mathbf{\Phi}(u)\right\} = \boldsymbol{0}.$$

Proof: First, we show that

$$\mathbf{\Phi}_{\mathscr{F}}(u)^\top \odot (y-u) = \sup\left\{\widehat{\mathbf{G}^w}^\top \odot (y-u) \ominus_{gH} c\|y-u\| : (\widehat{\mathbf{G}^w}, c) \in \partial^w \mathbf{\Phi}(u)\right\}$$

by which the desired equality can be easily proved. By gH-Fréchet differentiability of $\mathbf{\Phi}$ and by taking the supremum on the inequality (3.32), we obtain

$$\begin{aligned} \sup_{(\widehat{\mathbf{G}^w}, c) \in \partial^w \mathbf{\Phi}(u)} \{\widehat{\mathbf{G}^w}^\top \odot (y-u) \ominus_{gH} c\|y-u\|\} &\preceq \sup_{(\widehat{\mathbf{G}^w}, c) \in \partial^w \mathbf{T}(u)} \{\mathbf{\Phi}_{\mathscr{F}}(u)^\top \odot (y-u)\} \\ &= \mathbf{\Phi}_{\mathscr{F}}(u)^\top \odot (y-u). \end{aligned}$$

Since $(\mathbf{\Phi}_{\mathscr{F}}(u), 0) \in \partial^w \mathbf{\Phi}(y)$,

$$\mathbf{\Phi}_{\mathscr{F}}(u) \odot (y-u) \in \left\{ \widehat{\mathbf{G}^w}^\top \odot (y-u) \ominus_{gH} c\|y-u\| : (\widehat{\mathbf{G}^w}, c) \in \partial^w \mathbf{\Phi}(u) \right\}$$

and hence the result follows. □

3.5 Optimality for the difference of two IVFs

In this section, we consider the constrained IOP as below:

$$\min_{y \in \mathcal{Y}} \{\mathbf{\Phi}_2(y) \ominus_{gH} \mathbf{\Phi}_1(y)\}, \tag{3.33}$$

where $\emptyset \neq \mathcal{Y} \subseteq \mathbb{R}^n$ and $\mathbf{\Phi}_1, \mathbf{\Phi}_2 : \mathcal{Y} \to I(\mathbb{R})$ are two IVFs. We are going to study weak efficiency conditions for the IOP (3.33) under some additional assumptions.

Theorem 3.11 *Let $\emptyset \neq \mathcal{Y} \subseteq \mathbb{R}^n$. Let $\mathbf{\Phi}_1, \mathbf{\Phi}_2 : \mathcal{Y} \to I(\mathbb{R})$ be gH-weak subdifferentiable at u, which is a weak-efficient point of $\mathbf{\Phi}_2 \ominus_{gH} \mathbf{\Phi}_1$. If $\mathbf{\Phi}_1(u) = \mathbf{\Phi}_2(u)$, then*

$$\partial^w \mathbf{\Phi}_1(u) \subset \partial^w \mathbf{\Phi}_2(u).$$

Proof: The gH-weak subdifferentiability of $\mathbf{\Phi}_1$ at u implies that $\partial^w \mathbf{\Phi}_1(u)$ is nonempty. Hence, there exists $(\widehat{\mathbf{U}^w}, c) \in I(\mathbb{R}) \times \mathbb{R}_+$ such that

$$\widehat{\mathbf{U}^w}^\top \odot (y-u) \ominus_{gH} c\|y-u\| \preceq \mathbf{\Phi}_1(y) \ominus_{gH} \mathbf{\Phi}_1(u) \text{ for all } y \in \mathcal{Y}. \tag{3.34}$$

Since $\mathbf{\Phi}_2 \ominus_{gH} \mathbf{\Phi}_1$ gets the weak efficiency value $\mathbf{0}$ at u, for any $y \in \mathcal{Y}$, we have

$$\begin{aligned}
&\mathbf{0} \preceq (\mathbf{\Phi}_2 \ominus_{gH} \mathbf{\Phi}_1)(y) \\
\Longrightarrow\ &\mathbf{0} \preceq \mathbf{\Phi}_2(y) \ominus_{gH} \mathbf{\Phi}_1(y) \\
\Longrightarrow\ &\mathbf{\Phi}_1(y) \preceq \mathbf{\Phi}_2(y) \text{ by Lemma 2.1(ii) of [64]} \\
\Longrightarrow\ &\mathbf{\Phi}_1(y) \ominus_{gH} \mathbf{\Phi}_1(u) \preceq \mathbf{\Phi}_2(y) \ominus_{gH} \mathbf{\Phi}_2(u) \text{ by Note 2 of [68].}
\end{aligned} \tag{3.35}$$

Consequently, the inequality (3.35) implies that

$$\widehat{\mathbf{U}^w}^\top \odot (y-u) \ominus_{gH} c\|y-u\| \preceq \mathbf{\Phi}_2(y) \ominus_{gH} \mathbf{\Phi}_2(u).$$

This means $(\widehat{\mathbf{U}^w}, c) \in \partial^w \mathbf{\Phi}_2(y)$. Hence, the result follows.

□

Note 3.1 *If we had taken an efficient solution of* $\mathbf{\Phi}_2 \ominus_{gH} \mathbf{\Phi}_1$ *instead of a weak efficient solution, the additional condition* $\mathbf{\Phi}_1(u) = \mathbf{\Phi}_2(u)$ *becomes essential for Theorem 3.11 to hold. For instance, let two IVFs* $\mathbf{\Phi}_1 : \left[-\frac{1}{2}, \frac{1}{2}\right] \to I(\mathbb{R})$ *and* $\mathbf{\Phi}_2 : \left[-\frac{1}{2}, \frac{1}{2}\right] \to I(\mathbb{R})$ *be defined as*

$$\mathbf{\Phi}_1(y) = [2|y|, |y| + 1] \text{ and } \mathbf{\Phi}_2(y) = [|y|, 2y^2 + |y|],$$

respectively. Now, according to Theorem 3.11, $(\mathbf{\Phi}_2 \ominus_{gH} \mathbf{\Phi}_1)(y) = [2y^2 - 1, -|y|]$*, and* 0 *is an efficient point of* $(\mathbf{\Phi}_2 \ominus_{gH} \mathbf{\Phi}_1)$ *because* $(\mathbf{\Phi}_2 \ominus_{gH} \mathbf{\Phi}_1)(y)$ *and* $(\mathbf{\Phi}_2 \ominus_{gH} \mathbf{\Phi}_1)(0)$ *are not comparable for all* $y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$*. Note that*

$$\partial^w \mathbf{\Phi}_1(0) = \{(\boldsymbol{K}_1^w, c_1) : [-2, -1] \preceq (\boldsymbol{K}_1^w \oplus c_1), (\boldsymbol{K}_1^w \ominus_{gH} c_1) \preceq [1, 2]\}$$
$$\text{and } \partial^w \mathbf{\Phi}_2(0) = \{(\boldsymbol{K}_2^w, c_2) : [-1, -1] \preceq (\boldsymbol{K}_2^w \oplus c_2), (\boldsymbol{K}_2^w \ominus_{gH} c_2) \preceq [1, 1]\}$$

Hence, $\partial^w \mathbf{\Phi}_1(0) \not\subset \partial^w \mathbf{\Phi}_2(0)$*. So,* $\mathbf{\Phi}_1(u) = \mathbf{\Phi}_2(u)$ *is an essential condition.*

As the restriction $\mathbf{\Phi}_1(u) = \mathbf{\Phi}_2(u)$ is a bit restrictive, in the next result, we give more flexible condition for which the inclusion in Theorem 3.11 holds.

Theorem 3.12 *Let* $\emptyset \neq \mathcal{Y} \subseteq \mathbb{R}^n$*. Let* $\mathbf{\Phi}_1, \mathbf{\Phi}_2$ *have* gH*-weak subdifferential at* $u \in \mathcal{Y}$*, and* $\mathbf{\Phi}_2 \ominus_{gH} \mathbf{\Phi}_1$ *attains weak efficient solution at* u*. Then,*

$$\partial^w \mathbf{\Phi}_1(u) \subset \partial^w \mathbf{\Phi}_2(u), \tag{3.36}$$

provided that $w(\mathbf{\Phi}_1(y)) \geq w(\mathbf{\Phi}_2(y))$ *for* $y \in \mathcal{Y}$ *or* $w(\mathbf{\Phi}_1(y)) \leq w(\mathbf{\Phi}_2(y))$ *for* $y \in \mathcal{Y}$*, where* $w(\boldsymbol{A})$ *is the width of the interval* $\boldsymbol{A} \in I(\mathbb{R})$*.*

Proof: The gH-weak subdifferentiability of $\mathbf{\Phi}_1$ at u implies that $\partial^w \mathbf{\Phi}_1(u)$ is nonempty. Hence, there exists $(\widehat{\mathbf{U}^w}, c) \in I(\mathbb{R}) \times \mathbb{R}_+$ such that

$$\widehat{\mathbf{U}^w}^\top \odot (y - u) \ominus_{gH} c\|y - u\| \preceq \mathbf{\Phi}_1(y) \ominus_{gH} \mathbf{\Phi}_1(u) \text{ for all } y \in \mathcal{Y}. \tag{3.37}$$

Since u is a weak efficient point of $(\mathbf{\Phi}_2 \ominus_{gH} \mathbf{\Phi}_1)$,

$$(\mathbf{\Phi}_2 \ominus_{gH} \mathbf{\Phi}_1)(u) \preceq (\mathbf{\Phi}_2 \ominus_{gH} \mathbf{\Phi}_1)(y) \; \forall \; y \in \mathcal{Y}. \tag{3.38}$$

- Case 1. If $w(\mathbf{\Phi}_1(u)) \geq w(\mathbf{\Phi}_2(u))$, then from the inequality (3.38), for all $y \in \mathcal{Y}$, we have

$$\begin{aligned} & [\overline{\phi}_2(u) - \overline{\phi}_1(u), \underline{\phi}_2(u) - \underline{\phi}_1(u)] \preceq [\overline{\phi}_2(y) - \overline{\phi}_1(y), \underline{\phi}_2(y) - \underline{\phi}_1(y)] \\ \implies \quad & \overline{\phi}_1(y) - \overline{\phi}_1(u) \leq \overline{\phi}_2(y) - \overline{\phi}_2(u) \\ & \& \; \underline{\phi}_1(u) - \underline{\phi}_1(u) \leq \underline{\phi}_2(y) - \underline{\phi}_2(u) \end{aligned} \tag{3.39}$$

Now there arise two subcases.

- Subcase 1. If $\underline{\phi}_1(y) - \underline{\phi}_1(u) \leq \overline{\phi}_1(y) - \overline{\phi}_1(u)$, $\underline{\phi}_1(y) - \underline{\phi}_1(u) \leq \min\{\underline{\phi}_2(y) - \underline{\phi}_2(u), \overline{\phi}_2(y) - \overline{\phi}_2(u)\}$ and $\overline{\phi}_1(y) - \overline{\phi}_1(u) \leq \max\{\underline{\phi}_2(y) - \underline{\phi}_2(u), \overline{\phi}_2(y) - \overline{\phi}_2(u)\}$. Clearly, we have $[\underline{\phi}_1(y) - \underline{\phi}_1(u), \overline{\phi}_1(y) - \overline{\phi}_1(u)] \preceq [\min\{\underline{\phi}_2(y) - \underline{\phi}_2(u), \overline{\phi}_2(y) - \overline{\phi}_2(u)\}, \max\{\underline{\phi}_2(y) - \underline{\phi}_2(u), \overline{\phi}_2(y) - \overline{\phi}_2(u)\}]$.
- Subcase 2. If $\overline{\phi}_1(y) - \overline{\phi}_1(u) \leq \underline{\phi}_1(y) - \underline{\phi}_1(u)$, $\overline{\phi}_1(y) - \overline{\phi}_1(u) \leq \min\{\underline{\phi}_2(y) - \underline{\phi}_2(u), \overline{\phi}_2(y) - \overline{\phi}_2(u)\}$ and $\underline{\phi}_1(y) - \underline{\phi}_1(u) \leq \max\{\underline{\phi}_2(y) - \underline{\phi}_2(u), \overline{\phi}_2(y) - \overline{\phi}_2(u)\}$. Clearly we have $[\overline{\phi}_1(y) - \overline{\phi}_1(u), \underline{\phi}_1(y) - \underline{\phi}_1(u)] \preceq [\min\{\underline{\phi}_2(y) - \underline{\phi}_2(u), \overline{\phi}_2(y) - \overline{\phi}_2(u)\}, \max\{\underline{\phi}_2(y) - \underline{\phi}_2(u), \overline{\phi}_2(y) - \overline{\phi}_2(u)\}]$.

Combining Subcase 1 and Subcase 2, we have

$$\mathbf{\Phi}_1(y) \ominus_{gH} \mathbf{\Phi}_1(u) \preceq \mathbf{\Phi}_2(y) \ominus_{gH} \mathbf{\Phi}_2(u). \tag{3.40}$$

- Case 2. If $w(\mathbf{\Phi}_2(u)) \geq w(\mathbf{\Phi}_1(u))$, then from the inequality (3.38), for all $y \in \mathcal{Y}$, we have

$$\begin{aligned} & [\underline{\phi}_2(u) - \underline{\phi}_1(u), \overline{\phi}_2(u) - \overline{\phi}_1(u)] \preceq [\underline{\phi}_2(y) - \underline{\phi}_1(y), \overline{\phi}_2(y) - \overline{\phi}_1(y)] \\ \implies \; & \underline{\phi}_1(y) - \underline{\phi}_1(u) \leq \underline{\phi}_2(y) - \underline{\phi}_2(u) \;\&\; \overline{\phi}_1(y) - \overline{\phi}_1(u) \leq \overline{\phi}_2(y) - \overline{\phi}_2(u) \end{aligned} \tag{3.41}$$

By a similar manner as in Case 1, we have

$$\mathbf{\Phi}_1(y) \ominus_{gH} \mathbf{\Phi}_1(u) \preceq \mathbf{\Phi}_2(y) \ominus_{gH} \mathbf{\Phi}_2(u).$$

Hence, in all cases, we have

$$\mathbf{\Phi}_1(y) \ominus_{gH} \mathbf{\Phi}_1(u) \preceq \mathbf{\Phi}_2(y) \ominus_{gH} \mathbf{\Phi}_2(u). \tag{3.42}$$

In view of (3.37) and from (3.42), we get

$\widehat{\mathbf{U}^w}^\top \odot (y-u) \ominus_{gH} c\|y-u\| \preceq \mathbf{\Phi}_2(y) \ominus_{gH} \mathbf{\Phi}_2(u)$ for all $y \in \mathcal{Y}$, by Lemma 2.3 (ii) of [68].

which implies $(\widehat{\mathbf{U}^w}, c) \in \partial^w \mathbf{\Phi}_2(u)$. Hence, the result follows. □

Note 3.2 *If we had taken an efficient solution of* $\mathbf{\Phi}_2 \ominus_{gH} \mathbf{\Phi}_1$ *instead of a weak efficient solution, the additional condition* $w(\mathbf{\Phi}_1(y)) \geq w(\mathbf{\Phi}_2(y))$ *or* $w(\mathbf{\Phi}_1(y)) \leq w(\mathbf{\Phi}_2(y))$ *becomes essential for Theorem 3.12 to hold. For instance, consider the IVFs* $\mathbf{\Phi}_1 : [-1,1] \to I(\mathbb{R})$ *and* $\mathbf{\Phi}_2 : [-1,1] \to I(\mathbb{R})$ *which are defined by*

$$\mathbf{\Phi}_1(y) = \begin{cases} [y^3, y], & \text{if } 0 \leq y \leq 1 \\ [4y, y], & \text{if } -1 \leq y < 0 \end{cases} \quad \text{and} \quad \mathbf{\Phi}_2(y) = \begin{cases} [y^3, 5y], & \text{if } 0 \leq y \leq 1 \\ [3y, 2y], & \text{if } -1 \leq y < 0, \end{cases}$$

respectively. Now, according to Theorem 3.12,

$$(\mathbf{\Phi}_2 \ominus_{gH} \mathbf{\Phi}_1)(y) = \begin{cases} [0, 4y], & \text{if } 0 \leq y \leq 1 \\ [y, -y], & \text{if } -1 \leq y < 0 \end{cases}$$

gets efficient solution at 0 *because* $(\mathbf{\Phi}_2 \ominus_{gH} \mathbf{\Phi}_1)(0) \preceq (\mathbf{\Phi}_2 \ominus_{gH} \mathbf{\Phi}_1)(y)$ *for all* $y \in [0,1]$ *and* $(\mathbf{\Phi}_2 \ominus_{gH} \mathbf{\Phi}_1)(0)$ *is not comparable with the values* $(\mathbf{\Phi}_2 \ominus_{gH} \mathbf{\Phi}_1)(y)$ *for all* $y \in [-1,0]$. *It is not difficult to check that*

$$\begin{aligned} \partial^w \mathbf{\Phi}_1(0) &= \{(\boldsymbol{K}_1^w, c_1) : [1,4] \preceq (\boldsymbol{K}_1^w \oplus c_1), \boldsymbol{K}_1^w \ominus_{gH} c_1 \preceq [0,1]\} \\ \text{and} \quad \partial^w \mathbf{\Phi}_2(0) &= \{(\boldsymbol{K}_2^w, c_2) : [2,3] \preceq \boldsymbol{K}_2^w \oplus c_2, \boldsymbol{K}_2^w \ominus_{gH} c_2 \preceq [0,5]\}. \end{aligned}$$

Here, we see that $\partial^w \mathbf{\Phi}_1(0)$ *and* $\partial^w \mathbf{\Phi}_2(0)$ *are not comparable and at the same time, we notice that* $w(\mathbf{\Phi}_2(y)) \geq w(\mathbf{\Phi}_1(y))$ *on* $[0,1]$ *and* $w(\mathbf{\Phi}_1(y)) \geq w(\mathbf{\Phi}_2(y))$ *on* $[-1,0]$.

Remark 3.5.1 *In Theorem 3.12, the inclusion (3.36) is a necessary but not sufficient condition for weak efficient point of* $\mathbf{\Phi}_2 \ominus_{gH} \mathbf{\Phi}_1$. *For instance, consider the IVFs* $\mathbf{\Phi}_1 : [-1,1] \to I(\mathbb{R})$ *and* $\mathbf{\Phi}_2 : [-1,1] \to I(\mathbb{R})$ *that are defined by*

$$\mathbf{\Phi}_1(y) = \begin{cases} [y^3, y], & \text{if } 0 \leq y \leq 1 \\ [3y, 1.5y], & \text{if } -1 \leq y < 0 \end{cases} \quad \text{and} \quad \mathbf{\Phi}_2(y) = \begin{cases} [y^3 + y^2, 2y^2 + y], & \text{if } 0 \leq y \leq 1 \\ [3y, 2y], & \text{if} -1 \leq y < 0. \end{cases}$$

We notice that $w(\mathbf{\Phi}_2(y)) \geq w(\mathbf{\Phi}_1(y))$ *on* $[0,1]$ *and* $w(\mathbf{\Phi}_2(y)) \leq w(\mathbf{\Phi}_1(y))$ *on* $[-1,0]$. *Note that*

$$\begin{aligned} \partial^w \mathbf{\Phi}_1(0) &= \{(\boldsymbol{K}_1^w, c_1) : [1.5,3] \preceq \boldsymbol{K}_1^w \oplus c_1, \boldsymbol{K}_1^w \ominus_{gH} c_1 \preceq [0,1]\} \\ \text{and } \partial^w \mathbf{\Phi}_2(0) &= \{(\boldsymbol{K}_2^w, c_2) : [2,3] \preceq \boldsymbol{K}_2^w \oplus c_2, \boldsymbol{K}_2^w \ominus_{gH} c_2 \preceq [0,1]\}. \end{aligned}$$

Hence, $\partial^w \mathbf{\Phi}_1(0) \subset \partial^w \mathbf{\Phi}_2(0)$ *but* 0 *is not a weak efficient point of* $\mathbf{\Phi}_2 \ominus_{gH} \mathbf{\Phi}_1$ *on* $[-1,1]$.

Next, we study a relation between the augmented normal cone and gH-weak subdifferential. So, let us define the augmented normal cone to $\mathcal{Y}$ as below.

Definition 3.4 (Augmented normal cone). *An* augmented normal cone *to $\mathcal{Y}$ at u is*

$$\mathcal{N}^c_{\mathcal{Y}}(u) = \left\{(\widehat{\boldsymbol{G}}, c) \in I(\mathbb{R})^n \times \mathbb{R}_+ : \widehat{\boldsymbol{G}}^\top \odot (y-u) \ominus_{gH} c\|y-u\| \preceq \boldsymbol{0} \;\forall\; y \in \mathcal{Y}\right\}.$$

Theorem 3.13 (Optimality condition via augmented normal cone). *An IVF $\boldsymbol{\Psi} : \mathcal{Y} \to I(\mathbb{R})$ attains weak efficient solution at u if and only if $(\boldsymbol{0}, 0) \in \partial^w \boldsymbol{\Psi}(u) \oplus \mathcal{N}^c_{\mathcal{Y}}(u)$, where $(\boldsymbol{0}, 0)$ denotes the zero of $I(\mathbb{R}) \times \mathbb{R}_+$.*

Proof: Since u is a weak efficient point of $\boldsymbol{\Psi}$ on $\mathcal{Y}$,

$$\begin{aligned}
&\boldsymbol{\Psi}(u) \preceq \boldsymbol{\Psi}(y) \;\forall\; y \in \mathcal{Y} \\
\Longrightarrow\; &\mathbf{0} \preceq \boldsymbol{\Psi}(y) \ominus_{gH} \boldsymbol{\Psi}(u) \;\forall\; y \in \mathcal{Y} \;\text{ by Lemma 2.1(ii) of [64]} \\
\Longrightarrow\; &(\mathbf{0}, 0) \in \partial^w \boldsymbol{\Psi}(u).
\end{aligned}$$

Let $\delta_{\mathcal{Y}} : \mathcal{Y} \to I(\mathbb{R})$ be an indicator function, defined by $\delta_{\mathcal{Y}}(y) = \begin{cases} \mathbf{0}, & \text{for } y \in \mathcal{Y} \\ \infty, & \text{for } y \notin \mathcal{Y} \end{cases}$.
Since

$$(\boldsymbol{\Psi} \oplus \delta_{\mathcal{Y}})(y) = \begin{cases} \boldsymbol{\Psi}(y) & \text{if } y \in \mathcal{Y} \\ \infty & \text{if } y \notin \mathcal{Y}, \end{cases}$$

$(\mathbf{0}, 0) \in \partial^w \boldsymbol{\Psi}(u) = \partial^w (\boldsymbol{\Psi} \oplus \delta_{\mathcal{Y}})(u)$. It needs to show that $\partial^w (\boldsymbol{\Psi} \oplus \delta_{\mathcal{Y}})(u) \subset \partial^w \boldsymbol{\Psi}(u) \oplus \mathcal{N}^c_{\mathcal{Y}}(u)$. To prove this, let $\widehat{\mathbf{G}^w} \in \partial^w (\boldsymbol{\Psi}_1 \oplus \delta_{\mathcal{Y}})(u)$. Then,

$$\begin{aligned}
&\widehat{\mathbf{G}^w}^\top \odot (y-u) \ominus_{gH} c\|y-u\| \preceq (\boldsymbol{\Psi} \oplus \delta_{\mathcal{Y}})(y) \ominus_{gH} (\boldsymbol{\Psi} \oplus \delta_{\mathcal{Y}})(u) \\
\Longrightarrow\; &\widehat{\mathbf{G}^w}^\top \odot (y-u) \ominus_{gH} c\|y-u\| \preceq (\boldsymbol{\Psi}(y) \oplus \delta_{\mathcal{Y}}(y)) \ominus_{gH} (\boldsymbol{\Psi}(u) \oplus \delta_{\mathcal{Y}}(u)) \\
\Longrightarrow\; &\widehat{\mathbf{G}^w}^\top \odot (y-u) \ominus_{gH} c\|y-u\| \preceq \boldsymbol{\Psi}(y) \ominus_{gH} \boldsymbol{\Psi}(u),
\end{aligned}$$

which implies $\widehat{\mathbf{G}^w} \in \partial^w \boldsymbol{\Psi}(u) \subset \partial^w \boldsymbol{\Psi}(u) \oplus \partial^w \delta_{\mathcal{Y}}(u)$, where $\{(\mathbf{0}, 0)\} \subset \partial^w \delta_{\mathcal{Y}}(u)$. Hence, $\widehat{\mathbf{G}^w} \in \partial^w \boldsymbol{\Psi}(u) \oplus \partial^w \delta_{\mathcal{Y}}(u) = \partial^w \boldsymbol{\Psi}(u) \oplus \mathcal{N}^c_{\mathcal{Y}}(u)$.
To show the converse part, let $(\mathbf{0}, 0) \in \partial^w \boldsymbol{\Psi}(u) \oplus \mathcal{N}^c_{\mathcal{Y}}(u) = \partial^w (\boldsymbol{\Psi}(u) \oplus \delta_{\mathcal{Y}}(u))$. Now, for any $y \in \mathcal{Y}$,

$$\begin{aligned}
&\mathbf{0} \odot (y-u) \ominus_{gH} 0\|y-u\| \preceq (\boldsymbol{\Psi}(y) \oplus \delta_{\mathcal{Y}}(y)) \ominus_{gH} (\boldsymbol{\Psi}(u) \oplus \delta_{\mathcal{Y}}(u)) \\
\text{or, } &\mathbf{0} \preceq \boldsymbol{\Psi}(y) \ominus_{gH} \boldsymbol{\Psi}(u) \\
\text{or, } &\boldsymbol{\Psi}(u) \preceq \boldsymbol{\Psi}(y) \text{ by Lemma 2.1(ii) of [64]}.
\end{aligned}$$

So, u is a weak efficient solution of $\boldsymbol{\Psi}$. □

3.6 gH-directional derivative and gH-weak subdifferential for IVFs

In this section, we explore a connection between gh-directional derivative and gH-weak subdifferential for IVFs using the supremum relation, which aids in assessing the existence of efficient solutions for nonconvex IVFs. Leveraging this proposed relation, we introduce the $\mathcal{W}$-gH-weak subgradient method in obtaining a weakly efficient solution for unconstrained IOPs in the subsequent section.

Lemma 3.3 *Let $\mathcal{Y} \subseteq \mathbb{R}^n$ be starshaped at $u \in \mathcal{Y}$. Let at u, the IVF $\mathbf{\Phi} : \mathcal{Y} \to I(\mathbb{R})$ admits gH-Directional derivative in every direction $y - u$ for any $y \in \mathcal{Y}$, satisfying*

$$\mathbf{\Phi}_{\mathscr{D}}(u; y-u) \preceq \mathbf{\Phi}(y) \ominus_{gH} \mathbf{\Phi}(u) \;\forall\; y \in \mathcal{Y}. \tag{3.43}$$

Then, u is a weak efficient point of $\mathbf{\Phi}$ over $\mathcal{Y}$ if and only if

$$\mathbf{0} \preceq \mathbf{\Phi}_{\mathscr{D}}(u; y-u) \;\forall\; y \in \mathcal{Y}. \tag{3.44}$$

Proof: Assume that condition (3.44) holds. Thus, by using (3.43), we have $\mathbf{0} \preceq \mathbf{\Phi}(y) \ominus_{gH} \mathbf{\Phi}(u)$ for all $y \in \mathcal{Y}$, which leads to that u is a weak efficient point of $\mathbf{\Phi}$ over $\mathcal{Y}$. It is provided that for all $y \in \mathcal{Y}$, $\mathbf{\Phi}_{\mathscr{D}}(u; y-u)$ exists. Then,

$$\mathbf{\Phi}_{\mathscr{D}}(u; y-u) = \lim_{\beta \to 0} \frac{1}{\beta} \odot [\mathbf{\Phi}(u + \beta(y-u) \ominus_{gH} \mathbf{\Phi}(u))]. \tag{3.45}$$

Because u is a weak efficient point of $\mathbf{\Phi}$ on $\mathcal{Y}$, we can conclude that $\mathbf{0} \preceq \mathbf{\Phi}_{\mathscr{D}}(u; y-u)$. □

Theorem 3.14 *Let all the suppositions of Lemma 3.3 are met. In addition, Let at u, the gH-Directional derivative $\mathbf{\Phi}_{\mathscr{D}}(u, \cdot)$ be gH-lower semicontinuous on $\mathcal{K} = cone(\mathcal{Y} - u)$, with the property that*

$$-\boldsymbol{\infty} \prec \inf\{\mathbf{\Phi}_{\mathscr{D}}(u; h) : h \in \mathcal{K} \cap \mathcal{U}\}, \tag{3.46}$$

where $\mathcal{U} = \{v \in \mathbb{R}^n : \|v\| = 1\}$. Then, $\mathbf{\Phi}$ is gH-weak subdifferentiable at u on $\mathcal{Y}$, i.e., $\partial^w_{\mathcal{Y}} \mathbf{\Phi}(u)$ is nonempty and

$$\mathbf{\Phi}_{\mathscr{D}}(u; h) = \sup\{\widehat{\boldsymbol{G}^w}^\top \odot h \ominus_{gH} c\|h\| : (\widehat{\boldsymbol{G}^w}, c) \in \partial^w_{\mathcal{Y}} \mathbf{\Phi}(u),\; c > 0\} \;\forall\; h \in \mathcal{K}. \tag{3.47}$$

Proof: For simplicity, we suppose

$$\mathbf{\Psi}(h) = \mathbf{\Phi}_{\mathscr{D}}(u; h) \;\forall\; h \in \mathcal{K}$$

Clearly, for $\alpha \geq 0$,

$$\begin{aligned}\mathbf{\Psi}(\alpha h) &= \mathbf{\Phi}_{\mathscr{D}}(u;\alpha h) = \lim_{\beta\to 0}\frac{1}{\beta}\odot[\mathbf{\Phi}(u+(\beta\alpha)h)\ominus_{gH}\mathbf{\Phi}(u)]\\ &= \alpha\odot\lim_{\beta\to 0}\frac{1}{\beta\alpha}\odot[\mathbf{\Phi}(u+(\beta\alpha)h)\ominus_{gH}\mathbf{\Phi}(u)] = \alpha\odot\mathbf{\Phi}_{\mathscr{D}}(u;h) = \alpha\odot\mathbf{\Phi}(h).\end{aligned}$$

So, $\mathbf{\Psi}$ is a homogeneous, nonnegative IVF and $\mathbf{\Psi}(0) = \mathbf{0}$. By the hypothesis, $\mathbf{\Psi}$ admits lower bounded on $\mathcal{K}\cap\mathcal{U}$. Due to this fact, for any given $\widehat{\mathbf{G}^w} \in I(\mathbb{R})^n$, the relation

$$\widehat{\mathbf{G}^w}^\top \odot h \ominus_{gH} c\|h\| \preceq \mathbf{\Psi}(h)\ominus_{gH}\mathbf{\Psi}(0)\ \forall\ h\in\mathcal{K}\cap\mathcal{U} \tag{3.48}$$

will follow for sufficiently large c. The inequality (3.48) indicates that $(\widehat{\mathbf{G}^w}, c) \in \partial^w_{\mathcal{Y}-u}\mathbf{\Psi}(0)$, that means $\mathbf{\Psi}$ is gH-weak subdifferentiable on $\mathcal{Y}-u$ at 0. So, $\partial^w_{\mathcal{Y}-u}\mathbf{\Psi}(0)$ is nonempty. Now it is remaining to show that

$$\partial^w_{\mathcal{Y}}\mathbf{\Phi}(\bar{y}) = \partial^w_{\mathcal{Y}-u}\mathbf{\Psi}(0). \tag{3.49}$$

Let $(\widehat{\mathbf{G}^w}, c) \in \partial^w_{\mathcal{Y}-u}\mathbf{\Psi}(0)$. Thus, from (3.43) and (3.48), it implies that (3.1) is fulfilled, i.e., $(\widehat{\mathbf{G}^w}, c) \in \partial^w_{\mathcal{Y}}\mathbf{\Phi}(u)$. To prove the converse equality (3.49), let us start by taking $(\widehat{\mathbf{G}^w}, c) \in \partial^w_{\mathcal{Y}}\mathbf{\Phi}(u)$. Then for any fixed $y \in \mathcal{Y}$, we get

$$\begin{aligned}\mathbf{\Psi}(y-u) &= \mathbf{\Phi}_{\mathscr{D}}(u;y-u)\\ &= \lim_{\beta\to 0+}\frac{1}{\beta}\odot[\mathbf{\Phi}(u+\beta(y-u)\ominus_{gH}\mathbf{\Phi}(u)]\\ &= \lim_{\beta\to 0+}\frac{1}{\beta}\odot\Bigg[\min\Big\{\underline{\phi}(u+\beta(y-u))-\underline{\phi}(u), \overline{\phi}(u+\beta(y-u))-\overline{\phi}(\bar{y})\Big\},\\ &\qquad\max\Big\{\underline{\phi}(u+\beta(y-u))-\underline{\phi}(u), \overline{\phi}(u+\beta(y-u))-\overline{\phi}(u)\Big\}\Bigg].\end{aligned} \tag{3.50}$$

Let $(y-u)$ has the first m nonnegative components and the remaining $n-m$ negative components. Then, according to the definition of weak subgradient on $\underline{\phi}$ and $\overline{\phi}$, we have $c_1 > 0$ and $c_2 > 0$ such that

$$\sum_{i=1}^{m}\beta(y_i-u_i)\underline{g}_i + \sum_{j=m+1}^{n}\beta(y_j-u_j)\underline{g}_j - \lambda c_1\|y-u\| \leq \underline{\phi}(u+\beta(y-u))-\underline{\phi}(u) \tag{3.51}$$

and

$$\sum_{i=1}^{m}\beta(y_i-u_i)\overline{g}_i+\sum_{j=m+1}^{n}\beta(y_j-u_j)\overline{g}_j-\lambda c_2\|y-u\|\leq\overline{\phi}(u+\beta(y-u))-\overline{\phi}(u). \tag{3.52}$$

With the aid of (3.51) and (3.52), (3.50) splits into two cases.

- Subcase 1.

$$\begin{aligned}
&\lim_{\beta\to 0+}\frac{1}{\beta}\odot[\underline{\phi}(u+\beta(y-u))-\underline{\phi}(u),\overline{\phi}(u+\beta(y-u))-\overline{\phi}(u)]\\
&=\mathbf{\Psi}(y-u)\\
&\Longrightarrow\lim_{\beta\to 0+}\frac{1}{\beta}\odot\left[\sum_{i=1}^{m}\beta(y_i-u_i)\underline{g}_i+\sum_{j=m+1}^{n}\beta(y_j-u_j)\underline{g}_j-\beta c_1\|y-u\|,\right.\\
&\left.\sum_{i=1}^{m}\beta(y_i-u_i)\overline{g}_i+\sum_{j=m+1}^{n}\beta(y_j-u_j)\overline{g}_j-\beta c_2\|y-u\|\right]\preceq\mathbf{\Psi}(y-u)\\
&\Longrightarrow\bigoplus_{i=1}^{m}[\underline{g}_i,\overline{g}_i]\odot(y_i-u_i)\bigoplus_{j=m+1}^{n}[\overline{g}_j,\underline{g}_j]\odot(y_j-u_j)\ominus_{gH}\max\{c_1,c_2\}\|y-u\|\\
&\preceq\ \mathbf{\Psi}(y-u).
\end{aligned}\tag{3.53}$$

- Subcase 2.

$$\lim_{\beta\to 0^+}\frac{1}{\beta}\odot[\underline{\phi}(u+\beta(y-u))-\underline{\phi}(u),\overline{\phi}(u+\beta(y-u))-\overline{\phi}(u)] \tag{3.54}$$

$$\begin{aligned}
&=\mathbf{\Psi}(y-u)\\
&\Longrightarrow\bigoplus_{i=1}^{m}[\underline{g}_i,\overline{g}_i]\odot(y_i-u_i)\bigoplus_{j=m+1}^{n}[\overline{g}_j,\underline{g}_j]\odot(y_j-u_j)\ominus_{gH}\max\{c_1,c_2\}\|y-u\|\\
&\preceq\ \mathbf{\Psi}(y-u).
\end{aligned}\tag{3.55}$$

By assembling (3.53) and (3.55), we obtain

$$\bigoplus_{i=1}^{m}(y_i-u_i)^\top\odot\mathbf{G}_i^w\bigoplus_{j=m+1}^{n}(y_j-u_j)^\top\odot\mathbf{G}_j^w\ominus_{gH}c\|y-u\|\preceq\mathbf{\Psi}(y-u),$$
$$\text{where } c=\max\{c_1,c_2\},$$

which implies

$$\widehat{\mathbf{G}^w}^\top\odot(y-u)\ominus_{gH}c\|y-u\|\preceq\mathbf{\Psi}(y-u)$$

that leads to (3.48); that is $(\widehat{\mathbf{G}^w}, c) \in \partial^w_{\mathcal{Y}-u}\mathbf{\Psi}(0)$.

Now we prove that

$$\mathbf{\Psi}(h) = \sup\{\widehat{\mathbf{G}^w}^\top \odot h \ominus_{gH} c\|h\| : (\widehat{\mathbf{G}^w}, c) \in \partial^w_{\mathcal{Y}-u}\mathbf{\Psi}(0),\ c \geq 0\}\ \forall\ h \in \mathcal{K}. \tag{3.56}$$

Suppose $h = 0$; the equality in (3.56) is trivially satisfied. Hence we focus on the case $h \neq 0$. Let $h \in \mathcal{K}$ be a point on the boundary of the unit sphere, implying that, $\|h\| = 1$; that is, $h \in \mathcal{K} \cap \mathcal{U}$. Let $\epsilon \geq 0$ be arbitrary. Our task is to prove that

$$(\mathbf{\Psi}(h) \ominus_{gH} \epsilon \oplus c) \odot h^T z \ominus_{gH} c\|z\| \preceq \mathbf{\Psi}(z)\ \forall\ z \in \mathcal{K} \cap \mathcal{U} \tag{3.57}$$

is valid for sufficiently large numbers c. Now, we proceed by assuming the contrary that there exist two sequences $\{c_n\}$ and $\{z_n\}$ with $c_n \to \infty$ and $z_n \in \mathcal{K} \cap \mathcal{U}$ such that

$$\begin{aligned} \mathbf{\Psi}(z_n) \preceq&\ (\mathbf{\Psi}(h) \ominus_{gH} \epsilon \oplus c_n) \odot h^T z_n \ominus_{gH} c_n\|z_n\| \text{ for all } n = 1, 2, \ldots \\ =&\ (\mathbf{\Psi}(h) \ominus_{gH} \epsilon) \odot h^T z_n \oplus c_n \odot (h^T z_n - 1) \text{ for all } n = 1, 2, \ldots. \end{aligned} \tag{3.58}$$

Since $\mathcal{K} \cap \mathcal{U}$ is both closed and bounded, $\{z_n\}$ must have a convergent subsequence. Without loss of generality, presume that z_n converges to $z \in \mathcal{K} \cap \mathcal{U}$.

Let $z \neq h$ and $\|h\| = 1$. Then $h^\top z \leq h^\top h = \|h\|^2 = 1$ follows. Thus, letting c_n approaches to ∞ in (3.58), we have $\mathbf{\Psi}(z) = -\infty$, which contradicts (3.46).

Thus, $z = h$ which ensures that $\|h\|^2 = 1$. By gH-lower semicontinuity of gH-Directional derivative $\mathbf{\Phi}_{\mathscr{D}}(u; h)$, we have

$$\mathbf{\Psi}(h) \preceq \liminf_{n\to\infty} \mathbf{\Psi}(z_n) \preceq (\mathbf{\Psi}(h) \ominus_{gH} \epsilon)\|h\|^2 = \mathbf{\Psi}(h) \ominus_{gH} \epsilon, \tag{3.59}$$

which leads to a contradiction.
Note that the inequality (3.57) holds true for some $c \geq 0$. Let $\widehat{\mathbf{G}^w} = (\mathbf{\Psi}(h) \ominus_{gH} \epsilon \oplus c) \odot h^\top$. Applying inequality (3.57), we conclude that $(\widehat{\mathbf{G}^w}, c) \in \partial^w_{\mathcal{Y}-u}\mathbf{\Psi}(0)$. It is evident that

$$(\mathbf{\Psi}(h) \ominus_{gH} \epsilon \oplus c) \odot h^\top h \ominus_{gH} c\|h\| \preceq \sup\{\widehat{\mathbf{G}^w}^\top \odot h \ominus_{gH} c\|h\| : (\widehat{\mathbf{G}^w}, c) \in \partial^w_{\mathcal{Y}-u}\mathbf{\Psi}(0),\ c \geq 0\}.$$

For $\|h\| = 1$, we have

$$\mathbf{\Psi}(h) \ominus_{gH} \epsilon \preceq \sup\{\widehat{\mathbf{G}^w}^\top \odot h \ominus_{gH} c\|h\| : (\widehat{\mathbf{G}^w}, c) \in \partial^w_{\mathcal{Y}-u}\mathbf{\Psi}(0),\ c \geq 0\}.$$

Since this inequality is true for every $\epsilon > 0$, we infer that

$$\mathbf{\Psi}(h) \preceq \sup\{\widehat{\mathbf{G}^w}^\top \odot h \ominus_{gH} c\|h\| : (\widehat{\mathbf{G}^w}, c) \in \partial^w_{\mathcal{Y}-u}\mathbf{\Psi}(0),\ c \geq 0\}, \tag{3.60}$$

$\widehat{\mathbf{G}^w}^\top \odot h \ominus_{gH} c\|h\| \preceq \mathbf{\Psi}(h)$ for all $(\widehat{\mathbf{G}^w}, c) \in \partial^w_{\mathcal{Y}-u}\mathbf{\Psi}(0)$, which yields

$$\mathbf{\Psi}(h) = \sup\{\widehat{\mathbf{G}^w}^\top \odot h \ominus_{gH} c\|h\| : (\widehat{\mathbf{G}^w}, c) \in \partial^w_{\mathcal{Y}-u}\mathbf{\Psi}(0),\ c \geq 0\}. \tag{3.61}$$

Thus, (3.56) is true. Then, (3.48) is followed by (3.49) and (3.57), which completes the proof of the theorem. □

3.7 $\mathcal{W}$-gH-weak subgradient method

In this section, we present a $\mathcal{W}$-gH-weak subgradient method for achieving a weakly efficient solution to the following unconstrained IOP:

$$\min_{y \in \mathbb{R}^n} \mathbf{\Phi}(y), \tag{3.62}$$

where $\mathbf{\Phi} : \mathbb{R}^n \to I(\mathbb{R})$ is a nonsmooth, nonconvex gH-Lipschitz continuous IVF. To develop the method, we first define the weak efficient direction of an IVF.

Definition 3.5 (Weak efficient-direction). *A direction $d \in \mathbb{R}^n$ is considered a weak efficient-direction of an IVF $\mathbf{\Phi} : \mathbb{R}^n \to I(\mathbb{R})$ at $u \in \mathbb{R}^n$ if there exists a $\delta > 0$ such that*

(i) $\mathbf{\Phi}(u + \lambda d) \preceq \mathbf{\Phi}(u)$ *for all* $\lambda \in (0, \delta)$, *and*

(ii) there also exists a point $y' = u + \alpha d$ with $\alpha \in (0, \delta)$ and a positive real number $\delta' \leq \alpha$ satisfying

$$\mathbf{\Phi}(y') \preceq \mathbf{\Phi}(y' + \lambda d) \text{ for all } \lambda \in (-\delta', \delta').$$

The point y' is referred to as a weak efficient solution of $\mathbf{\Phi}$ in the direction d.

In the method proposed, like to the existing result for gH-differentiable IVF [66, Theorem 5.4], we choose the weak efficient direction $-\mathcal{W}(\widehat{\mathbf{G}^w})$, where $(\widehat{\mathbf{G}^w}, c) \in \partial^w \mathbf{\Phi}(u)$ at any point $u \in \mathbb{R}^n$ and the mapping $\mathcal{W} : I(\mathbb{R})^n \to \mathbb{R}^n$ is defined by

$$\mathcal{W}(\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_n) = (w_1\underline{b}_1 + w_2\overline{b}_1, w_1\underline{b}_2 + w_2\overline{b}_2, \ldots, w_1\underline{b}_n + w_2\overline{b}_n)^\top$$

for two given numbers $w_1, w_2 \in [0, 1]$ with $w_1 + w_2 = 1$ and $\mathbf{B}_j = [\underline{b}_j, \overline{b}_j] \in I(\mathbb{R})$. We employ the $\mathcal{W}$-map to identify the weak efficient solution. Applying $\mathcal{W}$-map, the weak efficient solution of IOP (3.62) can be determined by solving the following problem:

$$\min_{y\in\mathbb{R}^n} w_1\underline{\phi}(y) + w_2\overline{\phi}(y). \tag{3.63}$$

The reason is given below:
Clearly, $(w_1\underline{g}^w + w_2\overline{g}^w, c) \in \partial^w(w_1\underline{\phi}(y) + w_2\overline{\phi}(y))$ for any $y \in \mathbb{R}^n$, where $(\underline{g}^w, c) \in \partial^w\underline{\phi}(y)$ and $(\overline{g}^w, c) \in \partial^w\overline{\phi}(y)$. It is observed that $w_1\underline{g}^w + w_2\overline{g}^w \in [\underline{g}^w, \overline{g}^w]$, which implies $(w_1\underline{g}^w + w_2\overline{g}^w, c) \in \partial^w\mathbf{\Phi}(y)$. To confirm this, we will prove a Theorem to show that $(w_1\underline{g}^w + w_2\overline{g}^w, c) \in \partial^w\mathbf{\Phi}(y)$ is correct.

Since $\mathbf{\Phi}$ is gH-Lipschitz continuous IVF with Lipschitz constant L, it is also gH-lower Lipschitz at any $u \in \mathbb{R}^n$ as well. Therefore, the gH-weak subdifferential set of $\mathbf{\Phi}$, denoted as $\partial^w\mathbf{\Phi}(u)$, is nonempty. Additionally, assuming L is a positive real number, we define

$$\partial^w_L\mathbf{\Phi}(u) = \{(\widehat{\mathbf{G}^w}, c) \in \partial^w\mathbf{\Phi}(u) : c \leq L, j \in \mathbb{N}\} \neq \emptyset,$$

is clearly found to be compact set and $\|\widehat{\mathbf{G}^w}\| \leq l + L$ for every $(\widehat{\mathbf{G}^w}, c) \in \partial^w_L\mathbf{\Phi}(u)$. This compactness of $\partial^w_L\mathbf{\Phi}(u)$ will be utilised to develop an algorithm for the computing weak efficient direction using the computation of gH-weak subgradients at any given point. To approximate gH-weak subgradients, we make use the relation between gH-Direction derivative and gH-weak subdifferential (see Theorem 3.14) assuming that all assumptions, in Lemma 3.3 and Theorem 3.14, are true.

To describe the algorithm for computing gH-weak subgradient, we first consider the following set and sequence for using the relation (3.47):

$Q = \{\vartheta = (\vartheta_1, \vartheta_2, \ldots, \vartheta_n) \in \mathbb{R}^n : |\vartheta_j| = 1, j = 1, 2, \ldots, n\}$. For $\vartheta \in Q$, consider the sequence of n vectors $\vartheta^j = \vartheta^j(\mu), j = 1, 2, \ldots, n$ with $\mu \in (0, 1]$, where $\vartheta^j = (\mu\vartheta_1, \mu^2\vartheta_2, \ldots, \mu^j\vartheta_j, 0, \ldots, 0)$.
From the compactness of gH-weak subdifferential set $\partial^w_L\mathbf{\Phi}(u)$ and the relation (3.47), there exists a gH-weak subgradients $(\widehat{\bar{\mathbf{G}}^w}, \bar{c})$ such that

$$\mathbf{\Phi}_{\mathscr{D}}(u; \vartheta^j(\mu)) = \widehat{\bar{\mathbf{G}}^w}^{\top} \odot \vartheta^j(\mu) \ominus_{gH} \bar{c}\|\vartheta^j(\mu)\|.$$

Then, the set $\mathcal{G}_c = \{\widehat{\mathbf{G}^w} \in I(\mathbb{R})^n : (\widehat{\mathbf{G}^w}, \bar{c}) \in \partial^w_L\mathbf{\Phi}(u)\}$ is nonempty. Suppose that there is a set $\boldsymbol{\mathcal{A}} \subset \mathcal{G}_c$ such that

$$\mathbf{\Phi}_{\mathscr{D}}(u; \vartheta^j(\mu)) = \sup\{\widehat{\mathbf{G}^w}^{\top} \odot \vartheta^j(\mu) \ominus_{gH} \bar{c}\|\vartheta^j(\mu)\| : \widehat{\mathbf{G}^w} \in \mathcal{G}_c\}.$$

Next, we reconstruct a few following auxiliary sets similar to existing construction for weak subgradients (see [189] Remark 3.1): For any $\vartheta \in Q$ and $\mu > 0$,

$\boldsymbol{\mathcal{R}}_0(\vartheta) = \boldsymbol{\mathcal{A}}$,
$\boldsymbol{\mathcal{R}}_j(\vartheta) = \{\widehat{\mathbf{M}^w} = (\mathbf{M}_1^w, \mathbf{M}_2^w, \ldots, \mathbf{M}_n^w) \in \mathcal{A} : \vartheta_j \odot \mathbf{M}_j^w = \sup\{\vartheta_j \odot \mathbf{G}_j^w : \widehat{\mathbf{G}^w} = (\mathbf{G}_1^w, \mathbf{G}_2^w,$
$\ldots, \mathbf{G}_n^w) \in \boldsymbol{\mathcal{R}}_{j-1}$
$(\vartheta)\}\}$ and $\boldsymbol{\mathcal{R}}(u, \vartheta^j(\mu)) = \{\widehat{\mathbf{M}^w} \in \boldsymbol{\mathcal{A}} : \vartheta^j(\mu) \odot \widehat{\mathbf{M}} = \sup\{\vartheta^j(\mu) \odot \widehat{\mathbf{G}^w} : \widehat{\mathbf{G}^w} \in \boldsymbol{\mathcal{A}}\}$ for all $j = 1, 2, \ldots, n$.

By using this construction, we have that, for every $\vartheta^j(\mu), j = 1, 2, \ldots, n$, there is an element $\widehat{\bar{\mathbf{G}}^w} \in \boldsymbol{\mathcal{R}}(u, \vartheta^j(\mu))$ such that

$$\mathbf{\Phi}_{\mathscr{D}}(u; \vartheta^j(\mu)) = \widehat{\bar{\mathbf{G}}^w}^\top \odot \vartheta^j(\mu) \ominus_{gH} \bar{c}\|\vartheta^j(\mu)\|. \tag{3.64}$$

In the sequel, like to the existing definition in p. 1527 of [41], we are ready to define a vector $\widehat{\mathbf{G}^w}(\vartheta, \mu, \lambda) \in I(\mathbb{R})^n$ and a set $\mathcal{U}(\vartheta, \mu)$ as follows: For any given $\vartheta \in Q$, $\lambda > 0$ and $\mu > 0$, consider the following points:

$$y_0 = u, \quad y_j = y_0 + \lambda\vartheta^j(\mu), \quad j = 1, 2, \ldots, n.$$

Then, clearly $y_j = y_{j-1} + (0, \ldots, 0, \lambda\mu^j\vartheta_j, 0, \ldots)$ for every $j = 1, 2, \ldots, n$. Let $\widehat{\mathbf{G}^w} = \widehat{\mathbf{G}^w}(\vartheta, \mu, \lambda) \in I(\mathbb{R})^n$ be a vector with n coordinates:

$$\widehat{\mathbf{G}_j^w}(\vartheta, \mu, \lambda) = \frac{1}{\lambda\mu^j\vartheta_j} \odot \{\mathbf{\Phi}(y_j) \ominus_{gH} \mathbf{\Phi}(y_{j-1})\} + \frac{\bar{c}}{\vartheta_j}, \ j = 1, 2, \ldots, n.$$

For any fixed $\vartheta \in Q$ and $\mu > 0$, we define the set:

$$\mathcal{U}(\vartheta, \mu) = \{(\widehat{\mathbf{M}^w}, \bar{c}) \in I(\mathbb{R})^n \times \mathbb{R}_+ : \exists(\lambda_k \to +0, k \to +\infty), \ \widehat{\mathbf{M}^w} = \lim_{k \to \infty} \widehat{\mathbf{G}^w}(\vartheta, \mu, \lambda_k)\}.$$

We claim that $(\widehat{\mathbf{G}_j^w}, \bar{c})$ is an approximate gH-weak subgradient of $\mathbf{\Phi}$ at u, which need to satisfy the relation (3.64). To show $(\widehat{\mathbf{G}_j^w}, \bar{c})$ certainly satisfies the relation (3.64), it is sufficient to prove Theorem 3.15. This theorem will also show that $(\mathcal{W}(\widehat{\mathbf{G}_j^w}), \bar{c})$ is also an approximate gH-weak subgradient of $\mathbf{\Phi}$ at u. So, it indicates that $-\mathcal{W}(\widehat{\mathbf{G}_j^w})$ is an appropriate choice for weak efficient direction in the proposed $\mathcal{W}$-gH-weak subgradient method. Therefore, this method easily reduces to the conventional weak subgradient method of optimization problems in [189].

For establishing Theorem 3.15, we need the following two lemmas.

Lemma 3.4 *For any $\vartheta \in Q$, the set $\boldsymbol{\mathcal{R}}_n(\vartheta)$ is singleton.*

Proof: The proof is analogous to the proof of Proposition 3.1 for real-valued functions of real variables (see p. 1525 of [189]). □

Lemma 3.5 *There exist $\mu_0 > 0$ and $\widehat{\boldsymbol{M}^w} \in \boldsymbol{\mathcal{R}}_j(\vartheta)$ such that*

$$\mathbf{\Phi}_{\mathscr{D}}(u, \vartheta^j(\mu)) = \mathbf{\Phi}_{\mathscr{D}}(u, \vartheta^{j-1}(\mu)) \oplus \mu^j \vartheta_j \odot \widehat{\boldsymbol{M}^w} \ominus_{gH} \bar{c}\mu^j$$

for all $\mu \in (0, \mu_0]$ and for every $j = 1, 2, \ldots, n$.

Proof: The proof is analogous to the proof of Corollary 3.4 for real-valued functions of real-variables (see p. 1527 of [189]). □

In order to show $(\widehat{\mathbf{G}_j^w}, \bar{c})$ is an approximate gH-weak subgradient of $\mathbf{\Phi}$ at u, we establish a relationship between the sets $\mathcal{U}(\vartheta, \mu)$ and $\partial_L^w \mathbf{\Phi}(u)$ via the following theorem.

Theorem 3.15 *There exists $\mu_0 > 0$ such that $\mathcal{U}(\vartheta, \mu) \subseteq \partial_L^w \mathbf{\Phi}(u)$ $\forall$ $\mu \in (0, \mu_0]$.*

Proof: Let $\widehat{\mathbf{G}_j^w}(\vartheta, \mu, \lambda) = [\underline{g}_j^w(\vartheta, \mu, \lambda), \overline{g}_j^w(\vartheta, \mu, \lambda)] = \frac{1}{\lambda \mu^j \vartheta_j} \odot \{\mathbf{\Phi}(y_j) \ominus_{gH} \mathbf{\Phi}(y_{j-1})\} \oplus \frac{\bar{c}}{\vartheta_j}$. It implies that

$$\widehat{\mathbf{G}_j^w}(\vartheta, \mu, \lambda) \subseteq \frac{1}{\lambda \mu^j \vartheta_j} \odot \{\{\mathbf{\Phi}(y_j) \ominus_{gH} \mathbf{\Phi}(u)\} \ominus_{gH} \{\mathbf{\Phi}(y_{j-1}) \ominus_{gH} \mathbf{\Phi}(u)\}\}.$$

Since $\mathbf{\Phi}_{\mathscr{D}}(u, \vartheta^j(\mu)) = \lim_{\lambda \to +0} \frac{1}{\lambda} \odot \{\mathbf{\Phi}(y_j) \ominus_{gH} \mathbf{\Phi}(u)\}$, we have

$$\begin{aligned}
&\widehat{\mathbf{G}_j^w}(\vartheta, \mu, \lambda) \\
\subseteq &\frac{1}{\lambda \mu^j \vartheta_j} \odot \{\lambda \odot \mathbf{\Phi}_{\mathscr{D}}(u, \vartheta^j(\mu)) \ominus_{gH} \lambda \odot \mathbf{\Phi}_{\mathscr{D}}(u, \vartheta^{j-1}(\mu)) \oplus o(\lambda, \vartheta^j(\mu)) - o(\lambda, \vartheta^{j-1}(\mu))\} + \frac{\bar{c}}{\vartheta_j},
\end{aligned}$$

where $\lambda^{-1} o(\lambda, \vartheta^i) \to 0, \lambda \to +0, i = j-1, j$. Due to nonemptiness of $\boldsymbol{\mathcal{R}}_j(\vartheta)$ for all $j = 1, 2, \ldots, n$, we let $\mathbf{M} = (\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_n) = ([\underline{m}_1, \overline{m}_1], [\underline{m}_2, \overline{m}_2], \ldots, [\underline{m}_n, \overline{m}_n]) \in \boldsymbol{\mathcal{R}}_n(\vartheta)$. By Lemma 3.4, $\mathbf{M}$ is unique element of $\boldsymbol{\mathcal{R}}_n(\vartheta)$. From the definition $\boldsymbol{\mathcal{R}}_j(\vartheta)$ for all $j = 1, 2, \ldots, n$, it is clear that $\boldsymbol{\mathcal{R}}_n(\vartheta) \subseteq \boldsymbol{\mathcal{R}}_j(\vartheta)$ for all $j = 1, 2, \ldots, n$. Then, from this inclusion and Lemma 3.5, we have that there exist $\mu_0 > 0$ such that

$$\begin{aligned}
\widehat{\mathbf{G}_j^w}(\vartheta, \mu, \lambda) \subseteq &\frac{1}{\lambda \mu^j \vartheta_j} \odot \{\lambda \odot (\mu^j \vartheta_j \odot \widehat{\mathbf{M}_j^w} \ominus_{gH} \bar{c}\mu^j) + o(\lambda, \vartheta^j(\mu)) - o(\lambda, \vartheta^{j-1}(\mu))\} + \frac{\bar{c}}{\vartheta_j} \\
= &\widehat{\mathbf{M}_j^w} \ominus_{gH} \frac{\bar{c}}{\vartheta_j} \oplus \frac{o(\lambda, \vartheta^j(\mu)) - o(\lambda, \vartheta^{j-1}(\mu))}{\lambda \mu^j \vartheta^j} \oplus \frac{\bar{c}}{\vartheta_j}
\end{aligned}$$

$$= \widehat{\mathbf{M}_j^w} \oplus \frac{o(\lambda, \vartheta^j(\mu)) - o(\lambda, \vartheta^{j-1}(\mu))}{\lambda \mu^j \vartheta^j}$$

for all $\mu \in (0, \mu_0]$. Then, for any $\mu \in (0, \mu_0]$, we have

$$\lim_{\lambda \to +0} \widehat{\mathbf{G}_j^w}(\vartheta, \mu, \lambda) \ominus_{gH} \widehat{\mathbf{M}_j^w} \subseteq \{\mathbf{0}\} \implies \lim_{\lambda \to +0} \widehat{\mathbf{G}_j^w}(\vartheta, \mu, \lambda) = \widehat{\mathbf{M}_j^w}.$$

Consequently, $\lim_{\lambda \to +0} \widehat{\mathbf{G}^w}(\vartheta, \mu, \lambda) = \widehat{\mathbf{M}^w} \in \mathcal{G}_c$.
On the other hand,

$$\begin{aligned}
\mathcal{W}(\widehat{\mathbf{G}_j^w}(\vartheta, \mu, \lambda)) &= w_1 \underline{g}_j^w(\vartheta, \mu, \lambda) + w_2 \overline{g}_j^w(\vartheta, \mu, \lambda) \\
&= \frac{(w_1 \underline{\phi}(y^j) + w_2 \overline{\phi}(y^j)) - (w_1 \underline{\phi}(y^{j-1}) + w_2 \overline{\phi}(y^{j-1}))}{\lambda \mu^j \vartheta_j} + \frac{(w_1 + w_2)\bar{c}}{\vartheta_j} \\
&= w_1 \left\{ \frac{(\underline{\phi}(y^j) - \underline{\phi}(y^{j-1}))}{\lambda \mu^j \vartheta_j} + \frac{\bar{c}}{\vartheta_j} \right\} + w_2 \left\{ \frac{(\overline{\phi}(y^j) - \overline{\phi}(y^{j-1}))}{\lambda \mu^j \vartheta_j} + \frac{\bar{c}}{\vartheta_j} \right\} \\
&= \frac{w_1 \lambda \{ \underline{\phi}_{\mathscr{D}}(\lambda, \vartheta^j(\mu)) - \underline{\phi}_{\mathscr{D}}(\lambda, \vartheta^{j-1}(\mu)) \} + o(\lambda, \vartheta^j(\mu)) - o(\lambda, \vartheta^{j-1}(\mu))}{\lambda \mu^j \vartheta_j} + \frac{\bar{c}}{\vartheta_j} \\
&\quad + \frac{w_2 \lambda \{ \overline{\phi}_{\mathscr{D}}(\lambda, \vartheta^j(\mu)) - \overline{\phi}_{\mathscr{D}}(\lambda, \vartheta^{j-1}(\mu)) \} + o(\lambda, \vartheta^j(\mu)) - o(\lambda, \vartheta^{j-1}(\mu))}{\lambda \mu^j \vartheta_j} + \frac{\bar{c}}{\vartheta_j} \\
&= \frac{w_1 \lambda \{ \underline{m}_j \mu^j \vartheta_j - \bar{c} \mu^j \} + o(\lambda, \vartheta^j(\mu)) - o(\lambda, \vartheta^{j-1}(\mu))}{\lambda \mu^j \vartheta_j} + \frac{\bar{c}}{\vartheta_j} \\
&\quad + \frac{w_2 \lambda \{ \overline{m}_j \mu^j \vartheta_j - \bar{c} \mu^j \} + o(\lambda, \vartheta^j(\mu)) - o(\lambda, \vartheta^{j-1}(\mu))}{\lambda \mu^j \vartheta_j} + \frac{\bar{c}}{\vartheta_j} \\
&= \frac{\lambda \{ (w_1 \underline{m}_j + w_2 \overline{m}_j) \mu^j \vartheta_j - \bar{c} \mu^j \} + o(\lambda, \vartheta^j(\mu)) - o(\lambda, \vartheta^{j-1}(\mu))}{\lambda \mu^j \vartheta_j} + \frac{\bar{c}}{\vartheta_j}.
\end{aligned}$$

Similarly, $\lim_{\lambda \to +0} w_1 \underline{g}_j^w(\vartheta, \mu, \lambda) + w_2 \overline{g}_j^w(\vartheta, \mu, \lambda) = (w_1 \underline{m}_j + w_2 \overline{m}_j)$. Since $w_1 \underline{g}_j^w(\vartheta, \mu, \lambda) + w_2 \overline{g}_j^w(\vartheta, \mu, \lambda) \in \widehat{\mathbf{G}_j^w}(\vartheta, \mu, \lambda)$, is closed and bounded interval, then each point of $\widehat{\mathbf{G}_j^w}(\vartheta, \mu, \lambda)$ is a limit point of $\widehat{\mathbf{G}_j^w}(\vartheta, \mu, \lambda)$ and $\lim_{\lambda \to +0} w_1 \underline{g}_j^w(\vartheta, \mu, \lambda) + w_2 \overline{g}_j^w(\vartheta, \mu, \lambda) = (w_1 \underline{m}_j + w_2 \overline{m}_j) \in \widehat{\mathbf{M}_j^w}$. Therefore, $\lim_{\lambda \to +0} w_1 \underline{g}^w(\vartheta, \mu, \lambda) + w_2 \overline{g}^w(\vartheta, \mu, \lambda) = (w_1 \underline{m} + w_2 \overline{m}) \in \widehat{\mathbf{M}^w} \in \mathcal{G}_c$. □

In the Algorithm 1 below, we describe a step-by-step procedure for computing gH-Weak subgradient $(\widehat{\mathbf{G}^w}, c)$ approximately of the given IVF $\mathbf{\Phi}$ at the point $u \in \mathbb{R}^n$ based on the above assumptions, lemmas and theorems. From Algorithm 1, we obtain gH-weak subgradients of the objective IVF $\mathbf{\Phi}$ at every iteration. Algorithm 2 is initialized by choosing a point. We take the function value at this initial point and name this value UB. Algorithm 2 uses one of gH-weak subgradients obtained from Algorithm 1 for computing weak efficient direction at every iterative step and attempts to find a weak efficient solution by sequentially moving along the weak efficient direction for the

Algorithm 1 Approximate estimating of the gH-weak subgradient $(\widehat{\mathbf{G}^w}, c) \in \partial_L^w \mathbf{\Phi}(u)$.

Let $\vartheta \in Q = \{\vartheta = (\vartheta_1, \vartheta_2, \ldots, \vartheta_n) \in \mathbb{R}^n : |\vartheta_j| = 1, j = 1, 2, \ldots, n\}$ and $\lambda > 0, \mu \in (0, 1], u \in \mathbb{R}^n$.
Set $\vartheta^j(\mu) = (\vartheta_1\mu, \vartheta_2\mu^2, \ldots, \vartheta_j\mu^j, 0, \ldots, 0), j = 1, 2, \ldots, n$.
Let $y^0 = u$.
Select a number $c > 0$.
$j \leftarrow 1$.
while $j \leq n$ **do**
  $y_j = y_0 + \lambda\vartheta^j(\mu)$
  $\widehat{\mathbf{G}_j^w} = \frac{1}{\lambda\mu^j\vartheta_j} \odot \{\mathbf{\Phi}(y_j) \ominus_{gH} \mathbf{\Phi}(y_{j-1})\} + \frac{c}{\vartheta_j}$.
  $j = j + 1$
end while

diminishing stepsize. This algorithm will not stop until the function value at any point of the sequence $\{y_k\}_{k=1}^{\infty}$ is not less than UB. In the below, we present a step-by-step procedure via Algorithm 2 for finding weak efficient points for a given IOP (3.62) with the help of the above process.

Algorithm 2 $\mathcal{W}$-gH-weak subgradient method

Input: Given an initial solution $y_0 \in \mathbb{R}^n$, $w_1, w_2 \in [0, 1]$ such that $w_1 + w_2 = 1$, let the current upper bound be UB $= \mathbf{\Phi}(y_0)$, and weak efficient solution be $y_{eff} = y_0$.
Define the initial iteration and let $k \leftarrow 1$.
while $k \leq n$ **do**
  From Algorithm 1, choose a $(\widehat{\mathbf{G}_k^w}, c) \in \partial_L^w \mathbf{\Phi}(y_k)$ such that $\mathcal{W}(\widehat{\mathbf{G}_k^w}) \neq 0$ and an α_k such that

$$\alpha_k > 0, \quad \lim_{k\to\infty} \alpha_k = 0 \text{ and } \sum_{k=1}^{\infty} \alpha_k = \infty.$$

  Calculate

$$y_{k+1} = y_k - \alpha_k \mathcal{W}(\widehat{\mathbf{G}_k^w}).$$

  if $\mathbf{\Phi}(y_{k+1}) \prec$ UB **then**
  UB $= \mathbf{\Phi}(y_{k+1})$
  $y_{eff} = y_{k+1}$.
  end if
  Set $k = k + 1$
end while
return : the weak efficient solution

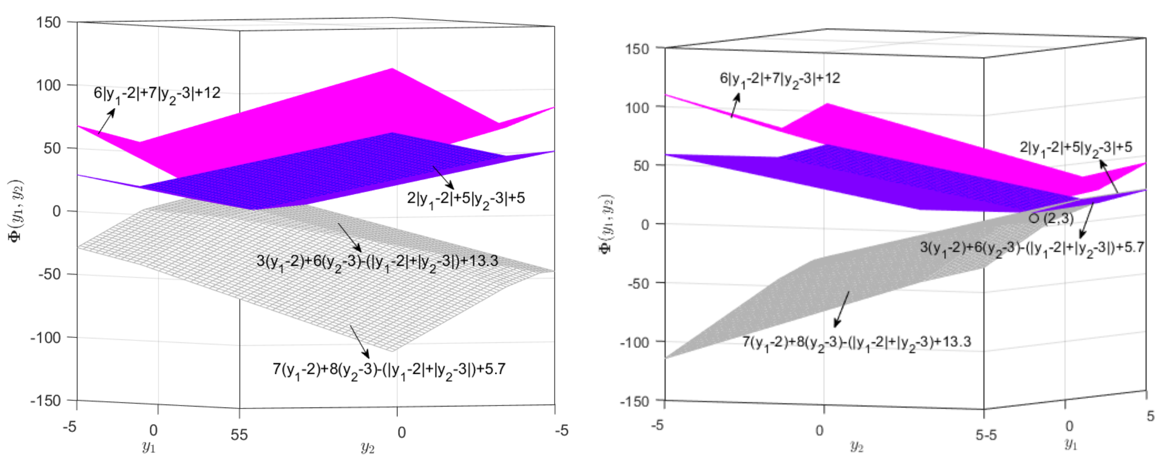
In the numerical example below, we apply the proposed Algorithm 1 to calculate a gH-weak subgradient of the objective function to the IOP.

Example 3.7.1 *Consider the following IOP:*

$$\min_{(y_1,y_2)\in\mathbb{R}^2} \mathbf{\Phi}(y_1,y_2) = [2,6]\odot|y_1-2|\oplus[5,7]\odot|y_2-3|\oplus[5,12]. \tag{3.65}$$

We solve this IOP (3.65) by the method of gH-weak subgradient method. For this, we first start Algorithm 1 with initial point $u = [2.1, 3.1]$ and perform two iterations with the parameters $\vartheta = (1,1), \lambda = 0.1, \mu = 0.5, c = 1$. Thereafter, we obtain pair of two gH-weak subgradients $(\widehat{\boldsymbol{G}^w}, c) = ((\boldsymbol{G}_1^w, \boldsymbol{G}_2^w), c) = (([3,7],\ [6,8]), 1) \in \partial^w\mathbf{\Phi}((2.1, 3.1))$ in two successive iterations.

Geometrically, $(\widehat{\boldsymbol{G}^w}, c)$ represents that there exists a concave and gH-continuous IVF $\boldsymbol{H}(y_1,y_2) = [3,7]\odot(y_1-2)\oplus[6,8]\oplus(y_2-3)-(|y_1-2|+|y_2-3|\oplus\mathbf{\Phi}(2.1,3.1) = [3,7]\odot(y_1-2)\oplus[6,8]\odot(y_2-3)-(|y_1-2|+|y_2-3|)\oplus[5.7,13.3]$, is a conic surface that coincides with some section of $\underline{\phi}(y_1,y_2) = 2|y_1-2|+5|y_2-3|+5$ and also intersects $\underline{\phi}$ at least the point $(2,3)$ from bottom.



(a) The IVFs $\mathbf{\Phi}$ and $\mathbf{H}$

(b) Weak efficient point of the IOP (3.65) in Example 3.7.1

Figure 3.2: Visualization of the IVF $\mathbf{\Phi}$ with its supporting below conic surface $\mathbf{H}$ in Example 3.7.1

Taking a gH-weak subgradient $\widehat{\boldsymbol{G}_1^w}$ in first iteration of Algorithm 1, we start Algorithm 2 with diminishing step length $\alpha_k = \frac{1}{k}$ at k-th iteration, we compute an unique weak efficient point $(2,3)$ (shown in Figure 3.2) of IOP (3.65) after four iterations for seven different combinations of w_1 and w_2 with different initial points, depicted in Table 3.1.

Table 3.1: Result of Algorithm 2 to find efficient solutions of IOP (3.65)

w_1	w_2	Initial point	Weak efficient solution
0.1	0.9	$(3.95, 4.95)$	$(2, 3)$
0.3	0.7	$(3.85, 4.85)$	$(2, 3)$
0.4	0.6	$(3.80, 4.80)$	$(2, 3)$
0.5	0.5	$(3.75, 4.75)$	$(2, 3)$
0.6	0.4	$(3.70, 4.70)$	$(2, 3)$
0.9	0.1	$(3.55, 4.55)$	$(2, 3)$
0.7	0.3	$(3.65, 4.65)$	$(2, 3)$

3.7.1 Convergence analysis of $\mathcal{W}$-gH-weak subgradient algorithm

$\mathcal{W}$-gH-weak subgradient algorithm generates the sequence of points $\{y_k\}_{k=1}^{\infty} \subseteq \mathbb{R}^n$, given by

$$y_{k+1} = y_k - \mu_k \mathcal{W}(\widehat{\mathbf{G}_k^w}), \text{ where } (\widehat{\mathbf{G}_k^w}, c_k) \in \partial^w \mathbf{\Phi}(y_k).$$

Towards the convergence of $\mathcal{W}$-gH-weak subgradient method, we need the following lemma.

Lemma 3.6 *Let $\{y_k\}$ be the sequence generated by $\mathcal{W}$-gH-weak subgradient method. Then, for all $k \geq 0$, we have*

$$\|y_{k+1} - y^*\|^2 \leq \|y_k - y^*\|^2 - 2\mu_k\{\mathcal{W}(\mathbf{\Phi}(y_k)) - \mathcal{W}(\mathbf{\Phi}(y^*)) - c_k\|y^* - y_k\|\} + \mu_k^2\|\mathcal{W}(\widehat{\boldsymbol{G}_k^w})\|^2.$$

Proof: From Definition 3.1, we have for every $(\widehat{\mathbf{G}_k^w}, c_k)$ that

$$\begin{aligned}
&\widehat{\mathbf{G}_k^w}^\top \odot (y^* - y_k) \ominus_{gH} c_k\|y^* - y_k\| \preceq \mathbf{\Phi}(y^*) \ominus_{gH} \mathbf{\Phi}(y_k)\\
\implies &\{\mathbf{\Phi}(y_k) \ominus_{gH} \mathbf{\Phi}(y^*)\} \ominus_{gH} c_k\|y^* - y_k\| \preceq \widehat{\mathbf{G}^w}^\top \odot (y_k - y^*)\\
\implies &\mathbf{\Phi}(y_k) \ominus_{gH} \mathbf{\Phi}(y^*) \preceq \widehat{\mathbf{G}^w}^\top \odot (y_k - y^*) \oplus c_k\|y^* - y_k\|\\
\implies &\mathcal{W}(\{\mathbf{\Phi}(y_k) \ominus_{gH} \mathbf{\Phi}(y^*)\}) \preceq \mathcal{W}(\widehat{\mathbf{G}^w}^\top \odot (y_k - y^*) \oplus c_k\|y^* - y_k\|). \qquad (3.66)
\end{aligned}$$

We note that

$$\mathbf{\Phi}(y_k) \ominus_{gH} \mathbf{\Phi}(y^*) = [\min\{\underline{\phi}(y_k) - \underline{\phi}(y^*), \overline{\phi}(y_k) - \overline{\phi}(y^*)\}, \max\{\underline{\phi}(y_k) - \underline{\phi}(y^*), \overline{\phi}(y_k) - \overline{\phi}(y^*)\}].$$

We now consider the following two cases.

- Case 1. If $\underline{\phi}(y_k) - \underline{\phi}(y^*) < \overline{\phi}(y_k) - \overline{\phi}(y^*)$, then

$$\begin{aligned}\mathcal{W}(\mathbf{\Phi}(y_k) \ominus_{gH} \mathbf{\Phi}(y^*)) =& w_1(\underline{\phi}(y_k) - \underline{\phi}(y^*)) + w_2(\overline{\phi}(y_k) - \overline{\phi}(y^*))\\ =& (w_1\underline{\phi}(y_k) + w_2\overline{\phi}(y_k)) - (w_1\underline{\phi}(y^*) + w_2\overline{\phi}(y^*))\\ =& \mathcal{W}(\mathbf{\Phi}(y_k)) \ominus_{gH} \mathcal{W}(\mathbf{\Phi}(y^*)).\end{aligned}$$

- Case 2. If $\overline{\phi}(y_k) - \overline{\phi}(y^*) < \underline{\phi}(y_k) - \underline{\phi}(y^*)$, then

$$\begin{aligned}\mathcal{W}(\mathbf{\Phi}(y_k) \ominus_{gH} \mathbf{\Phi}(y^*)) =& w_1(\overline{\phi}(y_k) - \overline{\phi}(y^*)) + w_2(\underline{\phi}(y_k) - \underline{\phi}(y^*))\\ =& (w_1\overline{\phi}(y_k) + w_2\underline{\phi}(y_k)) - (w_1\overline{\phi}(y^*) + w_2\underline{\phi}(y^*))\\ =& \mathcal{W}(\mathbf{\Phi}(y_k)) \ominus_{gH} \mathcal{W}(\mathbf{\Phi}(y^*)).\end{aligned}$$

Accumulating the above two cases, we have from (3.66) that

$$\begin{aligned}&\mathcal{W}(\mathbf{\Phi}(y_k)) \ominus_{gH} \mathcal{W}(\mathbf{\Phi}(y^*)) \preceq \mathcal{W}(\widehat{\mathbf{G}^w}^\top \odot (y_k - y^*) \oplus c_k\|y^* - y_k\|)\\ \implies &\mathcal{W}(\mathbf{\Phi}(y_k)) \ominus_{gH} \mathcal{W}(\mathbf{\Phi}(y^*)) \preceq \mathcal{W}(\widehat{\mathbf{G}^w})^\top (y_k - y^*) \oplus c_k\|y^* - y_k\|. \qquad (3.67)\end{aligned}$$

Using (3.67), we obtain:

$$\begin{aligned}&\|y_{k+1} - y^*\|^2 = \|y_k - \mu_k\mathcal{W}(\widehat{\mathbf{G}^w}) - y^*\|^2\\ =& \|y_k - y^*\|^2 - 2\mu_k\mathcal{W}(\widehat{\mathbf{G}^w})^\top (y_k - y^*) + \mu_k^2\|\mathcal{W}(\widehat{\mathbf{G}^w})\|^2\\ \leq& \|y_k - y^*\|^2 - 2\mu_k\{\mathcal{W}(\mathbf{\Phi}(y_k)) - \mathcal{W}(\mathbf{\Phi}(y^*)\} - c_k\|y^* - y_k\|\} + \mu_k^2\|\mathcal{W}(\widehat{\mathbf{G}_k^w})\|^2.\end{aligned}$$

□

Theorem 3.16 (Convergence analysis of $\mathcal{W}$-gH-weak subgradient method for the constant stepsize). *For the sequence $\{y_k\}$ generated by $\mathcal{W}$-gH-weak subgradient method with constant stepsize μ, we have*

(i) if $\mathbf{\Phi}(y^*) = -\boldsymbol{\infty}$, then

$$\liminf_{k\to\infty} \mathbf{\Phi}(y_k) = -\boldsymbol{\infty}, \ \text{and} \qquad (3.68)$$

(ii) if $-\boldsymbol{\infty} \prec \mathbf{\Phi}(y^*)$, then

$$\liminf_{k\to\infty} \mathbf{\Phi}(y_k) \preceq \mathbf{\Phi}(y^*) \oplus \mu\frac{(l+L)^2}{2} \oplus \liminf_{k\to\infty} c_k d_{\mathcal{Y}}, \qquad (3.69)$$

where $d_{\mathcal{Y}}$ is the diameter of $\mathcal{Y}$, denoted by $d_{\mathcal{Y}} = diam(\mathcal{Y}) = \max_{y_1,y_2\in\mathcal{Y}} \|y_1 - y_2\|$.

Proof: The statements (3.68) and (3.69) can be proven simultaneously. If possible, let there exist an $\epsilon > 0$ such that

$$\mathbf{\Phi}(y^*) \oplus \mu\frac{(l+L)^2}{2} \oplus \liminf_{k\to\infty} c_k d_{\mathcal{Y}} \oplus \epsilon \prec \liminf_{k\to\infty} \mathbf{\Phi}(y_k),$$

and let k_0 be sufficiently large such that for all $k \geq k_0$ we have

$$\begin{aligned}
&\mu\frac{(l+L)^2}{2} \oplus \epsilon \prec (\mathbf{\Phi}(y_k) \ominus_{gH} \mathbf{\Phi}(y^*)) \ominus_{gH} c_k d_{\mathcal{Y}} \\
\implies &\mu\frac{(l+L)^2}{2} \oplus \epsilon \prec \mathcal{W}((\mathbf{\Phi}(y_k) \ominus_{gH} \mathbf{\Phi}(y^*)) \ominus_{gH} c_k d_{\mathcal{Y}}) \\
\implies &\mu\frac{(l+L)^2}{2} \oplus \epsilon \prec \mathcal{W}(\mathbf{\Phi}(y_k)) \ominus_{gH} \mathcal{W}(\mathbf{\Phi}(y^*)) \ominus_{gH} c_k d_{\mathcal{Y}} \text{ by Lemma 3.6.}
\end{aligned}$$

Since $\|y_k - y^*\| \leq d_{\mathcal{Y}}$, we have, from Lemma 3.6, that

$$\begin{aligned}
&\|y_{k+1} - y^*\|^2 \\
\leq\ &\|y_k - y^*\|^2 - 2\mu\{\mathcal{W}(\mathbf{\Phi}(y_k)) - \mathcal{W}(\mathbf{\Phi}(y^*)) - c_k\|y^* - y_k\|\} + \mu^2\|\mathcal{W}(\widehat{\mathbf{G}_k^w})\|^2 \\
\leq\ &\|y_k - y^*\|^2 - 2\mu\{\mathcal{W}(\mathbf{\Phi}(y_k)) - \mathcal{W}(\mathbf{\Phi}(y^*)) - c_k d_{\mathcal{Y}}\|\} + \mu^2\|\mathcal{W}(\widehat{\mathbf{G}_k^w})\|^2 \\
\leq\ &\|y_k - y^*\|^2 - 2\mu\left[\mu\frac{(l+L)^2}{2} \oplus \epsilon\right] + \mu^2\|\widehat{\mathbf{G}_k^w}\|^2 \\
\leq\ &\|y_k - y^*\|^2 - \mu^2(l+L)^2 - 2\mu\epsilon + \mu^2(l+L)^2 \\
=\ &\|y_k - y^*\| - 2\mu\epsilon \\
\leq\ &\|y_{k-1} - y^*\| - 4\mu\epsilon \\
\leq\ &\cdots \leq \|y_{k_0} - y^*\|^2 - 2(k+1-k_0)\mu\epsilon,
\end{aligned}$$

which may not hold for k large enough, so it is a contradiction. □

Theorem 3.17 (Convergence analysis of $\mathcal{W}$-gH-weak subgradient method for the diminishing stepsize). *Let the stepsize μ_k be such that*

$$\lim_{k\to\infty} \mu_k = 0, \sum_{k=0}^{\infty} \mu_k = \infty.$$

Then, for sequence $\{y_k\}$ generated by the $\mathcal{W}$-gH-weak subgradient method with the diminishing stepsize μ_k, we have:

$$\liminf_{k\to\infty} \mathbf{\Phi}(y_k) \preceq \mathbf{\Phi}(y^*) \oplus \liminf_{k\to\infty} c_k d_{\mathcal{Y}}.$$

Proof: On contrary, if possible let there exist an $\epsilon > 0$ such that

$$\mathbf{\Phi}(y^*) \oplus \liminf_{k\to\infty} c_k d_{\mathcal{Y}} \oplus \epsilon \prec \liminf_{k\to\infty} \mathbf{\Phi}(y_k).$$

Let k_0 be sufficiently large so that for all $k \geq k_0$ we have

$$\epsilon \prec (\mathbf{\Phi}(y_k) \ominus_{gH} \mathbf{\Phi}(y^*)) \ominus_{gH} c_k d_{\mathcal{Y}}.$$

By using Lemma 3.6 and following similar steps used in the proof of Theorem 3.16, we obtain:

$$\|y_{k+1} - y^*\|^2 \leq \|y_k - y^*\|^2 - 2\mu_k \epsilon + \mu_k(\mu_k \|\mathcal{W}(\widehat{\mathbf{G}_k^w})\|^2).$$

Since $\mu_k \to 0$ and $\{\widehat{\mathbf{G}_k^w}\}$ is bounded, then k_0 is large enough so that $\mu_k \|\widehat{\mathbf{G}_k^w}\|^2 < \epsilon$ for all $k \geq k_0$. Consequently, $\mu_k \|\mathcal{W}(\widehat{\mathbf{G}_k^w})\|^2 < \epsilon$ for all $k \geq k_0$. This implies that

$$\begin{aligned}
\|y_{k+1} - y^*\|^2 &\leq \|y_k - y^*\|^2 - 2\mu_k \epsilon + \mu_k \epsilon \\
&= \|y_k - y^*\|^2 - \mu_k \epsilon \leq \|y_{k-1} - y^*\|^2 - (\mu_{k-1} + \mu_k)\epsilon \\
&\leq \cdots \leq \|y_{k_0} - y^*\|^2 - \epsilon \sum_{j=k_0}^{k} \mu_j.
\end{aligned}$$

Since $\sum_{k=0}^{\infty} \mu_k = \infty$, this relation may not hold for k sufficient large, so it leads to a contradiction. $\square$

3.8 Conclusion

In this paper, the concepts of gH-weak subdifferentials and gH-weak subgradients (Definition 3.1) for IVFs with illustrative examples have been provided. The gH-weak subdifferential set of an IVF has been found to be convex (Theorem 3.1) and closed (Theorem 3.2). We have further introduced a necessary and sufficient condition (Theorem 3.4) for the set of gH-weak subgradients to be non-empty. We have derived the necessary optimality condition (Theorem 3.10) involving gH- Fréchet differential and gH-weak subdifferential for IVFs. We have derived a necessary optimality criterion for the difference of two IVFs (Theorem 3.11 and Theorem 3.13). We have provided a necessary and sufficient condition for a weak efficient solution in terms of two notions of augmented normal cone and gH-weak subdifferential. Towards the end of the paper, we have proposed the $\mathcal{W}$-gH-weak subgradient method and its algorithmic implementations (Algorithm 1 and Algorithm 2) to obtain efficient solutions to an unconstrained IOP with the nonconvex and nonsmooth objective IVF. The convergence of the proposed method using the constant and diminishing stepsize is explained (Theorem 3.16 and Theorem 3.17).

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