Chapter 1

Introduction

1.1 Background

Neurons indeed do not operate in isolation; they are integrated into assemblies and networks, where they interact and influence each other. Artificial neural networks draw inspiration from biological neural networks and are designed as information or signal processing systems. These networks consist of a substantial number of basic processing elements referred to as nodes or neurons. Neural networks are frequently used in digital communication, such as channel identification and equalization [1], pattern recognition and classification [2], hydrology for real-time forecasting [3], quantum computation [4]. CVNN is used to model and forecast the wind signal components [5]. An OVNN is used to predict the rainfall run-off rates using rainfall time series from a raingauge near a Canadian watershed [6]. An artificial neural network has been used to forecast the carbon emission intensity in the five countries by learning through the data collected from 1980 to 2015 [7].

William James was the first to publish a number of facts related to brain structure and function [8]. He also stated the correlation learning and associative memory. After half a century later, McCulloch and Pitts published one of the famous articles, "Neural Network," in which they had derived theorems related to neuronal system models in the early 1940s [9]. Then the key contribution to the journey of neural networks was made by Donald Hebb. The development of the initial functional neuro-computer took place between 1957 and 1958, spearheaded by notable researchers such as Frank Rosenblatt and Charles Wightman, among others. Rosenblatt is widely recognized as a seminal figure in the field of neurocomputing, often referred to as the "father" of this discipline. The author's initial publication on neurocomputing was titled 'Neurodynamics.' In 1987, a significant event in the history of neural networks occurred with the organization of the first open conference on neural networks in modern times. This conference was named the IEEE International Conference on Neural Networks and took place in San Diego. As a result of this conference, the International Neural Network Society (INNS) was established.

The field of neural networks has grown since then. One can find a variety of journals and the conferences related to neural networks. The neural networks are today applied in many fields viz., in voice recognition, forecasting stock prices [10], geotechnical engineering [11], decision making and control in medical applications [12]. Therefore, it becomes an interesting area of research from application point of view. Also, it will be shown that the neural network models are expressed by the differential equations. So, the mathematical analysis of these will be of interest to mathematicians and the related people.

Stability and synchronization are important aspects of the neural networks that will be looked out in this thesis. To give a glimpse, some stability definitions are included in this section. The methods used to obtain the results are also described here.

1.2 Artificial Neural Network

Neurons are the basic functional unit of the nervous system. These help in carrying messages to different parts of the body in the form of electric signals. A neuron has three main parts viz., the cell body or soma, the dendrites and the axon.The cell body, also known as the soma, serves as the core of a neuron. It houses the nucleus and plays a pivotal role in maintaining protein synthesis. Additionally, a neuron often features multiple dendrites that extend outward in a tree-like pattern. These dendritic branches serve as receivers, gathering information from other neurons in the neural network. The synapse is the area of contact between two neurons. The shape of the neuron varies with place and functioning. In a neural network, the neurons are connected with each other through synaptic weights. Each neuron receives the weighted information by these and produces the output by passing this weighted sum through the activation function. Furthermore, the network is fully connected if every neuron is connected to every other. It is said to be partially connected if at least one pair of neurons does not have the connection between them.

The most important feature of neural networks over computers is that it can learn from its environment and updates itself. Among the many learning rules, the one that is frequently employed is the back propagation algorithm. Artificial neural networks have been applied to the problems ranging from speech recognition to weather forecasting. Human brain is unique in the sense that it learns the things by itself. Unlike electronic devices, we do not have to update the software to learn something new. By applying methods to mimic the functioning of real neurons, one can make the network to solve many problems. Neural network composed of large number of neurons working in parallel is used to solve a particular problem. Neural networks learn through examples. They can't be programmed to work for a specific task. A trained neural network can be thought of an expert in the category of information that has been given to analyse. The biological inspired methods of computing are great challenges for the computing industry. Even a simple animal brain can do far better computation than the computer. Computers are good at rote learning. But computers have great trouble even in recognising simple patterns.

The artificial neural network consists of hundreds of nodes connected with each other to form a complex communication network. Each node represents a neuron like that in the biological neural network. Each node fires when a sufficiently strong input from other nodes to which it is connected. Neural network is more than just a bunch of neurons. It is now known that even the brains of snails are well structured. Most of the neural network consists of at least three layers viz., input, hidden and output. The input layer receives the data from the device and the output layer sends the data to other device. In between them are the hidden layers, that are responsible for all the processing of the data. The neurons of the hidden layer receives the data from the input layer and after processing of the data, it send the data to the output layer.

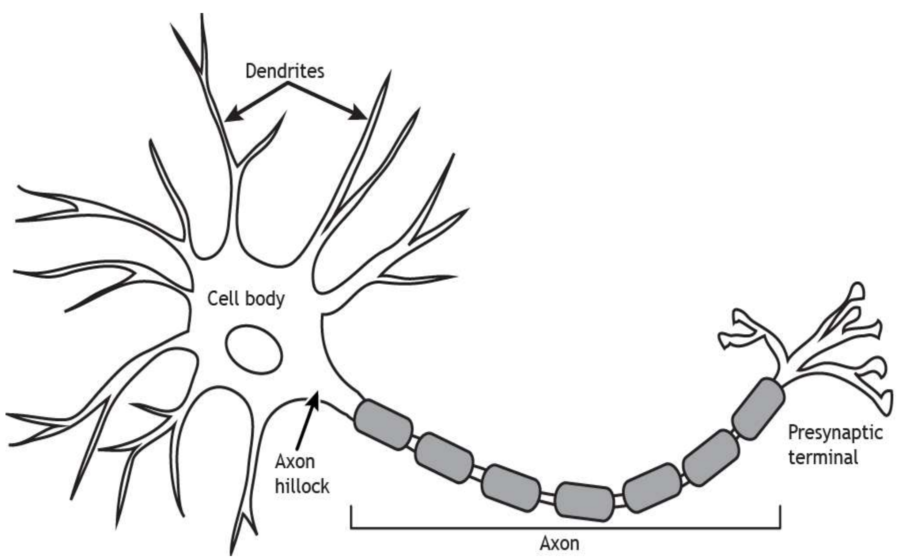

FIGURE 1.1: Basic structure of a neuron.

1.2.1 Training of a Neural Network

Once a neural network is designed for a particular task, it is ready to be trained. To start with, the random weights are chosen for that neural network. There are some approaches of training- supervised, unsupervised and reinforced learning.

1.2.1.1 Supervised Learning

In the supervised learning of the neural networks, both the inputs and the outputs are given to the neural networks. The network processes the inputs and compares the obtained outputs with the desired outputs. Then the error obtained is minimized by changing the weights. It is not the case that for every set of inputs-outputs and the neural network, we will be able to train the neural network. Sometimes, there are not enough data sets that help us to reach to the desired level of accuracy in the output or there may be problems with the neural network structure. Interestingly, some neural networks never learn at all.

One of the key algorithm of the supervised learning is the Back-propagation algorithm. This algorithm uses the gradient descent method to find the minimum of the error function. Mostly, sigmoidal function is employed while using the back-propagation algorithm.

1.2.1.2 Unsupervised Learning

In this kind of leaning for the neural networks, they are provided with the input sets but not the desired outputs. We can take the examples of working of robots as they face the changing environments and the conditions. Although, unsupervised learning is not adapted frequently. Its use is very limited.

1.2.1.3 Reinforced Learning

This type of learning is a mixture of the two types of learning. The neural network gets some outputs corresponding to the some given inputs. Then the desired outputs are rewarded as good whereas others are as bad.

1.2.2 Activation functions

Activation functions are generally used in the artificial neural network to convert a weighted input to the output of that neural network. Activation functions play a crucial role in the process of putting the input value inside the desired range. Nonlinear activation functions are predominantly utilized in consideration of their practical applicability in the real world. A neural network lacking an activation function can be seen as a linear regression model with constrained efficacy. The selection of an appropriate activation function is crucial in order to imbue the network with dynamism and equip it with the capability to extract intricate and intricate information from the provided data. This can be done by adopting a nonlinear activation function. Although the predefined activation functions are used, but the research on the trainable activation function is on the way [13]. The activation functions that are predefined fall under the category of fixed-shape activation functions. There are generally three types of activation functions that are mostly used.

1.2.2.1 Sigmoidal function

The function having a S-shaped curve is called sigmoidal function. This function is frequently used in the recent research articles. It transforms the values in the range 0 to 1. It is a bounded differentiable function. It can be defined as $\sigma(x) = \frac{1}{1+e^{-x}}, \quad -\infty < x < \infty$.

1.2.2.2 Tanh function

This hyperbolic function is defined by $tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad -\infty < x < \infty$. This is similar to sigmoidal function but symmetric around the origin.

1.2.2.3 RELU function

The acronym RELU is derived from the term "rectified linear unit." The non-linear activation function discussed herein is extensively employed in neural networks. The mathematical definition of the function can be expressed as $f(x) = max(0, x)$. The utilization of the Rectified Linear Unit (ReLU) function offers several benefits, including enhanced training speed and reduced issues related to saturation. The revised iterations of the rectified linear unit (RELU) functions encompass the leaky RELU, parametric RELU, and randomized RELU. These functions overcome the property of the RELU function of retaining very little or very high information in the storage resulting in an explosion or the dealt of the neural network [14].

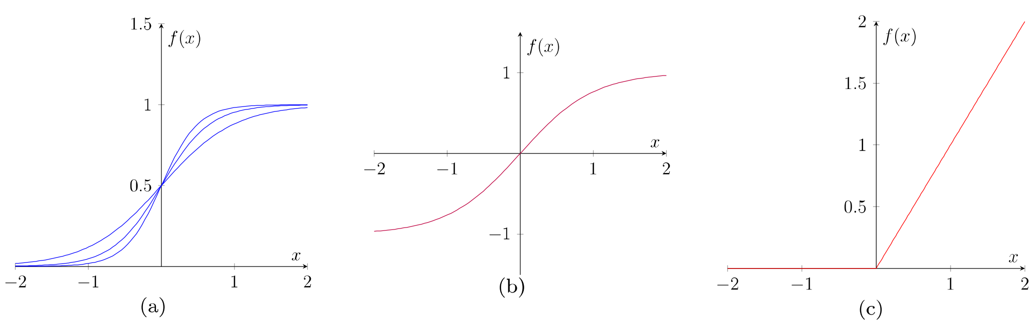

FIGURE 1.2: (a) Sigmoidal function, (b) tanh function (c) RELU function

1.2.3 Mathematical Model

Neural network consists of a number of simple node elements, which are connected together to form either a single layer or multiple layers. The relative strengths of the input signals and also of the connections between the layers are then decided as the network learns of the specific task. The activation function is applied to the weighted sum of the different inputs. The image under the activation function becomes the output for the current neuron. Sometimes, a biased as an external input is added to the weighted sum of the inputs.

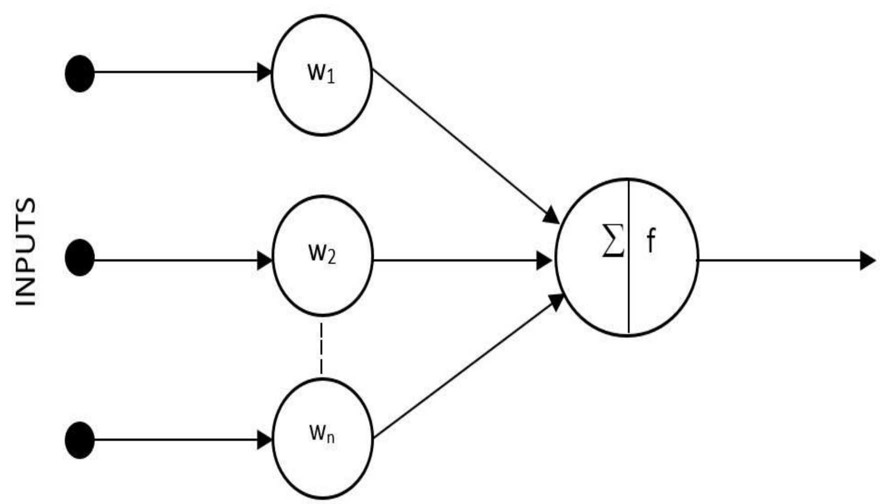

FIGURE 1.3: McCulloch-Pitts model of neural network.

1.3 Types of Neural Network

As stated above, the working of an artificial neural network is an insight into the functioning of human brain. There are various types of neural network models that are classified based on the value of the state variable. These include real valued neural networks (RVNN), complex valued neural networks (CVNN), quaternion valued neural networks (QVNN), and others. The literature contains descriptions of different models of neural networks, which vary in terms of their structure and usefulness. Below, we provide an overview of some of these models.

1.3.1 Feedforward Neural Network

Feedforward neural network is the simplest neural network. Many complex neural networks are constructed by taking this as the foundational structure. As the name suggests, in this type of neural network, the information flows in one direction only. The outlook of this kind with 3 neurons is given in Fig. 1.4. This kind of model is used in time series forcasting, mechanics etc. [15, 16].

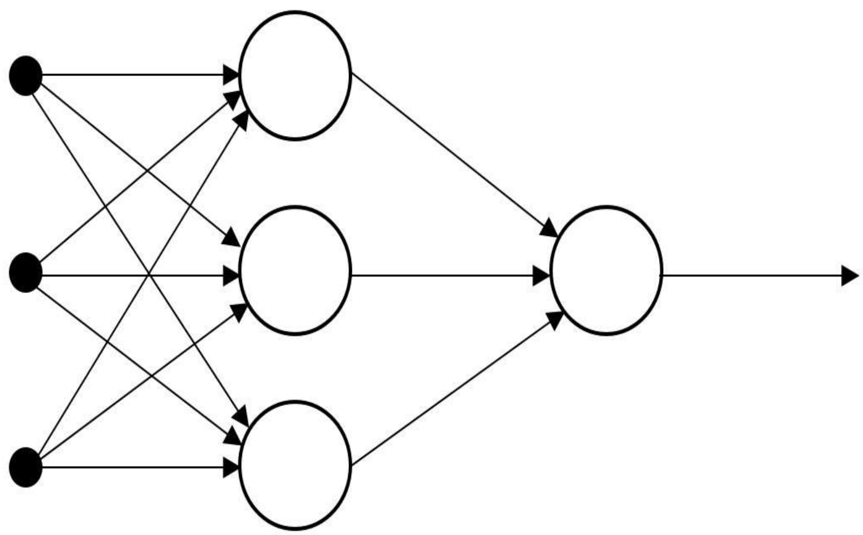

FIGURE 1.4: Feedforward Neural Network.

1.3.2 Recurrent Neural Network

This is a kind of neural network in which the information is sent back to the neurons as the input. Some examples of recurrent neural network include BAM, Hopfield, Boltzmann Machine etc. Some applications of the neural networks include control systems, robotics, language learning, robotic behavior etc. [17]

These networks can further be classified into fully connected-and partially connected

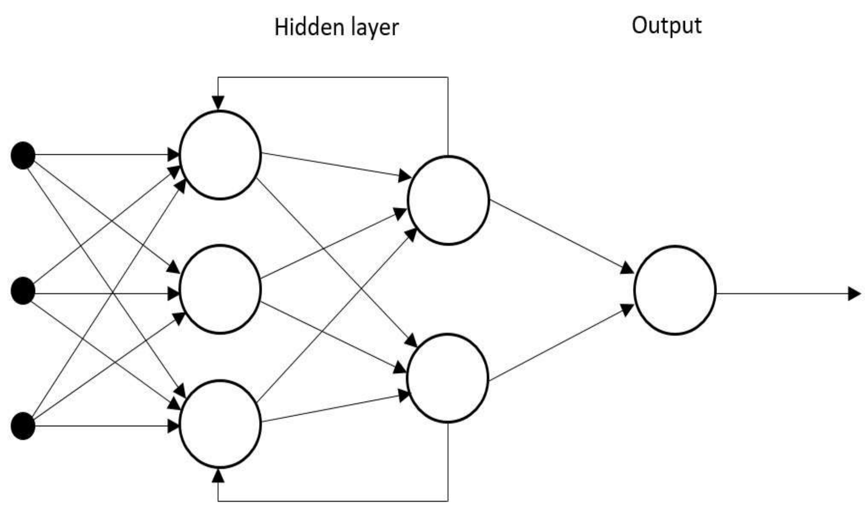

FIGURE 1.5: Recurrent Neural Network.

neural networks. In the fully connected neural network, every node is connected to every other node in the preceding and succeeding layers. While in partially connected neural networks, there is at least one pair of nodes which are not connected to each other.

1.3.2.1 Cohen-Grossberg Neural Network

The Cohen-Grossberg Neural Networks were initially proposed and investigated in 1983, as documented in [18]. This neural network framework incorporates various models derived from evolutionary theory, population biology, and neurobiology. It's worth noting that the Cohen-Grossberg neural network includes the Hopfield neural network as a particular case. It is specifically designed for the pattern recognition, learning and adaptation.

1.3.2.2 Inertial Neural Network

The inertial neural network was first proposed by Babrock and Westervelt [19]. It is given by

$$x''(t) = f(x(t), x'(t)), \tag{1.1}$$

where $x(t)$ denotes the state vector of the neural network and the second derivative of the state variable is called the inertial term.

Furthermore, the neural networks can be classified on the basis of the state vector. If the state vectors are real, these are called real-valued neural networks (RVNNs). Similarly, for complex, quaternion and octonion state variables these are called Complex-valued neural networks (CVNNs), Quaternion-valued neural networks (QVNNs) Octonion-valued neural networks (OVNNs), respectively. We are mostly aware of the addition and multiplication rules for the real, complex and quaternion numbers. But for the octonion numbers, there is a process of learning many things.

The addition and scalar multiplication in octonion numbers are defined as

$$\sum_{i=0}^{7} u^i \varrho_i + \sum_{i=0}^{7} v^i \varrho_i = \sum_{i=0}^{7} (u^i + v^i) \varrho_i, \quad \alpha(\sum_{i=0}^{7} u^i \varrho_i) = \sum_{i=0}^{7} (\alpha u^i) \varrho_i. \tag{1.2}$$

However, the multiplication of two octonion numbers is defined as

$$\Big(\sum_{i=0}^{7} u^i \varrho_i\Big)\Big(\sum_{i=0}^{7} v^i \varrho_i\Big) = \sum_{0\leq i,j\leq 7} (u^i v^j)\varrho_i\varrho_j, \tag{1.3}$$

where the multiplication of octonion units is given in Table 1, is in accordance with the article [20].

$\cdot$	ϱ_0	ϱ_1	ϱ_2	ϱ_3	ϱ_4	ϱ_5	ϱ_6	ϱ_7
ϱ_0	ϱ_0	ϱ_1	ϱ_2	ϱ_3	ϱ_4	ϱ_5	ϱ_6	ϱ_7
ϱ_1	ϱ_1	$-\varrho_0$	ϱ_3	$-\varrho_2$	ϱ_5	$-\varrho_4$	$-\varrho_7$	ϱ_6
ϱ_2	ϱ_2	$-\varrho_3$	$-\varrho_0$	ϱ_1	ϱ_6	ϱ_7	$-\varrho_4$	$-\varrho_5$
ϱ_3	ϱ_3	ϱ_2	$-\varrho_1$	$-\varrho_0$	ϱ_7	$-\varrho_6$	ϱ_5	$-\varrho_4$
ϱ_4	ϱ_4	$-\varrho_5$	$-\varrho_6$	$-\varrho_7$	$-\varrho_0$	ϱ_1	ϱ_2	ϱ_3
ϱ_5	ϱ_5	ϱ_4	$-\varrho_7$	ϱ_6	$-\varrho_1$	$-\varrho_0$	$-\varrho_3$	ϱ_2
ϱ_6	ϱ_6	ϱ_7	ϱ_4	$-\varrho_5$	$-\varrho_2$	ϱ_3	$-\varrho_0$	$-\varrho_1$
ϱ_7	ϱ_7	$-\varrho_6$	ϱ_5	ϱ_4	$-\varrho_3$	$-\varrho_2$	ϱ_1	$-\varrho_0$

Table 1: Multiplication of octonion units.

1.4 Delay Differential Equations

In the ordinary differential equations, the dependent variable and all of its derivatives are calculated at the same instant of time. The delay differential equations are the differential equations in which the derivative of the state variable at the current instant depends not only on the value of state variable and its lower derivatives at current time instant but also at the previous time. For example,

$$x'(t) = -3x(t-1) \tag{1.4}$$

A list of real life examples of Delay differential equations can be found in chapter 5 of [21]. The order of delay differential equations is defined in the same way as of ordinary differential equations. It has been noticed frequently that the stability of an equilibrium point may be lost when delays are increased and there are very less cases in which the re-stabilization is obtained after further increasing the delays [22]. Delays are the inherent part of the real world problems.

1.4.1 Types of delay

There are various kinds of delays there have discussed in the literature e.g., discrete delay, time varying delay, distributed delay etc. Even a combination of these is also used in the neural network models. The introduction of the delays in the differential equations makes it more realistic.

1.5 Impulsive system

The Impulsive dynamical system is one in which the trajectory of the system undergoes sudden changes at discrete point. Impulsive systems, due to their ability to incorporate impulses occurring at different points in time, can be categorised as time-varying systems in a generic sense. The impulsive system comprises three primary components: an ordinary differential equation that governs the continuous evolution of the system between impulses, a difference equation that determines the abrupt change in the system state at the times of impulses, and an impulse law that dictates the timing of the impulses.

1.5.0.1 Types of impulses

The heterogeneous impulses are of three types; desynchronizing impulses increase the value of the state variable at the impulsive points; synchronizing impulses reduce the value of the state variable at the impulsive point; and the last are the impulses which do not alter the values of the state variables. Now, we will be defining the concept of the average impulsive interval (AII) and average impulsive gain (AIG).

Suppose $N_{\mathcal{S}}(s,t)$ is the number of impulsive times of the impulsive sequence $\mathcal{S} = \{t_k : k \in \mathbb{N}\}$ in the time interval (s,t). Then the following inequality estimates $N_{\mathcal{S}}(s,t)$ if $\exists\, T_a > 0$ and $N_0 > 0$, such that

$$\frac{t-s}{T_a} - N_0 \leq N_{\mathcal{S}}(s,t) \leq \frac{t-s}{T_a} + N_0, \quad \forall\, t \geq s > 0, \tag{1.5}$$

where T_a is AII of the impulsive sequence $\mathcal{S}$ [23].

The average impulsive gain AIG of all impulses $\{\mu_1, \mu_2, ...\}$ is designed as

$$\mu = \lim_{t\to\infty} \frac{|\mu_1| + |\mu_2| + ... + |\mu_{N_{\mathcal{S}}(t,s)}|}{N_{\mathcal{S}}(t,s)} > 0,$$

where μ_k, $k = 1, 2, ...$ denotes impulsive gain at $t = t_k$.

1.6 Stability

In this section, some definitions of the stability of a dynamical system are given. Before that, let us define an equilibrium point.

1.6.1 Equilibrium point

A point $x^\star$ is said to be equilibrium point of the system $\dot{x}(t) = f(x,t)$ if $0 = f(x^\star, t)$, i.e. the trajectories starting at the point remain static.

1.6.2 Asymptotic stability

An equilibrium point $x^\star = 0$ of $\dot{x}(t) = f(x,t)$ is asymptotically stable at $t = t_0$ if

- $x^\star$ is stable.
- $x^\star$ is locally attractive; i.e., there exists $\delta(t_0)$ such that $\|x(t_0)\| < \delta \implies \lim\limits_{t\to\infty} x(t) = 0$.

i.e., any solution that starts close enough not only remains close to equilibrium point but eventually converges to the equilibrium point.

1.6.3 Exponential stability

The equilibrium point $x^\star$ of the dynamical system $\dot{x}(t) = f(x,t)$ is said to be exponentially stable [24], if there exist constant $\lambda > 0$ and $M > 0$ such that the following holds for all $t > 0$.

$$\|x(t) - x^\star\| \leq M\|x(t_0) - x^\star\|e^{-\lambda(t-t_0)}. \tag{1.6}$$

1.6.4 Finite and Fixed time stability

Finite-time stability was first introduced in the year 1950 [25] and the Fixed-time synchronization was given by Polyakov in 2012 to eradicate the dependence of settling time on the initial conditions [26]. Compared with the asymptotic or exponential stability, where the solution curve reaches the equilibrium point in infinite time,

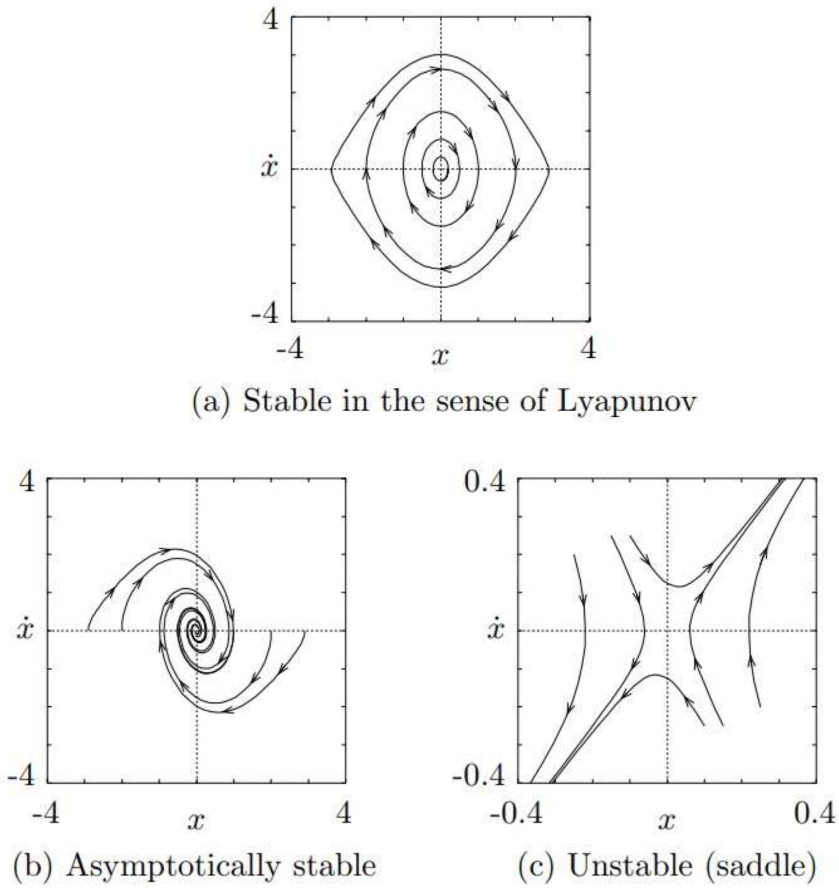

FIGURE 1.6: Phase portrait for stable and unstable equilibrium point

the trajectory of the solution curve reaches the equilibrium point in finite time. This time depends on the initial conditions in case of finite time stability, whereas this is not the case for fixed time stability. A well known example of the finite time convergence is the bang-bang time-optimal feedback control [27]. Some applications of finite and fixed time stabilities include robotics, hypersonic missiles, power systems, engineering systems and space technology [26, 28, 29]. The basic definitions of both of these stabilities are as follows

1.6.4.1 Finite time stablity

The equilibrium point of the dynamical system is said to be finite time stable [30] if for any solution $x(t)$ starting at x_0, it is asymptotically stable and $\lim_{t \to T(x_0)} \|x(t)\| = 0$ and $x(t) = 0$, $\forall t \geq T_0$, where T is called the settling time function.

1.6.4.2 Fixed time stability

The equilibrium point is called fixed time stable [30], if it is finite time stable and the settling time is bounded, i.e., $T(x_0) < K, \ \forall x_0 \in \mathbb{R}^n$, where K is a constant.

1.6.5 Predefined-time stability

The concept of fixed-time stability offers certain advantages over finite-time stability. However, it can be challenging to establish a direct relationship between the tuning gain and the fixed-time stabilization duration. A different category of dynamical systems, characterized by predefined-time stability, emerges to address this challenge. In such systems, the tuning gains provide an upper bound on the fixed-time stabilization duration. The predefined time is defined in the same way as the fixed-time stability, only difference is that the settling time depends on the systems' parameters. If suppose, ρ is the system parameter and for some constant $T_c = T_c(\rho) > 0$, the origin of the system $\dot{x} = f(x, \rho)$ is said to be predefined-time stable if it is fixed-time stable and the settling time is such that $T(x_0) \leq T_c, \ \forall x_0 \in \mathbb{R}^n$ [31]. In [32], a Lyapunov like characterization of the predefined time stability is given. The article [33], explores the altitude tracking problem of a rigid spacecraft using the pre-defined time stability theory.

1.7 Some useful methods

In this section, two important methods viz., the matrix measure method and the Lyapunov functional method have been discussed. These two methods are used in establishing the sufficient conditions for the stability or synchronization of the considered neural networks.

1.7.1 Matrix measure method

It is an important tool used in obtaining the sufficient conditions for the stability of neural network models. One important advantage of using this method is that we do not need to construct the Lyapunov functional. Let $\|x\|$ denotes the norm of the vector in $\mathbb{C}^n$ and let $\|A\|$ be the induced matrix norm. Then the matrix measure of A is denoted by $\mu_p(A)$ is given by [34]

$$\mu_p(A) = \lim_{h \to 0^+} \frac{\|A + hI\|_p - 1}{h}, \tag{1.7}$$

I is the identity matrix here and $p = 1, 2, \infty$.

The norms $\|.\|_p$ $(p = 1, 2, \infty)$ for vector $x \in \mathbb{R}^n$ and matrix $A \in \mathbb{R}^{n \times n}$ are defined as

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \qquad \|x\|_2 = \left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}}, \qquad \|x\|_\infty = \max_{1 \le i \le n} |x_i|, \tag{1.8}$$

and

$$\|A\|_1 = \max_j \sum_{i=1}^{n} |A_{ij}|, \qquad \|A\|_2 = \Big[\lambda_{max}(A^T A)\Big]^{\frac{1}{2}}, \qquad \|A\|_\infty = \max_i \sum_{j=1}^{n} |A_{ij}|. \tag{1.9}$$

Using equations (1.8) and (1.9), we obtain the corresponding matrix measures

$$\mu_1(A) = \max_j \left[a_{jj} + \sum_{i=1, i\neq j}^{n} |a_{ij}|\right], \qquad \mu_2(A) = \lambda_{max}[(A + A^T)/2],$$
$$\mu_\infty(A) = \max_i \left[a_{ii} + \sum_{j=1, j\neq i}^{n} |a_{ij}|\right], \tag{1.10}$$

where $\lambda_{max}(.)$ denotes the maximum eigenvalue of the matrix.

Also, the Coppel inequality has given the upper bound of the differential equation $\dot{y}(t) = A(t)y(t)$ in terms of the matrix measure as

$$\|y(t)\| \leq \|y(t_0)\| \exp\left(\int_{t_0}^{t} \mu(A(\tau))d\tau\right) \tag{1.11}$$

The matrix measure is well defined for any norm and satisfies many useful properties:

1. The matrix measure is convex.
2. $-\|A\|_p \leq \mu_p(A) \leq \|A\|_p$.
3. $\mu(A + B) \leq \mu(A) + \mu(B)$.
4. $\mu(\alpha\mathrm{A}) = \alpha\mu(A)$ for some non negative constant.

1.7.2 Lyapunov functional method

Lyapunov methods have demonstrated their effectiveness as valuable tools for analyzing and designing nonlinear control systems. Likewise, they have found extensive applications in estimating the convergence rates of systems that exhibit finite-time and fixed-time stabilities. This holds true for systems with predefined-time stability as well. Let $\mathbf{0}$ be the equilibrium point of the differential equation $\dot{x} = f(x)$ where $f : D \to \mathbb{R}^n$ is locally Lipschitz on domain $D \subset \mathbb{R}^n$. If there is other equilibrium point, we can use transformation to shift the equilibrium point to origin. Assume there exists a continuously differentiable function $V : D \to \mathbb{R}$ such that the followings are satisfied.

1. $V(0) = 0$ and $V(x) > 0$ for all $x \in D - \{0\}$,
2. $\dot{V}(x) \leq 0$ for all $x \in D$.

Then $\mathbf{x} = \mathbf{0}$ is stable in the sense of Lyapunov.

It should be noted here that if the derivative of V is negative, then the origin is asymptotically stable. These are the sufficient conditions for a system to be stable, if one is unable to find the Lyapunov function, it doesn't mean that the system is unstable. One such theorem to check the instability of the system is Chetaev instability theorem [35].

1.8 Synchronization

The term 'synchronization,' derived from the Greek words 'syn,' meaning 'together,' and 'chronos,' meaning 'time,' can take on varying meanings depending on the specific context in which it is used. Synchronization, an important phenomenon in nonlinear sciences, was first observed by Christiaan Huygens in 1665 [18]. Singing crickets, ticking clocks, applauding of audience are some of the examples in nature working in synchrony. Synchronization phenomena in neural networks have been extensively studied in the context of laser dynamics, electronic circuits, chemical and biological systems [36]. In [37], the authors have shown a connection between a neural network and the cryptography. The exchange of information has been synchronized with the help of neural networks. In [38], the neural network synchronization technique is used for intelligent image protection against illegal swiping and abuse. There are various kinds of synchronizations those have been explored in the literature viz., phase synchronization, complete synchronization, adaptive synchronization, exponential synchronization, quasi synchronization, lag synchronization [39, 40, 41]. In particular lag synchronization appears as a coincidence of shifted-in-time states of two systems. A detailed explanation to the concept of synchronization is given in the monograph [18]. The synchronization of master slave systems is the stability of the error system defined by subtracting the master system from the slave system depending on the type of synchronization. Some variants of the synchronization from the literature are discussed here.

1.8.1 Complete synchronization

Strong mutual coupling of chaotic oscillators lead to complete synchronization. This synchronization can be observed in any kind of systems rather than the phase synchronization which can be possible only in autonomous systems. The systems which demonstrate chaos and whose states are identical at each time moment are called completely synchronized.

1.8.2 Phase synchronization

This type of synchronization mainly occurs when the two systems which deviate from each other other due to their phase difference, are synchronized. For example, in case of two moving pendula in same direction. If these pendula attain the rightmost position at same time, they are called in-phase synchronized. If the pendula of two synchronized clocks move in opposite direction they are said to be anti-phase synchronized. In 1980, experimental evidence lent support to the hypothesis that phase synchronization, especially within the gamma frequency range, creates temporary connections between different brain regions, each representing specific attributes of a stimulus. This finding is discussed in the work by Fell et al.[42]. For example, if you see a red sports MotoGP bike, the information about its colour, category (bike), speed are all stored in the different parts of the brain. The phase synchronization acts as a binding function or we can say that phase synchronization helps in the neural communication. Memory processes depend on the neural communication.

1.8.3 Lag synchronization

The master system and response system having state variables $x(t)$ and $y(t)$ respectively, are said to be lag synchronized, if

$$||x(t) - y(t-\tau)|| \to 0, \quad t \to \infty, \tag{1.12}$$

where τ is a given positive time delay. In other way, we can define the lag synchronization as the states of the two interacting systems nearly coincide if one is shifted in time. The lag synchronization was first studied theoretically by Rosenblum et al. and Sosnovtseva et al. and experimentally by Taherion and Lai [43]. In [44], the authors has used the lag-synchronization for decision making in solving the multi-armed bandit problem of chaos in ring laser-network configuration. Some recent works on lag synchronization can be found [45, 46, 47]

1.9 Anti-synchronization

Anti-synchronization is a phenomenon observed when the combined signals of two systems approach a convergence to zero. This intriguing occurrence involves the state vectors of synchronized systems possessing identical amplitudes but opposite signs compared to those of the driving system. Leveraging anti-synchronization in lasers can lead to the generation of not only intensity drop-outs, as seen with typical low-frequency fluctuations, but also the production of brief, high-intensity pulses. This presents novel opportunities for creating pulses with distinctive characteristics. When applied to communication systems, anti-synchronization enables the continuous transformation between synchronization and anti-synchronization, enhancing security and confidentiality in the transmission of digital signals [48]. In a related study by the same author [49], synchronization and anti-synchronization of chaotic systems were investigated. These results were obtained through a combination of Lyapunov stability analysis and Matrix measure theory.

1.10 Outline of the Thesis

This thesis is organized into five chapters. The basic information about these chapters is given below.

The first chapter gives the introduction of the work carried out. It starts with the background and includes neural network models, types of neural networks and their learning, types of stability, methods to be used and a glimpse into the synchronization and anti-synchronization etc.

The second chapter discusses the fixed time synchronization of quaternion valued neural networks with time varying delay. This result has been obtained by constructing a suitable Lyapunov function and novel controllers. The QVNNs are separated into four RVNNs.

The third chapter is about the effects of impulses on synchronization of neural networks. Here, the neural networks with mixed time varying delay is considered. The study of heterogeneous impulses i.e., the impulses that vary not only with the nodes but also with time make the chapter more interesting. The concept of average impulsive gain is also used.

In the fourth chapter, the global exponential stability of the octonion-valued neural network has been studied. The Matrix measure method is used over the Lyapunov method. The good thing about the Matrix measure is that there is no need of constructing the Lyapunov function.

The fifth chapter is about the exponential and adaptive chaotic lag synchronization of inertial Cohen-Grossberg neural networks. Both discrete and distributed delays have been considered.

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