

Chapter 6

Proinov contraction in the non-triangular metric space

The main purpose of this chapter is to generalize the Proinov contraction in the non-triangular metric space from the metric space. Further, as an application, we find the existence and uniqueness of a solution of the homogeneous Fredholm integral equation in the non-triangular metric space using Proinov contraction.

6.1 Introduction

Analysis has consistently been of great importance in conceptualizing spaces and it is further interesting in generalizing metric spaces and studying their properties. It has also proven to be a fascinating area of research for mathematicians due to its applications not only in other fields of mathematics but also in a few other disciplines. According to Jleli and Samet [51], JS -metric spaces represent a generalization of metric spaces, which includes several metric spaces such as standard metric space, b -metric space, dislocated metric space, and modular space.

In 2020, Khojasteh and Khandani [59] have introduced non-triangular metric spaces. In addition, the introduction of non-triangular metric spaces has shown that there is no inherent necessity for the triangle inequality for several fixed point results to be true. Several useful properties in the sequel are established in non-triangular metric spaces. Recently Proinov [76] has introduced a remarkable result. In addition to the fact that his theorem has nicely and greatly surprised researchers in fixed point theory, he has also speculated that his theorem can be proved to be true by setting forward a series of conditions that were only weak at first but eventually became strong enough to allow him to derive existence and uniqueness of fixed point theorem. In a clear and

definite way, his contractivity condition is,

$$\psi(d(Tx, Ty)) \leq \phi(d(x, y)) \quad \text{for all } x, y \in X \text{ with } d(Tx, Ty) > 0, \quad (6.1)$$

although it is taken from the inequality given by Dutta and Choudhury [41]

$$\psi'(d(Tx, Ty)) \leq \psi'(d(x, y)) - \phi'(d(x, y)) \quad \text{for all } x, y \in X \text{ with } d(Tx, Ty) > 0.$$

In fact, Proinov has introduced new contraction using $\psi = \psi'$ and $\phi = \psi' - \phi'$. According to Proinov, one of the main contributions he made was to outline how such general functions as ψ and ϕ are capable of developing all the reasoning that lead to the final objective, and that these functions must satisfy certain conditions viz:

- (a₁) ψ is monotone increasing;
- (a₂) $\phi(s) < \psi(s)$ for all $s > 0$;
- (a₃) $\limsup_{s \rightarrow e^+} \phi(s) < \lim_{s \rightarrow e^+} \psi(s)$ for all $e > 0$.

The condition (a₁) is a restrictive condition imposed on the function ψ . Although many authors have assumed this condition in various fixed point results due to its crucial role in proofs. Some examples of fixed points can be shown using this monotone condition, and satisfying the above contractivity condition (6.1) with ψ not necessarily increasing. In addition, it is possible to find such examples where ψ is not strictly increasing throughout in $(0, \infty)$, this means that our main result using Proinov contractions on non-triangular metric space is not applicable.

So we generalize the Proinov contractions in terms of the auxiliary function ψ , but avoid the monotone condition on non-triangular metric space. With the constraint imposed on auxiliary functions, we can introduce a new family of intermediate contractions that map the Proinov contractions to the extended contractions derived from the constraints placed on them.

A sequence $\{x_l\}$ is said to be a Picard sequence of T based on $x_0 \in X$ if $x_{l+1} = Tx_l$ for all $l \in \mathbb{N}$. Note that $x_l = T_l x_0$ for all $l \in \mathbb{N}$, where $\{T_l : X \rightarrow X\}$ is the iteration of T defined by $T_0 = \text{identity operator } I$, $T_1 = T$ and $T_{l+1} = T * T_l$ for all $l \geq 2$.

A sequence $\{x_l\}$ is infinite if $x_l \neq x_k$ for all $l \neq k$, and $\{x_l\}$ is almost periodic if there exists $n_0, N \in \mathbb{N}$ such that

$$x_{n_0+l+Np} = x_{n_0+l} \quad \text{for all } p \in \mathbb{N} \quad \text{and all } l \in \{0, 1, 2, \dots, N-1\}.$$

Lemma 6.1. [76] *Let (X, d) be a non-triangular metric space and $\{x_l\}$ be a Picard sequence in X with $\{d(x_l, x_{l+1})\} \rightarrow 0$. If there are $l_1, l_2 \in \mathbb{N}$ with $l_1 < l_2$ and $x_{l_1} = x_{l_2}$, then there is a $l_0 \in \mathbb{N}$ and $y \in X$ such that $x_l = y$ for all $l \geq l_0$. Then the Picard sequence $\{x_l\}$ converges to the fixed point of T .*

Theorem 6.2. [59] *Suppose (X, d) be a complete non-triangular metric space and let $T : X \rightarrow X$ satisfy the following two conditions:*

- (i) *For any two sub-sequences $\{x_{n_l}\}$ and $\{x_{m_l}\}$ of $\{x_l\}$ if $\lim_{l \rightarrow \infty} d(x_{n_l}, x_{m_l}) = L$ and $\lim_{l \rightarrow \infty} d(x_{n_l-1}, x_{m_l-1}) = L$, where $L \geq 0$, and $d(x_{n_l}, x_{m_l}) > L$ for all $l \in \mathbb{N}$, then $L = 0$.*
- (ii) *T is orbitally continuous, i.e., if $\lim_{l \rightarrow \infty} T_{n_l}(x_0) = x$ for some $x_0 \in X$ implies $\lim_{l \rightarrow \infty} TT_{n_l}(x_0) = T(x)$ for each $x \in X$.*

If there exists $x_0 \in X$ such that $\{d(T_i(x_0), T_j(x_0)) : i, j \in \mathbb{N}\}$ is bounded, then T has at least a fixed point in X .

6.1.1 Delineation

The current chapter is structured as follows: In Section 6.2, we have presented Proinov contraction in the non-triangular metric space. Section 6.3 represents extended Proinov contraction in the non-triangular metric space. furthermore, in Section 6.4, we have discussed the existence and uniqueness of a solution of the homogeneous Fredholm integral equation in non-triangular metric space using Proinov contraction.

6.2 Proinov contraction in non-triangular metric space

Theorem 6.3. *Assume that (X, d) is a complete non-triangular metric space and $T : X \rightarrow X$ be a mapping such that*

$$\psi(d(Tx, Ty)) \leq \phi(d(x, y)) \quad \text{for all } x, y \in X \quad \text{with } d(Tx, Ty) > 0, \quad (6.2)$$

where $\psi, \phi : (0, \infty) \rightarrow \mathbb{R}$ satisfy the following three conditions:

(a₁) ψ is monotone increasing;

(a₂) $\phi(s) < \psi(s)$ for all $s > 0$;

(a₃) $\limsup_{s \rightarrow e^+} \phi(s) < \lim_{s \rightarrow e^+} \psi(s)$ for all $e > 0$.

Then, T has a unique fixed point in X .

Proof. First we construct the sequence $x_{l+1} = T(x_l)$ for all $l \in \mathbb{N}$. Let $\alpha_l = d(x_{l+1}, x_l)$, if $\alpha_l = 0$ for some $l \geq 0$, then it is trivial to get the conclusion. Suppose that $\alpha_l > 0$ for all $l \geq 0$. Now applying the relation (6.2) and taking into account the condition (a₂) we have

$$\psi(\alpha_l) \leq \phi(\alpha_{l-1}) < \psi(\alpha_{l-1}). \quad (6.3)$$

Since ψ is monotone increasing therefore $\alpha_l < \alpha_{l-1}$. Therefore $\{\alpha_l\}$ is a strictly decreasing sequence in \mathbb{R} and bounded below by 0 and let $\lim_{l \rightarrow \infty} \alpha_l = \alpha \geq 0$. Now assume that $\alpha > 0$ and so taking the limit as $l \rightarrow \infty$ in (6.3) we have

$$\psi(\alpha+) = \lim_{l \rightarrow \infty} \psi(\alpha_l) \leq \limsup_{l \rightarrow \infty} \phi(\alpha_{l-1}) \leq \limsup_{t \rightarrow \alpha^+} \phi(t),$$

which contradicts the condition (a₃). Therefore $\alpha = \lim_{l \rightarrow \infty} d(x_l, x_{l-1}) = 0$ for all $l \in \mathbb{N}$. Now we prove that $\{x_l\}$ is a d -Cauchy sequence in the non-triangular metric space X . Let $r_l = \sup\{d(T_i(x_0)), T_j(x_0) : i, j \in \mathbb{N} \text{ and } i, j \geq l\}$. Note that $0 \leq r_{l+1} \leq r_l$. Therefore $\{r_l\}$ is a monotone decreasing sequence and bounded below by 0, which implies convergent. Let $\lim_{l \rightarrow \infty} r_l = r$.

Now applying the relation (6.2) and taking into account the condition (a₂) we have

$$\psi\left(d(T_{\xi_l}(x_0), T_{w_l}(x_0))\right) \leq \phi\left(d(T_{\xi_{l-1}}(x_0), T_{w_{l-1}}(x_0))\right) < \psi\left(d(T_{\xi_{l-1}}(x_0), T_{w_{l-1}}(x_0))\right).$$

Since ψ is a monotone increasing function, therefore $\{d(T_{\xi_l}(x_0), T_{w_l}(x_0))\}$ is monotone decreasing sequence in \mathbb{R} and bounded below by 0, which implies convergence.

$$\lim_{l \rightarrow \infty} d(T_{\xi_l}(x_0), T_{w_l}(x_0)) = r.$$

Let $\beta_l = d(T_{\xi_l}(x_0), T_{w_l}(x_0))$, therefore $\lim_{l \rightarrow \infty} \beta_l = r$. If possible let $r > 0$, we have

$$\psi(r+) = \lim_{l \rightarrow \infty} \psi(\beta_l) \leq \limsup_{l \rightarrow \infty} \phi(\beta_{l-1}) \leq \limsup_{t \rightarrow r+} \phi(t),$$

which contradicts the property (a₃).

Therefore $r = 0$, which implies $\limsup_{l \rightarrow \infty} \{d(T_i(x_0), T_j(x_0)) : i, j \in \mathbb{N} \text{ and } i, j \geq l\} = 0$. As such, we conclude that $\{x_l\}$ is d -Cauchy sequence in the non-triangular metric space X . Since X is d -complete non-triangular metric space therefore $\{x_l\}$ is d -convergent to some element ξ in X .

Therefore, $\lim_{l \rightarrow \infty} x_{l+1} = \lim_{l \rightarrow \infty} T(x_l) = \xi$.

If $d(Tx_l, T\xi) = 0$ for infinitely many values of l , then ξ is a fixed point of T . Now suppose that $d(Tx_l, T\xi) > 0$, then applying the relation (6.2) we have

$$\psi(d(Tx_l, T\xi)) \leq \phi(d(x_l, \xi)).$$

Now using the condition (a₂) we have $\psi(d(Tx_l, T\xi)) < \psi(d(x_l, \xi))$.

Since ψ is an increasing function, therefore,

$$d(Tx_l, T\xi) < d(x_l, \xi). \tag{6.4}$$

Now, taking the limit as $l \rightarrow \infty$ on both sides of (6.4) we get $d(\xi, T\xi) \leq 0$, which implies ξ is a fixed point of T . For the uniqueness of the fixed point, let us consider T has two fixed points ξ and η , therefore, $T\xi = \xi$, $T\eta = \eta$ and $d(\xi, \eta) > 0$. Therefore, we have $\psi(d(T\xi, T\eta)) \leq \phi(d(\xi, \eta)) < \psi(d(\xi, \eta))$.

Since ψ is an increasing function, thereby $d(\xi, \eta) < d(\xi, \eta)$, which is a contradiction. Therefore $d(\xi, \eta) = 0$, i.e. $\xi = \eta$ implying T has a unique fixed point.

Hence the desired assertion. \square

Remark 6.4. If $\psi(t) = t$ and $\phi(t) = \lambda t$, where $\lambda \in [0, 1)$, then Theorem 6.3 reduces to the Theorem 1.14.

Example 6.5. Let $X = [0, 1]$, define

$$d(x, y) = \begin{cases} \frac{(x+y)^2}{(x+y)^2+1}, & \text{if } 0 \neq x \neq y \neq 0, \\ \frac{x}{2}, & \text{if } y = 0, \\ \frac{y}{2}, & \text{if } x = 0, \\ 0, & \text{if } x = y. \end{cases}$$

Clearly $d(x, x) = 0$ for all $x \in X$ and $d(x, y) = d(y, x)$ for all $x, y \in X$. Let $\{x_l\}$ be a sequence in X such that $d(x_l, x) \rightarrow 0$ and $d(x_l, y) \rightarrow 0$ as $l \rightarrow \infty$. It implies that

$$\lim_{l \rightarrow \infty} \frac{(x_l + y)^2}{(x_l + y)^2 + 1} = \lim_{l \rightarrow \infty} \frac{(x_l + y)^2}{(x_l + y)^2 + 1} = 0$$

and these hold good iff $\lim_{l \rightarrow \infty} x_l = -x = -y$ and so $x = y$. Therefore (X, d) is a non-triangular metric space.

Let T be an operator on X defined by $Tx = \frac{x}{2}$. Let us define $\psi, \phi : (0, \infty) \rightarrow \mathbb{R}$, by:

$$\psi(s) = \begin{cases} s, & \text{if } s \in (0, 2) \\ 1 + \frac{s}{2}, & \text{if } s \in [2, \infty) \end{cases}$$

and

$$\phi(s) = \frac{2\psi(s)}{3}.$$

(i) if $x = y$ then it is trivial,

(ii) if $0 \neq x \neq y \neq 0$ then $\psi(d(Tx, Ty)) = \psi(d(\frac{x}{2}, \frac{y}{2})) = \psi\left(\frac{(x+y)^2}{(x+y)^2+4}\right) = \frac{(x+y)^2}{(x+y)^2+4}$ and

$$\phi(d(x, y)) = \phi\left(\frac{(x+y)^2}{(x+y)^2+1}\right) = \frac{2\psi\left(\frac{(x+y)^2}{(x+y)^2+1}\right)}{3} = \frac{2(x+y)^2}{3(x+y)^2+3},$$

(iii) if $x \neq 0$ and $y = 0$, $\psi(d(Tx, Ty)) = \psi(\frac{x}{4}) = \frac{x}{4}$ and $\phi(d(x, y)) = \phi(\frac{x}{2}) = \frac{x}{3}$,

(iv) if $y \neq 0$ and $x = 0$, $\psi(d(Tx, Ty)) = \psi(\frac{y}{4}) = \frac{y}{4}$ and $\phi(d(x, y)) = \phi(\frac{y}{2}) = \frac{y}{3}$.

Therefore, from the above cases, we can conclude that,

$$\psi(d(Tx, Ty)) \leq \phi(d(x, y)) \quad \text{for all } x, y \in X, \quad \text{and}$$

- (a₁) ψ is monotone increasing;
- (a₂) $\phi(s) < \psi(s)$ for all $s > 0$;
- (a₃) $\limsup_{s \rightarrow e+} \phi(s) < \lim_{s \rightarrow e+} \psi(s)$ for all $e > 0$.

Therefore, ψ and ϕ satisfy all the conditions of Theorem 6.3. Hence, T has a fixed point 0 which is unique.

6.3 Extended Proinov contractions in the non-triangular metric space

The following is an introduction to the new contractions class in the setting of non-triangular metric spaces derived from the Proinov contraction mentioned above. These contractions are defined in the following.

Let \mathfrak{F} be the set of all pairs (ψ, ϕ) with $\psi, \phi : (0, \infty) \rightarrow \mathbb{R}$ as two functions satisfying the following conditions:

- (b₁) if $\{t_l\} \subset (0, \infty)$ is a sequence such that $\psi(t_{l+1}) \leq \phi(t_l)$ for all $l \in \mathbb{N}$, then $\{t_l\} \rightarrow 0$;
- (b₂) $\limsup_{s \rightarrow e+} \phi(s) < \lim_{s \rightarrow e+} \psi(s)$ for all $e > 0$;
- (b₃) if $\{t_l\}, \{s_l\} \subset (0, \infty)$ are two sequences such that $\{s_l\} \rightarrow 0$ and $\psi(t_l) \leq \phi(s_l)$ for all $l \in \mathbb{N}$, then $\{t_l\} \rightarrow 0$.

Example 6.6. If $\psi, \phi : (0, \infty) \rightarrow \mathbb{R}$ are defined by $\psi(s) = \lambda_1 s$ and $\phi(s) = \lambda_2 s$ for all $s > 0$, where $\lambda_1, \lambda_2 \in (0, \infty)$ are such that $\lambda_2 < \lambda_1$, then $(\psi, \phi) \in \mathfrak{F}$.

In the following lemma, we find that the pair (ψ, ϕ) of functions associated with a Proinov contraction belongs to \mathfrak{F} .

Lemma 6.7. Let $\psi, \phi : (0, \infty) \rightarrow \mathbb{R}$ be functions satisfying the following properties:

(a₁) ψ is monotone increasing;

(a₂) $\phi(s) < \psi(s)$ for all $s > 0$;

(a₃) $\limsup_{s \rightarrow e^+} \phi(s) < \lim_{s \rightarrow e^+} \psi(s)$ for all $e > 0$.

Then $(\psi, \phi) \in \mathfrak{F}$.

Proof. Let $\{t_l\}$ be a sequence in $(0, \infty)$ such that $\psi(t_{l+1}) \leq \phi(t_l)$ for all $l \in \mathbb{N}$. Since $t_l > 0$, therefore the property (a₂) leads to

$$\psi(t_{l+1}) \leq \phi(t_l) < \psi(t_l) \quad \text{for all } l \in \mathbb{N}. \quad (6.5)$$

As ψ is increasing, therefore $0 < t_{l+1} < t_l$ for all $l \in \mathbb{N}$. Therefore $\{t_l\}$ is a decreasing sequence in \mathbb{R} and bounded below by 0, which implies it is convergent and let $e = \lim_{l \rightarrow \infty} t_l$. It is obvious that $e < t_l$ for all $l \in \mathbb{N}$. Now we prove $e = 0$. If possible let, $e > 0$.

Therefore,

$$\psi(e) \leq \lim_{s \rightarrow e^+} \psi(s) = \lim_{l \rightarrow \infty} \psi(t_l) = \lim_{l \rightarrow \infty} \psi(t_{l+1}).$$

Using the inequality (6.5) we get

$$\lim_{l \rightarrow \infty} \phi(t_l) = \lim_{s \rightarrow e^+} \psi(s),$$

which contradicts (a₃) because,

$$\lim_{s \rightarrow e^+} \psi(s) = \lim_{l \rightarrow \infty} \phi(t_l) \leq \limsup_{s \rightarrow e^+} \phi(s) < \lim_{s \rightarrow e^+} \psi(s).$$

Therefore $e = 0$, which implies the condition (b₁) is satisfied.

The property (b₂) is the same as the property (a₃).

Finally, to check the property (b₃), let $\{t_l\}, \{s_l\}$ be two sequences in $(0, \infty)$ such that $\{s_l\} \rightarrow 0$ and $\psi(t_l) \leq \phi(s_l)$ for all $l \in \mathbb{N}$. Since $s_l > 0$, therefore from property (a₂) we have $\psi(t_l) \leq \phi(s_l) < \psi(s_l)$.

Since ψ is monotone increasing function (by property (a₁)) therefore, $0 < t_l < s_l$ for all $l \in \mathbb{N}$.

Hence by applying the Sandwich theorem, we can conclude that $\{t_l\} \rightarrow 0$.

Hence $(\psi, \phi) \in \mathfrak{F}$.

□

Theorem 6.8. *Suppose that (X, d) is a d -complete non-triangular metric space and $T : X \rightarrow X$ be a mapping such that there exists $(\psi, \phi) \in \mathfrak{F}$ with*

$$\psi(d(Tx, Ty)) \leq \phi(d(x, y)) \quad \text{for all } x, y \in X \text{ and } d(Tx, Ty) > 0. \quad (6.6)$$

Then, the Picard sequence converges to a fixed point of T .

Proof. Let $x \in X$ be arbitrary. Construct the sequence $\{x_l\}$ such that $x_1 = x$ and $x_{l+1} = Tx_l$ for all $l \in \mathbb{N}$. If there is a $l_0 \in \mathbb{N}$ such that $x_{l_0} = x_{l_0+1}$, then x_{l_0} is a fixed point of T . In such a case, $\{d(x_l, x_{l+1})\} \rightarrow 0$, for all $l \geq l_0$.

Suppose that $x_l \neq x_{l+1}$ for all $l \in \mathbb{N}$ then each x_l is not a fixed point of T . Therefore, $d(x_l, x_{l+1}) > 0$, which implies $d(Tx_l, Tx_{l+1}) > 0$, for all $l \in \mathbb{N}$. Now applying the contractive condition (6.6) we get

$$\psi(d(x_{l+1}, x_{l+2})) = \psi(d(Tx_l, Tx_{l+1})) \leq \phi(d(x_l, x_{l+1})).$$

If we define $s_l = d(x_l, x_{l+1})$ then $\psi(s_{l+1}) \leq \phi(s_l)$, therefore using the property (b₁), we can conclude that $\{s_l\} \rightarrow 0$. If there are $l_1, l_2 \in \mathbb{N}$ such that $l_1 < l_2$ and $x_{l_1} = x_{l_2}$ then Lemma 6.1 ensures that there exists $l_0 \in \mathbb{N}$ and $\xi \in X$ such that $x_l = \xi$ for all $l \geq l_0$. Therefore, ξ is a fixed point of T , and there is an assurance of the existence of a fixed point. Now let $x_{l_1} \neq x_{l_2}$ for all $l_1 \neq l_2$, that is, $\{x_l\}$ is an infinite sequence. In particular, $d(Tx_{l_1}, Tx_{l_2}) = d(x_{l_1+1}, x_{l_2+1}) > 0$ for all $l_1 \neq l_2$. Now we have to prove that $\{x_l\}$ is a d -Cauchy sequence in non-triangular metric space X .

Let $r_l = \sup\{d(T_i(x_0), T_j(x_0)) : i, j \in \mathbb{N} \text{ and } i, j \geq l\}$. Note that $0 \leq r_{l+1} \leq r_l$. Therefore $\{r_l\}$ is a monotone decreasing sequence and bounded below by 0, which implies convergence of the sequence.

Let $\lim_{l \rightarrow \infty} r_l = r$.

Again by definition of supremum, for every $l \in \mathbb{N}$ there exist w_l, ξ_l such that $\xi_l > w_l \geq l$ and

$$r_l - \frac{1}{l} < d(T_{\xi_l}(x_0), T_{w_l}(x_0)) \leq r_l.$$

Hence, by Sandwich theorem we can conclude that

$$\lim_{l \rightarrow \infty} d(T_{\xi_l}(x_0), T_{w_l}(x_0)) = r.$$

Let $\beta_l = d(T_{\xi_l}(x_0), T_{w_l}(x_0))$, therefore $\lim_{l \rightarrow \infty} \beta_l = r$. If possible let $r > 0$, we have

$$\psi(r+) = \lim_{l \rightarrow \infty} \psi(\beta_l) \leq \limsup_{l \rightarrow \infty} \phi(\beta_{l-1}) \leq \lim_{t \rightarrow r+} \phi(t),$$

which contradicts the property (b_2) .

Therefore $r = 0$, which implies $\lim_{l \rightarrow \infty} \sup\{d(T_i(x_0), T_j(x_0)) : i, j \in \mathbb{N} \text{ and } i, j \geq l\} = 0$. Therefore, we conclude that $\{x_l\}$ is d -Cauchy sequence in non-triangular metric space X . Since X is d -complete non-triangular metric space, therefore there exists $\xi \in X$ such that $\{x_l\}$ d -converges to ξ . Since $\{x_l\}$ is an infinite sequence, there is a $l_0 \in \mathbb{N}$ with $x_l \neq \xi$ and $Tx_l \neq T\xi$ for all $l \geq l_0$. Then we have

$$\psi(d(x_{l+1}, T\xi)) = \psi(d(Tx_l, T\xi)) \leq \phi(d(x_l, \xi)).$$

It follows from the property (b_3) that, $\{d(x_{l+1}, T\xi)\} \rightarrow 0$, so $T\xi = \xi$. This completes the proof. \square

Theorem 6.9. *Suppose that the family of pairs (ψ, ϕ) satisfies the hypothesis of Theorem 6.8 with the following hypothesis:*

(b_4) *there is a subset $\Omega \subseteq X$ with fixed point set of $T \subseteq X$ and $\psi(d(x, y)) > \phi(x, y)$ for all $x \neq y \in \Omega$.*

Then, T has a unique fixed point $x_0 \in X$ and the Picard sequence $\{T(x_l)\}$ converges to x_0 for all $x \in X$.

Proof. For the uniqueness of the fixed point of T , let $x_1, x_2 \in X$ be two distinct fixed points of T . Then $d(Tx_1, Tx_2) = T(x_1, x_2) > 0$. The contractive condition implies that

$$\psi(d(x_1, x_2)) = \psi(d(Tx_1, Tx_2)) \leq \phi(d(x_1, x_2)),$$

which contradicts the property (b_4) . Hence, T has a fixed point, which is unique. \square

Example 6.10. Let $X = [0, 1]$, define

$$d(x, y) = \begin{cases} |x - y|, & \text{if } 0 \neq x \neq y \neq 0, \\ \frac{x}{2}, & \text{if } y = 0, \\ \frac{y}{2}, & \text{if } x = 0, \\ 0, & \text{if } x = y. \end{cases}$$

Clearly, (X, d) is a non-triangular metric space.

Let T be an operator on X defined by $Tx = \frac{x}{2}$. Let us define $\psi, \phi : (0, \infty) \rightarrow \mathbb{R}$, by:

$$\psi(s) = \begin{cases} s, & \text{if } s \in (0, 2) \\ 1 + \frac{2}{s}, & \text{if } s \in [2, \infty) \end{cases}$$

and

$$\phi(s) = \frac{2\psi(s)}{3}.$$

Further, we find the following:

- (i) if $x = y$ then it is trivial,
- (ii) if $x \neq y \neq 0$ then $\psi(d(Tx, Ty)) = \psi(d(\frac{x}{2}, \frac{y}{2})) = \psi\left(\left|\frac{x-y}{2}\right|\right) = \left|\frac{x-y}{2}\right|$ and
 $\phi(d(x, y)) = \phi(|x - y|) = \frac{2\psi(|x-y|)}{3} = \frac{2|x-y|}{3}$,
- (iii) if $x \neq 0$ and $y = 0$ then $\psi(d(Tx, Ty)) = \psi(\frac{x}{4}) = \frac{x}{4}$ and $\phi(d(x, y)) = \phi(\frac{x}{2}) = \frac{x}{3}$,
- (iv) if $y \neq 0$ and $x = 0$ then $\psi(d(Tx, Ty)) = \psi(\frac{y}{4}) = \frac{y}{4}$ and $\phi(d(x, y)) = \phi(\frac{y}{2}) = \frac{y}{3}$.

Therefore, we can conclude that, $\psi(d(Tx, Ty)) \leq \phi(d(x, y))$ for all $x, y \in X$.

However, Theorem 6.3 is not applicable as ψ is not monotone increasing in $[2, \infty)$.

We claim that $(\psi, \phi) \in \mathfrak{F}$.

- (i) Let $\{t_l\} \subset (0, \infty)$ be such that $\psi(t_{l+1}) \leq \phi(t_l)$ for all $l \in \mathbb{N}$, that is, $\psi(t_{l+1}) \leq \frac{2\psi(t_l)}{3}$ for all $l \in \mathbb{N}$.

Therefore $\psi(t_l) \leq (\frac{2}{3})^l t_0$ for all $l \in \mathbb{N}$, which implies, $\{\psi(t_l)\} \rightarrow 0$, so $\{t_l\} \rightarrow 0$;

- (ii) $\limsup_{s \rightarrow e^+} \phi(s) < \lim_{s \rightarrow e^+} \psi(s)$ for all $e > 0$;
- (iii) let $\{t_l\}$ and $\{s_l\}$ be two sequences in $(0, \infty)$ such that $\{s_l\} \rightarrow 0$ and $\psi(t_l) \leq \phi(s_l)$ for all $l \in \mathbb{N}$.

Then,

$$\psi(t_l) \leq \phi(s_l) = \frac{2\psi(s_l)}{3} = \frac{2s_l}{3}.$$

Since $\{s_l\} \rightarrow 0$ therefore, $\{\psi(t_l)\} \rightarrow 0$ as $l \rightarrow \infty$. Thus, we can conclude that the sequence $\{t_l\}$ converges to 0.

Hence $(\psi, \phi) \in \mathfrak{F}$ and it satisfies all the conditions of Theorem 6.8 and so T has a fixed point.

6.4 An application to homogeneous Fredholm integral equation

As an application, we find the existence and uniqueness of a solution of the following homogeneous Fredholm integral equation in the non-triangular metric space:

$$y : [0, 1] \rightarrow [0, \infty)$$

$$x(t) = \int_0^1 k(x, t)y(t)dt, \quad (6.7)$$

where $k(x, t)$ is continuous on $[0, 1] \times [0, 1]$ and let $k(x, t) > 0$.

Let $\mathfrak{C}[0, 1]$ be the set of all real continuous functions defined on $[0, 1]$ with supremum norm $\|y\| = \sup\{|x(t)| : x \in [0, 1]\}$ and let $M = \sup\{k(x, t) : x, t \in [0, 1]\}$.

Now, we define the mapping $T : \mathfrak{C}[0, 1] \rightarrow \mathfrak{C}[0, 1]$ by,

$$T(x(t)) = \int_0^1 k(x, t)y(t)dt. \quad (6.8)$$

Now, we define the non-triangular metric d on $\mathfrak{C}[0, 1]$ as,

$$d(x(t), y(t)) = \begin{cases} \sup\{|x(t) - y(t)| : x \in [0, 1]\}, & \text{if } 0 \neq x(t) \neq y(t) \neq 0, \\ \frac{1}{2} \sup\{|x(t)| : x \in [0, 1]\}, & \text{if } y(t) = 0, \\ \frac{1}{2} \sup\{|y(t)| : x \in [0, 1]\}, & \text{if } x(t) = 0, \\ 0, & \text{if } x(t) = y(t). \end{cases}$$

Clearly, $(\mathfrak{C}[0, 1], d)$ is a d -complete non-triangular metric space.

Let us define $\psi, \phi : (0, \infty) \rightarrow \mathbb{R}$, by:

$$\psi(s) = s \quad \text{and} \quad \phi(s) = \frac{2s}{3}.$$

Further, we find the following:

- (i) if $x(t) = y(t)$ then, it is trivial,
- (ii) if $0 \neq x(t) \neq y(t) \neq 0$ then,

$$\begin{aligned} \psi\left(d(Tx(t), Ty(t))\right) &= d(Tx, Ty) = \sup\{|T(x(t)) - T(y(t))| : x \in [0, 1]\} \\ &\leq \sup_{x \in [0, 1]} \left\{ \int_0^1 |k(x, t)| | [y(t) - y(t)] dt \right\} \\ &\leq \sup_{x, t \in [0, 1]} |k(x, t)| \sup_{t \in [0, 1]} \{|y(t) - y(t)|\} \int_0^1 dt \\ &= Md(x(t), y(t)) \end{aligned}$$

and

$$\phi\left(d(x(t), y(t))\right) = \frac{2}{3}d(x(t), y(t)),$$

- (iii) if $x(t) \neq 0$ and $y(t) = 0$ then,

$$\begin{aligned} \psi\left(d(Tx, Ty)\right) &= d(Tx, Ty) = \frac{1}{2} \sup\{|T(x(t))| : x \in [0, 1]\} \\ &\leq \frac{1}{2} \sup_{x \in [0, 1]} \left\{ \int_0^1 |k(x, t)| |y(t)| dt \right\} \\ &\leq \sup_{x, t \in [0, 1]} |k(x, t)| \sup_{t \in [0, 1]} \frac{1}{2} \{|y(t)|\} \int_0^1 dt \\ &= Md(x(t), y(t)) \end{aligned}$$

and

$$\phi\left(d(x(t), y(t))\right) = \frac{2}{3}d(x(t), y(t)),$$

- (iv) if $y(t) \neq 0$ and $x(t) = 0$ then,

$$\begin{aligned} \psi\left(d(Tx, Ty)\right) &= d(Tx, Ty) = \frac{1}{2} \sup\{|T(y(t))| : x \in [0, 1]\} \\ &\leq \frac{1}{2} \sup_{x \in [0, 1]} \left\{ \int_0^1 |k(x, t)| |y(t)| dt \right\} \end{aligned}$$

$$\begin{aligned} &\leq \sup_{x,t \in [0,1]} |k(x,t)| \sup_{t \in [0,1]} \frac{1}{2} \{|y(t)|\} \int_0^1 dt \\ &= Md(x(t), y(t)) \end{aligned}$$

and

$$\phi(d(x(t), y(t))) = \frac{2}{3}d(x(t), y(t)).$$

Now if $M = \sup\{k(x,t) : x,t \in [0,1]\} \leq \frac{2}{3}$ then, $\psi(d(Tx, Ty)) \leq \phi(d(x, y))$ for all $x, y \in \mathfrak{C}[0,1]$. Therefore ψ and ϕ satisfy all the conditions of Theorem 6.3. Therefore, T defined in (6.8) has a unique fixed point. Hence, Fredholm integral equation (6.7) has a unique solution.

6.5 Concluding remarks

In this chapter, we have generalized Proinov contractions in the non-triangular metric space in light of Proinov's attractive results in metric space. Also, we have introduced a family \mathfrak{F} of auxiliary function (ψ, ϕ) in non-triangular metric space to avoid monotone condition on ψ . This idea may be extended to non-triangular fuzzy metric space [70] by suitable changes. Further, as an application, we find the existence and uniqueness of a solution of the homogeneous Fredholm integral equation in non-triangular metric space using Proinov contraction.

It will be interesting to see if the contractivity condition (6.1) could be replaced by a generalized contractivity condition. We recall that a self-map T on the non-triangular metric space (X, d) satisfies a generalized contractive type condition if

$$\psi(d(Tx, Ty)) \leq \phi(m(x, y)) \quad \text{for all } x, y \in X, \quad (6.9)$$

where $\psi, \phi : (0, \infty) \rightarrow \mathbb{R}$ and $m(x, y)$ is defined by

$$m(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\}.$$
