Chapter 1

Introduction

The fractional diffusion equations can be used to describe numerous physical phenomena but it is not always possible to find their exact solutions, so it is certainly a challenge to develop numerical methods for solving the involved equations accurately. As per the title of the thesis it is quite clear that the work is primarily focused on solving fractional diffusion equations and observing how the concentrations of solutes change with the variation of diffusion model parameters. It is not just that the author has focused only on solving diffusion models but also on developing numerical techniques that are more accurate than those previously used. It is necessary to discuss some of the basic terms used in the thesis before going to the discussion about solving the diffusion model.

1.1 Fractional Calculus

Fractional calculus is a generalization of the integer order derivatives and integrals to real or even complex order. Complex order derivatives have allowed the fractional order derivatives to be a function of independent variables such as space, time or some other variables. The origins of fractional calculus can be traced back to the 17^{th} century. Its inception was marked by an important observation made during the correspondence between G. de L'Hôpital and G. W. Leibniz in the year 1695. The following question was raised by Leibniz in a letter to L'Hôpital "Can the

meaning of derivatives with integer order be generalized to derivatives with non-integer orders?" L'Hôpital showed interest in that question and provided Leibniz with another question in response "What if the order will be 1/2"? Leibniz wrote his reply on September 30, 1695, that "It will lead to a paradox, from which one-day useful consequences will be drawn". So this date can be regarded as the birth date of fractional calculus. Over the past three centuries, from 1695 to the present day, numerous renowned mathematicians such as Laplace, Leonhard Euler, J. Fourier, S.F. Lacroix, J. Liouville, N.H. Abel, O. Heaviside, B. Riemann, G. Leibniz, H. Weyl, A. K. Grunwald, A.V. Letnikov, F. Riesz and many others have made significant contributions to this field, accomplishing a multitude of important works.

This topic was also discussed in many other letters after the 1695 letter. G.W. Leibniz wrote to J. Wallis and J. Bernoulli in 1697, mentioning the possibility of fractional order differentiation for non-integer value of n as

$$\frac{d^n e^{ax}}{dx^n} = a^n e^{ax}. \tag{1.1}$$

Although L'Hôpital and Leibniz discussed fractional order derivatives in 1695 but no works have been found in the literature in this area from the end of the 17^{th} to 18^{th} century related to arbitrary order derivatives. Leonhard Euler and Joseph Fourier mentioned about the derivatives of arbitrary order, but they did not take it into consideration in their further works. In the second decade of the 19^{th} century, S. F. Lacroix introduced the concept of the derivative of an arbitrary order for x^n in 1819 by employing the Gamma function as

$$\frac{d^\alpha x^n}{dx^\alpha} = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha}. \tag{1.2}$$

The first application of fractional calculus was made by N. H. Abel in 1823 during the solution of an integral equation arising in the formulation of the tautochrone problem [1]. In 1832, J. Liouville introduced two distinct definitions for fractional derivatives [2]. The first definition was formulated using series expansion, where he represented the function $f(x)$ in the form of a series as

$$f(x) = \sum_{m=0}^{\infty} c_m e^{a_m x}, \tag{1.3}$$

whose derivative of arbitrary order α is

$$D^{\alpha} f(x) = \sum_{m=0}^{\infty} c_m a_m^{\alpha} e^{a_m x}, \tag{1.4}$$

J. Liouville developed a second definition of arbitrary order derivative through the transformation of an integral to the gamma function, which is the following

$$D^{\alpha} x^{-\beta} = \frac{(-1)^{\alpha}\Gamma(\alpha+\beta)}{\Gamma(\beta)} x^{-\beta-\alpha}. \tag{1.5}$$

A disadvantage of the second definition is its unsuitability for a broad range of functions.

The primary distinction between Lacroix's and Liouville's definitions of fractional derivatives is that in Lacroix's definition, the fractional derivative of a constant results in a non-zero value, whereas in Liouville's definition, it gives zero. This gave rise to an interesting discussion in the 19^{th} century about the correct definition of fractional derivative given by whom.

When Bernhard Riemann was trying to generalize a Taylor series during his student days in 1853, he wrote the article that contributed most to fractional calculus advancements. It was proposed by Riemann that fractional integration could be

defined as follows

$$D_a^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} f(s) ds + \Phi(x), \tag{1.6}$$

where $\Phi(x)$ is the complementary function of the Riemann.

Consequently, In 1869, Sonin [3] did the work which leads us to today's Riemann-Liouville differentiation of arbitrary order. Sonin started with Cauchy's integral formula to define the differentiation with arbitrary order. In a very short period of time, Letnikov [4] in the year 1872 extended the idea of Sonin. In 1892, Hadamard [5] gave definitions for both fractional order integral and derivative. In 1917, Weyl [6] had given a definition for arbitrary order derivative that is similar to the Riemann-Liouvill definition but whose kernel function is different. Additionally, Grunwald and Letnikov [7] introduced another concept of fractional order derivative that is also widely used today. Later in 1927, Marchaud [8] introduced an integral form of the definition of Grunwald and Letnikov. In 1967, M. Caputo developed a definition known as the Caputo fractional derivative, which is obtained by calculating an ordinary derivative followed by the fractional integral. There are many other definitions which can be found in the literature [9].

In the second half of the 20^{th} century, work in the field of fractional calculus accelerated. B. Ross organized the first conference on fractional calculus and its applications at the University of New Haven in June 1974, after the completion of his PhD dissertation on fractional calculus and after that, several research articles in this field have been published. Oldham and Spanier [10] published the first book on fractional calculus in the same year. This has prompted the attention of researchers from different background and as a result different authors have published a number of books on fractional calculus afterward viz., Nishimoto [11], Miller and Ross [12],

Rubin [13], Podlubny [14], Hilfer [15], Kilbas et al. [16], etc. It is observed that the theory of fractional calculus is highly effective in describing numerous physical phenomena. The integer order differential operator is a local operator, while the fractional order differential operator is a non-local operator. This means that the fractional order operator considers that the future state depends not only on the current state but also on the history of its previous states. Therefore, it allows the modelling of many natural phenomena containing long memory, for example, the atmospheric diffusion of pollution, cellular diffusion processes, dynamics of a viscoelastic material, network traffic, electronics etc. All these systems have non-local dynamics that cannot be accurately modelled using classical calculus theory. So, fractional calculus plays an important role during the modelling of these systems. In the last few decades, there have been an enormous of research activities focusing on the applications of fractional calculus in a wide range of scientific and engineering fields, including fluid flow, rheology, diffusive transport, electrical networks, electromagnetic theory, control theory, image and signal processing etc.

1.2 Special Functions

A few definitions of special functions are provided in this section that are normally used in the theory of fractional calculus and in subsequent chapters when computing numerically.

1.2.1 Euler's Gamma Function

The Gamma function is a generalization of the factorial function from integers to non-integers and even complex values. It is expressed as

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad \text{for } z \in \mathbb{C} \quad Re(z) > 0. \tag{1.7}$$

The Gamma function follows the reduction formula as

$$\Gamma(z+1) = z \cdot \Gamma(z). \tag{1.8}$$

1.2.2 Mittag-Leffler Function

In 1903, the Swedish mathematician Mittag-Leffler introduced the Mittag-Leffler function. The fundamental Mittag-Leffler function is a generalization of the exponential function. It is known that the exponential function plays a significant role in obtaining the general solutions of differential equations. When differential or integral equations of fractional order are solved, the Mittag-Leffler function arises naturally.

The basic one-parameter Mittag-Leffler function is represented by $E_\alpha(z)$ and is defined as follows [17]

$$E_\alpha(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + 1)}, \quad \alpha > 0. \tag{1.9}$$

The Mittag-Leffler function of two parameters is denoted by $E_{\alpha,\beta}(z)$ and is defined as

$$E_{\alpha,\beta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \beta)}, \quad \alpha > 0, \ \beta > 0. \tag{1.10}$$

For $\beta = 1$,

$$E_\alpha(z) = E_{\alpha,1}(z) \tag{1.11}$$

1.2.3 Wright Function

The Wright function, also known as the Wright omega function or the Wright generalized Bessel function, is a special function that arises in mathematical physics, particularly in the study of differential equations and fractional calculus. It is denoted by $W(z;\alpha,\beta)$ and defined by the power series [18]

$$W(z;\alpha,\beta)=\sum_{n=0}^{\infty}\frac{z^n}{n!\Gamma(\alpha n+\beta)},\quad \alpha>-1,\ \beta\in\mathbb{C}. \tag{1.12}$$

1.3 Definitions of fractional order integral and derivative

There are various types of definitions of fractional order derivative and integral in the literature, but in this section only those definitions are discussed that have been used in the forthcoming chapters.

1.3.1 Riemann-Liouville integral

Let $f(x)$ be a locally integrable function then the Reimann-Liouville integral of fractional order $\alpha\geq 0$ is defined by [14, 19]

$$I_x^{\alpha}f(x)=\begin{cases} f(x), & \text{if } \alpha=0,\\ \frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}f(t)dt, & \text{if } \alpha>0,\end{cases} \tag{1.13}$$

where $x,\alpha\in\mathbb{R}$ and $\Gamma(\cdot)$ is the gamma function.

1.3.2 Riemann-Liouville derivative

The Riemann-Liouville fractional derivative of arbitrary order $\alpha \geq 0$ of a function $f(x)$ is defined by

$$D_x^\alpha f(x) = \begin{cases} f(x), & \text{if } \alpha = 0, \\ \frac{1}{\Gamma(k-\alpha)} \frac{d^k}{dt^k} \int_0^x (x-t)^{k-\alpha-1} f(t) dt, & \text{if } \alpha > 0, \end{cases} \tag{1.14}$$

where $k-1 \leq \alpha < k$, $k \in \mathbb{N}$ and $x, \alpha \in \mathbb{R}$.

1.3.3 Some properties of Reimann-Liouville operator

(i) $\frac{d}{dx}(I_x^{\alpha+1} f(x)) = I_x^\alpha f(x)$,

(ii) $I_x^\alpha (I_x^\beta f(x)) = I_x^{\alpha+\beta} f(x)$,

(iii) $D_x^\alpha (I_x^\alpha f(x)) = f(x)$,

(iv) $I_x^\alpha x^p = \frac{\Gamma(p+1)}{\Gamma(\alpha+1+p)} x^{\alpha+p}$.

1.3.4 Caputo fractional derivative

The fractional order derivative of function $f(x)$ in Caputo sence of order $\alpha > 0$ is

$$D^\alpha f(x) = \begin{cases} \frac{d^m f(x)}{dx^m}, & \text{if } \alpha = m \in \mathbb{N}, \\ \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^m(t)}{(x-t)^{\alpha+1-m}} dt, & \text{if } m-1 < \alpha < m \in \mathbb{N}. \end{cases} \tag{1.15}$$

1.3.5 Some properties of Caputo fractional derivative

(i) Caputo derivative follows the linearity property as integer order differentiation as

$$D^{\alpha}(\lambda f(x) + \mu g(x)) = \lambda D^{\alpha} f(x) + \mu D^{\alpha} g(x),$$

where λ and μ are constants.

(ii) $D^{\alpha} c = 0$, where c is a constant.

(iii) $D^{\alpha} x^{p} = \begin{cases} 0, & p \in 0, 1, 2, ..., \lceil \alpha \rceil - 1, \\ \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} x^{p-\alpha}, & p \geq \lceil \alpha \rceil, \end{cases}$

where the function $\lceil \alpha \rceil$ is used to denote the ceiling function.

(iv) $I^{\alpha}(D^{\alpha} f(x)) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^{k-\alpha}, \quad n-1 \leq \alpha < n,$

where n is an integer number.

1.4 Fractional derivative and integration of variable order

Samko and Ross [20] firstly introduced the concept of variable order differential and integral as well as some fundamental properties in 1993. Lorenzo and Hartley [21] provided a summary of research outcomes of the variable order fractional operators and then examined the definitions of variable order fractional operators in various forms. The fractional derivative of variable order is a non-local operator. These operators are helpful for describing a system's memory characteristics. Variable order fractional derivatives are useful in determining the memory effect in two ways. First one is memory change with respect to spatial and time coordinates. Second one

is connected with the memory of orders. This can be effective by the previous values of orders of derivatives. The hereditary property and self-similarity of a system can be derived by variable order differential equations.

1.4.1 Riemann-Liouville integral of variable order

The Riemann-Liouville fractional integral of variable order $\beta(x,t) > 0$ of a function $f(x,t)$ is defined by [22, 23]

$$I_t^{\beta(x,t)} f(x,t) = \frac{1}{\Gamma(\beta(x,t))} \int_0^t (t-s)^{\beta(x,t)-1} f(x,s) ds. \tag{1.16}$$

1.4.2 Caputo derivative of variable order

The Caputo fractional derivative of variable order $\beta(x,t)$ of a function $f(x,t)$ is defined as [22, 23]

$$D_t^{\beta(x,t)} f(x,t) = \begin{cases} \frac{1}{\Gamma(q-\beta(x,t))} \int_0^t (t-s)^{q-\beta(x,t)-1} \frac{\partial^q f(x,s)}{\partial s^q} ds, & q-1 < \beta(x,t) < q, \ \ q \in \mathbb{N}, \\ \frac{\partial^q f(x,t)}{\partial t^q} & \beta(x,t) = q. \end{cases} \tag{1.17}$$

1.4.3 Some properties of fractional variable order operators

The operator $D_t^{\beta(x,t)}$ has the following properties

$$D_t^{\beta(x,t)}(\lambda h(x,t) + \mu g(x,t)) = \lambda D_t^{\beta(x,t)} h(x,t) + \mu D_t^{\beta(x,t)} g(x,t), \tag{1.18}$$

$D_t^{\beta(x,t)} C = 0,\ C$ is a constant,

and

$$D_t^{\beta(x,t)} t^p = \begin{cases} 0, & p \in 0, 1, 2, ..., q-1, \\ \frac{\Gamma(p+1)}{\Gamma(p+1-\beta(x,t))} t^{p-\beta(x,t)}, & otherwise, \end{cases} \tag{1.19}$$

where $q - 1 < \beta(x, t) < q$.

1.5 Diffusion phenomena

The term diffusion in fact originated from the Latin language that means "to spread out". Diffusion is the fundamental process in which any matter or material atoms or molecules are transported or moved from one system to another with random molecular motion. This process is not due to any action of force. It is a result of random moments of atoms with resulting in the uniform distribution of that matter or material atoms. It occurs in all types of materials with a temperature above absolute zero. It occurs from the regions of higher concentration to regions of lower concentration even in the absence of driving force or concentration gradient. One can understand the diffusion process more accurately by an example, a flask may be taken with full of clear water where water at rest, when a drop of ink is added in top of the flask in such a way that no convection current is set up then initially it can be seen that a clear boundary between the ink and water but as the time passes it is seen that the ink gets faint towards the bottom, i.e., from area of higher to lower concentration and after some time the whole water will have the same colour. It is due to the spreading of ink molecules throughout the water by the process of diffusion. This spreading of molecules cannot be seen through natural way, but through a microscope, an individual molecule of ink can be seen. From where the movement of the individual molecule due to its kinetic energy and collision with the

other molecules of ink and water are observed. Another interesting example is if someone uses the perfume then the nearby person can smell it due to the diffusion process. The diffusion process is related to the Markov process. The Markov process has three properties: drift, a random process and a jump process. Thus, a diffusion process is a Markov process with continuous sample paths.

1.5.1 Derivation of diffusion equation

In the year 1855, Adolf Fick was the first who found the analogy between convection and diffusion. Let us consider the rate of transfer per unit area of any section is given by F, u is the concentration of diffusing substance and x is its spatial co-ordinate normal to the section then according to Fick's first law

$$F \propto \frac{\partial u}{\partial x},$$

$$\Rightarrow F = -D\frac{\partial u}{\partial x}, \tag{1.20}$$

where D is the diffusion coefficient. The negative sign in this equation corresponds to the occurrences in the opposite direction of increasing concentration. This equation is valid for isotropic mediums. The derivation of the diffusion equation is performed with the help of Fick's law. Let us consider a volume of rectangular parallelepiped having sides parallel to the axes and have lengths of $2dx$, $2dy$ and $2dz$ (see Figure 1.1) [24].

The concentration at center $P(x, y, z)$ of parallelepiped is u. Then the entering rate of diffusing substance through $ABCD$ is given by

$$4dydz\left(F_x - \frac{\partial F_x}{\partial x}dx\right),$$

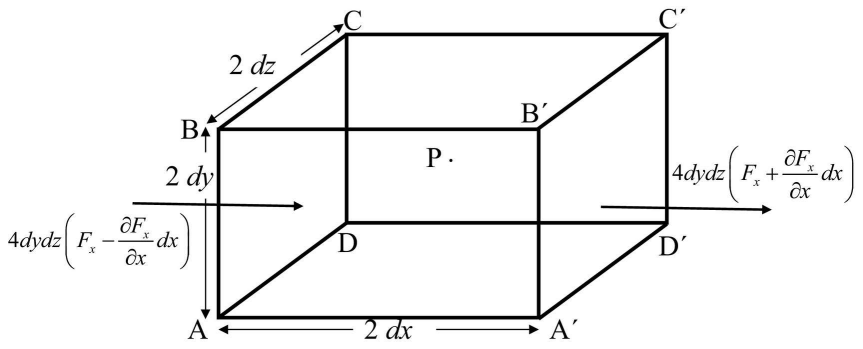

FIGURE 1.1: Geometry of considered rectangular volume

where F_x is known as rate of transfer in one unit area. Similarly, diffusing matter pass with a rate of

$$4dydz\left(F_x+\frac{\partial F_x}{\partial x}dx\right).$$

The net contribution in the X direction is given by

$$4dydz\left(F_x-\frac{\partial F_x}{\partial x}dx\right)-4dydz\left(F_x+\frac{\partial F_x}{\partial x}dx\right)=-8dxdydz\frac{\partial F_x}{\partial x}.$$

Thus in the similar way, the net contribution in Y and Z directions is obtained as

$$-8dxdydz\frac{\partial F_y}{\partial y} \text{ and } -8dxdydz\frac{\partial F_z}{\partial z},$$

respectively. The substance, which is diffusing, increases with a rate of

$$8dxdydz\frac{\partial u}{\partial t}.$$

Hence we have the following relation

$$8dxdydz\frac{\partial u}{\partial t}=-8dxdydz\frac{\partial F_x}{\partial x}-8dxdydz\frac{\partial F_y}{\partial y}-8dxdydz\frac{\partial F_z}{\partial z}.$$

$$\Rightarrow \frac{\partial u}{\partial t} + \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 0.$$

Considering diffusion coefficient as constants and using Fick's law, we have

$$\frac{\partial u}{\partial t} = D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right),$$

$$\frac{\partial u}{\partial t} = D\nabla^2 u. \tag{1.21}$$

The reaction-diffusion phenomenon involves both a reaction and diffusion process occurring simultaneously. In this common phenomenon, the concentration of the chemical substances changes with changes in space and time. And another local chemical reaction occurs with the transformation of substances into each other. This process is denoted by a diffusion equation with the addition of a reaction term $R(u)$ as

$$\frac{\partial u}{\partial t} = D\nabla^2 u + R(u). \tag{1.22}$$

1.6 Fractional order reaction-diffusion equation

When the concept of fractional order derivatives is taken into account, this gives advantage to the researchers to model any physical or chemical phenomenon more accurately, but this advantage comes with the complexity of the model. The fractional-order general form of the above classical reaction-diffusion equation (1.22) is given as follows

$$\frac{\partial^\alpha u}{\partial t^\alpha} = D\frac{\partial^\beta u}{\partial x^\beta} + R(u), \tag{1.23}$$

where $0 < \alpha \leq 1$ and $1 < \beta \leq 2$.

Experimental evidence suggests that the classical integer order diffusion equation

may not accurately model many physical and biological phenomena, this is where the fractional type diffusion model comes into the picture. Recently during the last two decades, many phenomena have been modelled with the help of fractional order diffusion equation, viz. Baeumer et al. [25] modelled the contaminant transport, Gorenflo et al. [26] gave a random walk model in the form of the fractional diffusion equation, Cartea and Negrete [27] developed a fractional order diffusion model to predict the option prices in markets with jumps, Magin et al. [28] modelled the porous biological tissues, Henry et al. [29] modelled the single processing in neuronal dendrites. Many researchers have modelled the cancer tumor cells with the help of fractional diffusion equation [30–34]. Although many phenomena can be modelled with a fractional diffusion equation, finding the solution of that model is not always easy due to the increase in complexity of the model. This becomes a challenging job for the researchers to develop an efficient method that can perform better on that type of complex model. Wang and Basu [35] developed a fast finite difference method to solve the space fractional diffusion model. Hanert and Piret [36] developed a numerical method with the help of Chebyshev polynomials and the Galerkin method to solve temporal fractional diffusion equation. Cheng [37] developed an efficient Eulerian-Lagrangian control volume method to solve the space-fractional diffusion equation. Li and Wu [38] have developed a numerical technique to solve distributed order fractional diffusion equations with the help of a classical quadrature formula. Wei et al. [39] have used the local radial basis function to develop a method to solve variable-order fractional diffusion equations for the two-dimensional irregular domain. Zaky and Tenreiro [40] have developed a spectral tau scheme with Legendre polynomial to solve multi-dimensional fractional diffusion equation. Verma and Kumar [41] have solved the multi-dimensional fractional diffusion equation with two step Adomian decomposition method. Obembe et al. [42] solved variable order fractional diffusion model describing fluid flow in a heterogeneous porous medium

with control volume finite difference approximations. Daihong Gu et al. [43] used Laplace transformation to solve the fractional diffusion model evaluating the performance of multiple fractured horizontal wells with stimulated reservoir volume in tight gas reservoirs. Bai et al. [44] used the finite difference method with the L_2 formula to solve the fractional diffusion model to measure the volatile organic compound concentrations of dry building materials. Obembe [45] solved the fractional diffusion model of single-phase, single-well simulation in hydrocarbon reservoirs with the block-centred finite-difference approximation method.

1.7 Numerical methods

Numerical methods provide an approximate solution to differential equations instead of an exact solution. The solution may be in the form of numeric data that is valid at the finitely considered points of the domain or in the form of some known functions that satisfy the equation approximately and are valid through the whole domain of computation. Derivation of the exact solution of differential equation has always been a challenging task. The analytical methods have their own limitations as those are unable to find out the solution of complex non-linear differential equation specially the fractional one. This problem can be tackled by the numerical methods. These methods are easy to apply and give better results even for complex fractional differential equation. Many numerical methods have been developed till now. Some of those are as follows:

(i) **Finite difference method**

This is one of the oldest methods to find the numerical solutions of differential equations. After discretization of the computation domain, the derivatives involved in differential equations are replaced by difference formulas. This

gives the recursive algorithm, and solution can be obtained by applying any one of the forward scheme, backward scheme or central difference scheme.

The forward space and time difference schemes are

$$\frac{\partial u(x_i,t_j)}{\partial x} \simeq \frac{u_{i+1}^j - u_i^j}{h}, \tag{1.24}$$

$$\frac{\partial u(x_i,t_j)}{\partial t} \simeq \frac{u_i^{j+1} - u_i^j}{k}, \tag{1.25}$$

and backward difference schemes are given for space and time as follows

$$\frac{\partial u(x_i,t_j)}{\partial x} \simeq \frac{u_i^j - u_{i-1}^j}{h}, \tag{1.26}$$

$$\frac{\partial u(x_i,t_j)}{\partial t} \simeq \frac{u_i^j - u_i^{j-1}}{k}. \tag{1.27}$$

The forward and backward difference approximations have a first order accuracy in both x and t. Another finite difference approximation with second order accuracy is the central difference scheme, which is defined by the following relations

$$\frac{\partial u(x_i,t_j)}{\partial x} \simeq \frac{u_{i+1}^j - u_{i-1}^j}{2h}, \tag{1.28}$$

$$\frac{\partial^2 u(x_i,t_j)}{\partial x^2} \simeq \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2}, \tag{1.29}$$

where h is the spatial step, k is the time step and u_i^j are the numerical estimates of the exact value of $u(x,t)$ at the point (x_i,t_j).

One can also approximate higher order derivative using finite difference approximation [46, 47].

(ii) **Non-standard finite difference method**

Unlike the standard finite difference method, non-standard finite difference method employs alternative discretization schemes to approximate for both time and spatial derivatives. These schemes may be chosen based on the specific characteristics of the problem, aiming to improve accuracy or stability. In standard finite difference methods, a uniform grid is often used, where the spatial domain is divided into equally spaced points. Non-standard approaches may involve non-uniform grids, where the spacing between points varies. This can be useful for refining the grid in regions of interest or adapting to the problem's characteristics.

According to this method, the discrete first derivative is defined as

$$\frac{du}{dt} = \frac{u_{r+1} - \psi(k)u_r}{\varphi(k)}, \tag{1.30}$$

where $\varphi(k)$ and $\psi(k)$ be the two functions depending on the step-size $k = \Delta t$ with

$$\psi(k) = 1 + o(k) \text{ and } \varphi(k) = k + o(k^2).$$

The two functions $\varphi(k)$ and $\psi(k)$ which depend on different parameters also appear in given differential equations. Moreover, $\varphi(k)$ is a continuous function which satisfies $0 < \varphi(k) < 1$, $k \to 0$. Some examples of $\varphi(k)$ which also satisfies these conditions viz.,

$$\varphi(k) = k, \ \varphi(k) = sinh(k) \ , \varphi(k) = exp(k) - 1 \text{ etc.}$$

There is no determined basis for the appropriate choices of the function $\varphi(k)$. More details of the non-standard finite difference method can be found in [48–50].

(iii) **Operational matrix method** [51–53]

The operational matrices method is a powerful technique for approximating solutions of integral and fractional differential equations. There are many equations including differential equations and integral equations which contain singularity and can not be solved by classical methods. The operational matrix method is effective in solving these equations and transforms the main problem into a set of algebraic equations. It not only simplifies the original problem but also accelerates the calculation process.

The operational matrices of the differentiation and integration operators can be evaluated in the following manner

$$\int_a^t \Psi(x)dx \approx P\Psi(t),$$

$$\frac{d\Psi(t)}{dt} \approx M\Psi(t),$$

where P and M are the $(n+1) \times (n+1)$ operational matrices of integration and differentiation respectively and $\Psi(t) = [\psi_0(t), \psi_1(t), \psi_2(t), ..., \psi_n(t)]$ is the orthogonal basis of any polynomial. More generally, one can write

$$\int_a^t ... \int_a^t \Psi(x)(dx)^k \approx P^k\Psi(t),$$

$$\frac{d^k\Psi(t)}{dt^k} \approx M^k\Psi(t).$$

(iv) **Finite element method** [54]

In this method, firstly the domain is divided into sub-domains referred as elements. The interpolation functions are selected to approximate the unknown function. The Ritz variational or Galerkin method is used to formulate a system of equations. After solving such a system, the solution can be obtained.

(v) **Cubic B-spline method** [55, 56]

The cubic B-spline method employs cubic B-spline functions to approximate the solution of a differential equation. B-splines or basis splines are piecewise-defined polynomial functions that are commonly used for curve and surface fitting or fitting of numerical data. Express the solution as a linear combination of cubic B-spline basis functions and calculate the derivatives of the cubic B-spline basis functions, which are needed to formulate the differential equation in terms of the B-spline representation. Substitute the B-spline representation of the solution into the differential equation and formulate a system of equations that can be solved for unknown coefficients.

(vi) **Sinc collocation method** [57, 58]

The Sinc collocation method is based on the use of the Sinc function, which is defined as the sine of a normalized variable divided by that variable. The Sinc collocation method is particularly effective for problems with singularities or discontinuities. When applied to differential equations, the method typically involves expressing derivatives of the solution in terms of Sinc functions and then enforcing the differential equation at the collocation points. This leads to a system of algebraic equations that can be solved for the unknown coefficients.

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