Chapter 2

Global exponential synchronization of complex valued recurrent neural networks in presence of uncertainty along with time-varying bounded and unbounded delay terms

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In this chapter, the global exponential synchronization of general class of complex valued recurrent neural networks (CVRNNs), introduced in previous section 1.3.2, with presence of uncertainty is investigated. The drive and response systems contain time-varying delay terms that have been given in the subsection 1.4.1. The concept of synchronization was first studied by Pecora and Carroll [84] towards the synchronization of master-slave or drive-response chaotic systems. Later on more general cases of synchronization, defined in the section introduction of the section 1.5, have been investigated in the literature [91, 92, 93, 94, 95, 96, 97, 98, 99].

2.1 Some preliminaries and Problem formulation

Notations

The standard notations are used in this thesis. The imaginary unit is denoted by i, where $i = \sqrt{-1}$. The symbols R, R^n and $R^{n\times n}$ denote the set of real numbers, set of n-dimensional real vectors and set of real matrices of $n \times n$ order, respectively. $w(t) \in C^n$ denotes the complex-valued function, where $w(t) = u(t) + iv(t), u(t)$ and $v(t) \in R^n$. C represents the set of complex number, C^n represents the complex-valued vectors of n-dimensional and $C^{n\times n}$ denotes the complex-valued matrices of $n \times n$ order. $\mu_p(M), p = 1, 2, \infty$ stands for the matrix measure of the square matrix $M \in R^{n\times n}$. $\|.\|_p, p = 1, 2, \infty$ represent vector norms.

2.1.1 Preliminaries

Assumption 1. *Let us consider* $z = z_1 + iz_2$, *where* $z_1, z_2 \in R$. $f_k(z)$ *and* $g_k(z)$ *are defined as*

$f_k(z) = f_k^R(z_1) + if_k^I(z_2)$ *and* $g_k(z) = g_k^R(z_1) + ig_k^I(z_2)$,

where $k = 1, 2, ..., n$, *and* $f_k^R(.), f_k^I(.), g_k^R(.), g_k^I(.) : R \to R$ *satisfies the Lipschitz conditions*

$$\|f_k^R(\nu) - f_k^R(\eta)\|_p \le r_k\|\nu - \eta\|_p,$$

$$\|f_k^I(\nu) - f_k^I(\eta)\|_p \le s_k\|\nu - \eta\|_p,$$

$$\|g_k^R(\nu) - g_k^R(\eta)\|_p \le m_k\|\nu - \eta\|_p,$$

$$\|g_k^I(\nu) - g_k^I(\eta)\|_p \le q_k\|\nu - \eta\|_p,$$

where $\|.\|_p$ *represents the vector norm,* $p = 1, 2, \infty$*, and* r_k, s_k, m_k*, and* q_k *are the Lipschitz constants,* ν *and* $\eta \in R^n$*.*

Lemma 2.1. *(Halanay inequality)[100] Let us consider* $v(t)$ *be a non-negative continuous function defined on* $[t_0 - \tau, +\infty)$*. Suppose two constants* k_1 *and* k_2*, with* $k_1 > k_2 > 0$*, such that the function v(t) satisfies the following inequality for* $t \geq t_0$*,*

$$D^+(v(t)) \leq -k_1 v(t) + k_2 \bar{v}(t),$$

where $\bar{v}(t) \triangleq sup_{t-\tau \leq s \leq t} v(s)$*. Then the inequality* $v(t) \leq \bar{v}(t_0)e^{-r(t-t_0)}$ *holds for* $t \geq t_0$*, where* $D^+v(t)$ *denotes the upper-right Dini derivative and defined as*

$$D^+v(t) = \overline{lim}_{h \to 0^+} \frac{v(t+h) - v(t)}{h},$$

where $h \to 0^+$ *represents that* $h \to 0$ *from the right-hand side and* r *is the exponential convergence rate, which is the unique positive solution of* $r = k_1 - k_2 e^{r\tau}$*.*

2.1.2 Problem formulation

Let us consider the CVRNN with time-varying delay with uncertain parameters as the master system as

$$\dot{w}(t) = -Cw(t) + (A + \Delta A)f(w(t)) + (B + \Delta B)g(w(t - \tau(t))) + L(t), \tag{2.1.1}$$

with the initial condition
$w(s) = \psi(s), \quad s \in [t_0 - \tau, t_0],$

where $\psi(s) = (\psi_1(s), \psi_2(s), .., \psi_n(s)) \in C^n$, in which $Re(\psi(s))$ and $Im(\psi(s))$ are continuous on $[t_0 - \tau, t_0]$ and $w(t) = (w_1(t), w_2(t), ..., w_n(t))^T \in C^n$ represents the state vector of n-neurons of the neural networks at time t, T denotes the transpose of the matrix, $C = diag(c_1, c_2, ..., c_n) \in R^{n\times n}$, $c_k > 0(k = 1, 2, .., n)$ is self-feedback connection weight matrix. $A = (a_{kj})_{n\times n}$ and $B = (b_{kj})_{n\times n}$, where $A, B \in C^{n\times n}$ are connection weight matrix and delayed connection weight matrix in the model (2.1.1), respectively. $\Delta A = \Delta a_{ij}$ and $\Delta B = \Delta b_{ij} \in C^{n\times n}$ denote the deviations of a_{ij} and b_{ij} respectively. $L = (l_1, l_2, ..., l_n)^T$ is the external input vector. $f(w(t)) =(f_1(w_1(t)), f_2(w_2(t)), ..., f_n(w_n(t)))^T : C^n \to C^n$, $g(w(t - \tau(t))) = (g_1(w_1(t-\tau(t))), g_2(w_2(t-\tau(t))), ..., g_n(w_n(t-\tau(t))))^T : C^n \to C^n$ denote the vector-valued without and with time-varying delay activation functions, where $\tau(t)$ is the transmission delay which satisfies $0 \leq \tau(t) \leq \tau(\tau > 0)$, where τ is a known constant. In equation (2.1.1), $\tau(t)$ is also considered in the form of unbounded function.

In the view of Assumption 1, if $w(t) = u(t) + iv(t)$ is considered, where $u(t), v(t) \in R^n$, then equation (2.1.1) can be rewritten as

$$\begin{aligned}
&\dot{u}(t) = -Cu(t) + (A^R + \Delta A^R)f^R(u(t)) - (A^I + \Delta A^I)f^I(v(t)) \\
&+ (B^R + \Delta B^R)g^R(u(t - \tau(t))) - (B^I + \Delta B^I)g^I(v(t - \tau(t))) + L^R(t), \\
&\dot{v}(t) = -Cv(t) + (A^I + \Delta A^I)f^R(u(t)) + (A^R + \Delta A^R)f^I(v(t)) \\
&+ (B^I + \Delta B^I)g^R(u(t - \tau(t))) + (B^R + \Delta B^R)g^I(v(t - \tau(t))) + L^I(t). \qquad (2.1.2)
\end{aligned}$$

The initial conditions of equation (2.1.2) will be

$$\begin{cases} u(s) = \psi^R(s), \\ v(s) = \psi^I(s), -\tau \leq s \leq 0, \end{cases}$$

where $\psi^R(s)$ denotes the real part and $\psi^I(s)$ denotes the imaginary part of the function $\psi(s)$, and $\|\psi^R\|_p = sup_{t_0-\tau\leq s\leq t_0}\|\psi^R(s)\|_p$ denotes the norm of the function

$\psi^R \in C([t_0-\tau, t_0], R^n)$ and $\|\psi^I\|_p = sup_{t_0-\tau\le s\le t_0}\|\psi^I(s)\|_p$ is the norm of the function $\psi^I \in C([t_0-\tau, t_0], R^n)$. The initial functions $u(s)$ and $v(s)$ are continuous functions.

Now considering the corresponding response system as

$$\dot{\widetilde{w}}(t) = -C\widetilde{w}(t) + (A+\Delta A)f(\widetilde{w}(t)) + (B+\Delta B)g(\widetilde{w}(t-\tau(t))) + L(t) + M(t), \tag{2.1.3}$$

where $M(t) = (M_1(t), M_2(t), .., M_n(t))^T \in C^n$ is the coupling control,

$\widetilde{w}(t) = (\widetilde{w}_1(t), \widetilde{w}_2(t), ..., \widetilde{w}_n(t))^T \in C^n$, and $C, A, B, \Delta A, \Delta B, L(t)$ and $\tau(t)$ are already been described in the system (2.1.1).

The initial condition of the system (2.1.3) is taken as

$\widetilde{w}(s) = \widetilde{\psi}(s), \quad s \in [t_0-\tau, t_0]$,

where $\widetilde{\psi}(s) = (\widetilde{\psi}_1(s), \widetilde{\psi}_2(s), ..., \widetilde{\psi}_n(s)) \in C^n, Re(\widetilde{\psi}(s))$ and $Im(\widetilde{\psi}(s))$ are continuous functions in $[t_0-\tau, t_0]$.

Now, separating equation (2.1.3) into real and imaginary parts, we get

$$\begin{aligned}\dot{\widetilde{u}}(t) &= -C\widetilde{u}(t) + (A^R+\Delta A^R)f^R(\widetilde{u}(t)) - (A^I+\Delta A^I)f^I(\widetilde{v}(t)) + (B^R\\ &+ \Delta B^R)g^R(\widetilde{u}(t-\tau(t))) - (B^I+\Delta B^I)g^I(\widetilde{v}(t-\tau(t))) + L^R(t) + M^R(t)\\ \dot{\widetilde{v}}(t) &= -C\widetilde{v}(t) + (A^I+\Delta A^I)f^R(\widetilde{u}(t)) + (A^R+\Delta A^R)f^I(\widetilde{v}(t)) + (B^I\\ &+ \Delta B^I)g^R(\widetilde{u}(t-\tau(t))) + (B^R+\Delta B^R)g^I(\widetilde{v}(t-\tau(t))) + L^I(t) + M^I(t)\end{aligned} \tag{2.1.4}$$

and the corresponding initial conditions are

$$\begin{cases}\widetilde{u}(s) = \widetilde{\psi}^R(s),\\ \widetilde{v}(s) = \widetilde{\psi}^I(s), -\tau \le s \le 0,\end{cases}$$

where $M^R(t), M^I(t)$ are the control input vectors of different forms. Here the control inputs are defined as the linear combinations of differences between the states of the systems (2.1.2) and (2.1.4).

Now, the control input vectors are defined as

$$M^R(t) = \Omega e^R(t) \tag{2.1.5}$$

$$and \quad M^I(t) = \Omega e^I(t), \tag{2.1.6}$$

where $M^R(t) = [M_1^R(t), M_2^R(t), ..., M_n^R(t)]^T$, $M^I(t) = [M_1^I(t), M_2^I(t), ..., M_n^I(t)]^T$, $e^R(t) = [e_1^R(t), e_2^R(t), ..., e_n^R(t)]^T$, $e^I(t) = [e_1^I(t), e_2^I(t), ..., e_n^I(t)]^T$, and $\Omega = \begin{pmatrix} \alpha_{11} \dots \alpha_{1n} \\ \vdots \ddots \vdots \\ \alpha_{n1} \dots \alpha_{nn} \end{pmatrix}$ stands for the controller gain matrix.

Let us define the error functions for synchronization of master system (2.1.2) and response system (2.1.4) as $e^R(t) = \tilde{u}(t) - u(t), e^I(t) = \tilde{v}(t) - v(t)$. The master and response systems will be synchronized if $e^R(t) \to 0$ and $e^I(t) \to 0$ as $t \to \infty$. From equations (2.1.5) and (2.1.6), we get the error systems as

$$\begin{aligned}
\dot{e}^R(t) &= -Ce^R(t) + (A^R + \Delta A^R)\tilde{f}^R(e(t)) - (A^I + \Delta A^I)\tilde{f}^I(e(t)) \\
&+ (B^R + \Delta B^R)\tilde{g}^R(e(t-\tau(t))) - (B^I + \Delta B^I)\tilde{g}^I(e(t-\tau(t))) + \Omega e^R(t), \\
\dot{e}^I(t) &= -Ce^I(t) + (A^R + \Delta A^R)\tilde{f}^I(e(t)) + (A^I + \Delta A^I)\tilde{f}^R(e(t)) \\
&+ (B^R + \Delta B^R)\tilde{g}^I(e(t-\tau(t))) + (B^I + \Delta B^I)\tilde{g}^R(e(t-\tau(t))) + \Omega e^I(t),
\end{aligned} \tag{2.1.7}$$

where

$e^R(t) = (e_1^R(t), e_2^R(t)...e_n^R(t))^T \in R^n, e^I(t) = (e_1^I(t), e_2^I(t)...e_n^I(t))^T \in R^n.$

$\widetilde{f}^R(e(t)) = f^R(\widetilde{w}(t)) - f^R(w(t)), \widetilde{f}^I(e(t)) = f^I(\widetilde{w}(t)) - f^I(w(t)),$

$\widetilde{g}^R(e(t-\tau(t))) = g^R(\widetilde{w}(t-\tau(t))) - g^R(w(t-\tau(t))),$

$\widetilde{g}^I(e(t-\tau(t))) = g^I(\widetilde{w}(t-\tau(t))) - g^I(w(t-\tau(t))).$

Definition 2.1.1. [99] After separating real and imaginary parts of systems (2.1.1) and (2.1.3), the master and response systems (2.1.2) and (2.1.4) are said to be exponentially synchronized if there exist scalars $k > 1$ and $\beta > 0$, such that $\|\widetilde{u}(t)-u(t)\|_p+\|\widetilde{v}(t)-v(t)\|_p \leq k\ sup_{[t_0-\tau,t_0]}(\|\widetilde{\psi}^R(t)-\psi^R(t)\|_p+\|\widetilde{\psi}^I(t)-\psi^I(t)\|_p)e^{-\beta(t-t_0)}$, $t \geq t_0$.

The exponential synchronization can be achieved between the systems (2.1.2) and (2.1.4) by choosing the controller gain matrix Ω according to Global exponential stability criteria.

2.2 Main results

Theorem 2.2. *If the controller gain matrix Ω satisfies the following inequality under Assumption 1 as*

$$\begin{aligned} 0 < (m_k + q_k)(\|B^R + \Delta B^R\|_p + \|B^I + \Delta B^I\|_p) < \\ -\{\mu_p(-C+\Omega) + (r_k + s_k)(\|A^R + \Delta A^R\|_p + \|A^I + \Delta A^I\|_p)\}, \end{aligned} \tag{2.2.1}$$

where $p = 1, 2, \infty$, then the CVRNNs (2.1.1) with uncertain parameters, bounded and unbounded time-varying delay terms will be globally exponentially synchronized with the considered response CVRNNs (2.1.3).

Proof. Let us define the following Lyapunov function as

$$V_1(e(t)) = \|e^R(t)\|_p + \|e^I(t)\|_p.$$

In the view of Taylor's formula with remainder of Peano, definition of $D^+V_1(e(t))$, and from equation (2.1.7), we get

$$\begin{aligned}
D^+V_1(e(t)) = &\overline{lim}_{\epsilon\to 0^+}\frac{\|e^R(t+\epsilon)\|_p + \|e^I(t+\epsilon)\|_p - \|e^R(t)\|_p - \|e^I(t)\|_p}{\epsilon}\\
=&\overline{lim}_{\epsilon\to 0^+}\Bigg[\frac{\|e^R(t)+\epsilon\dot{e}^R(t)+\mathcal{O}(\epsilon)\|_p + \|e^I(t)+\epsilon\dot{e}^I(t)+\mathcal{O}(\epsilon)\|_p}{\epsilon}\\
&-\frac{\|e^R(t)\|_p + \|e^I(t)\|_p}{\epsilon}\Bigg]\\
=&\overline{lim}_{\epsilon\to 0^+}\Bigg(\|e^R(t)+\epsilon(-Ce^R(t)+(A^R+\Delta A^R)\widetilde{f}^R(e(t))\\
&-(A^I+\Delta A^I)\widetilde{f}^I(e(t))+(B^R+\Delta B^R)\widetilde{g}^R(e(t-\tau(t)))\\
&-(B^I+\Delta B^I)\widetilde{g}^I(e(t-\tau(t)))+\Omega e^R(t))+\mathcal{O}(\epsilon)\|_p-\|e^R(t)\|_p\Bigg)\Bigg/\epsilon\\
&+\overline{lim}_{\epsilon\to 0^+}\Bigg(\|e^I(t)+\epsilon(-Ce^I(t)+(A^R+\Delta A^R)\widetilde{f}^I(e(t))\\
&+(A^I+\Delta A^I)\widetilde{f}^R(e(t))+(B^R+\Delta B^R)\widetilde{g}^I(e(t-\tau(t)))\\
&+(B^I+\Delta B^I)\widetilde{g}^R(e(t-\tau(t)))+\Omega e^I(t))+\mathcal{O}(\epsilon)\|_p-\|e^I(t)\|_p\Bigg)\Bigg/\epsilon\\
\le&\overline{lim}_{\epsilon\to 0^+}\frac{\|I+\epsilon(-C+\Omega)\|_p-1}{\epsilon}\|e^R(t)\|_p\\
&+\|A^R+\Delta A^R\|_p\|\widetilde{f}^R(e(t))\|_p+\|A^I+\Delta A^I\|_p\|\widetilde{f}^I(e(t))\|_p\\
&+\|B^R+\Delta B^R\|_p\|\widetilde{g}^R(e(t-\tau(t)))\|_p+\|B^I+\Delta B^I\|_p\|\widetilde{g}^I(e(t-\tau(t)))\|_p\\
&+\overline{lim}_{\epsilon\to 0^+}\frac{\|I+\epsilon(-C+\Omega)\|_p-1}{\epsilon}\|e^I(t)\|_p\\
&+\|A^R+\Delta A^R\|_p\|\widetilde{f}^I(e(t))\|_p+\|A^I+\Delta A^I\|_p\|\widetilde{f}^R(e(t))\|_p\\
&+\|B^R+\Delta B^R\|_p\|\widetilde{g}^I(e(t-\tau(t)))\|_p+\|B^I+\Delta B^I\|_p\|\widetilde{g}^R(e(t-\tau(t)))\|_p.
\end{aligned} \tag{2.2.2}$$

Now, we have

$$\begin{aligned}
\|\widetilde{f}^R(e(t))\|_p &\leq r_k\|e(t)\|_p \quad [by\ Assumption 1]\\
&= r_k\|e^R(t)+ie^I(t)\|_p\\
&\leq r_k(\|e^R(t)\|_p+\|e^I(t)\|_p), \quad [by\ Minkowski\ inequality]\\
\|\widetilde{f}^I(e(t))\|_p &\leq s_k\|e(t)\|_p \qquad\qquad [by\ Assumption 1]\\
&= s_k\|e^R(t)+ie^I(t)\|_p\\
&\leq s_k(\|e^R(t)\|_p+\|e^I(t)\|_p), \quad [by\ Minkowski\ inequality]\\
&Similarly,\\
\|\widetilde{g}^R(e(t-\tau(t)))\|_p &\leq m_k\|e(t-\tau(t))\|_p = m_k\|e^R(t-\tau(t))+ie^I(t-\tau(t))\|_p\\
&\leq m_k(\|e^R(t-\tau(t))\|_p+\|e^I(t-\tau(t))\|_p),\\
\|\widetilde{g}^I(e(t-\tau(t)))\|_p &\leq q_k\|e(t-\tau(t))\|_p = q_k\|e^R(t-\tau(t))+ie^I(t-\tau(t))\|_p\\
&\leq q_k(\|e^R(t-\tau(t))\|_p+\|e^I(t-\tau(t))\|_p).
\end{aligned} \tag{2.2.3}$$

Using inequalities (2.2.3) in the inequality (2.2.2), we have

$$\begin{aligned}
D^+V_1(e(t)) \leq &\overline{lim}_{\epsilon\to0^+}\frac{\|I+\epsilon(-C+\Omega)\|_p-1}{\epsilon}\|e^R(t)\|_p+\|A^R+\Delta A^R\|_p\|\widetilde{f}^R(e(t))\|_p\\
&+\|A^I+\Delta A^I\|_p\|\widetilde{f}^I(e(t))\|_p+\|B^R+\Delta B^R\|_p\|\widetilde{g}^R(e(t-\tau(t)))\|_p\\
&+\|B^I+\Delta B^I\|_p\|\widetilde{g}^I(e(t-\tau(t)))\|_p\\
&+\overline{lim}_{\epsilon\to0^+}\frac{\|I+\epsilon(-C+\Omega)\|_p-1}{\epsilon}\|e^I(t)\|_p+\|A^R+\Delta A^R\|_p\|\widetilde{f}^I(e(t))\|_p\\
&+\|A^I+\Delta A^I\|_p\|\widetilde{f}^R(e(t))\|_p+\|B^R+\Delta B^R\|_p\|\widetilde{g}^I(e(t-\tau(t)))\|_p\\
&+\|B^I+\Delta B^I\|_p\|\widetilde{g}^R(e(t-\tau(t)))\|_p
\end{aligned}$$

$$
\begin{aligned}
\leq & \overline{lim}_{\epsilon \to 0^+} \frac{\|I + \epsilon(-C + \Omega)\|_p - 1}{\epsilon} \|e^R(t)\|_p + (r_k \|A^R + \Delta A^R\|_p \\
& + s_k \|A^I + \Delta A^I\|_p)(\|e^R(t)\|_p + \|e^I(t)\|_p) + (m_k \|B^R + \Delta B^R\|_p \\
& + q_k \|B^I + \Delta B^I\|_p)(\|e^R(t - \tau(t))\|_p + \|e^I(t - \tau(t))\|_p) \\
& + \overline{lim}_{\epsilon \to 0^+} \frac{\|I + \epsilon(-C + \Omega)\|_p - 1}{\epsilon} \|e^I(t)\|_p + (s_k \|A^R + \Delta A^R\|_p \\
& + r_k \|A^I + \Delta A^I\|_p)(\|e^R(t)\|_p + \|e^I(t)\|_p) + (m_k \|B^R + \Delta B^R\|_p \\
& + q_k \|B^I + \Delta B^I\|_p)(\|e^R(t - \tau(t))\|_p + \|e^I(t - \tau(t))\|_p).
\end{aligned} \tag{2.2.4}
$$

In the view of Definition 1.4.4 and equation (2.2.4), we get

$$
\begin{aligned}
D^+ V_1(e(t)) \leq & \mu_p(-C + \Omega) \|e^R(t)\|_p + (r_k \|A^R + \Delta A^R\|_p \\
& + s_k \|A^I + \Delta A^I\|_p)(\|e^R(t)\|_p + \|e^I(t)\|_p) + (m_k \|B^R + \Delta B^R\|_p \\
& + q_k \|B^I + \Delta B^I\|_p)(\|e^R(t - \tau(t))\|_p + \|e^I(t - \tau(t))\|_p) + \mu_p(-C + \Omega) \|e^I(t)\|_p \\
& + (s_k \|A^R + \Delta A^R\|_p + r_k \|A^I + \Delta A^I\|_p)(\|e^R(t)\|_p + \|e^I(t)\|_p) \\
& + (m_k \|B^R + \Delta B^R\|_p + q_k \|B^I + \Delta B^I\|_p)(\|e^R(t - \tau(t))\|_p + \|e^I(t - \tau(t))\|_p) \\
\leq & \{\mu_p(-C + \Omega) + (r_k + s_k)(\|A^R + \Delta A^R\|_p + \|A^I + \Delta A^I\|_p)\} \\
& \times (\|e^R(t)\|_p + \|e^I(t)\|_p) + (m_k + q_k)(\|B^R + \Delta B^R\|_p + \|B^I + \Delta B^I\|_p)\} \\
& \times (\|e^R(t - \tau(t))\|_p + \|e^I(t - \tau(t))\|_p) \\
\leq & \{\mu_p(-C + \Omega) + (r_k + s_k)(\|A^R + \Delta A^R\|_p + \|A^I + \Delta A^I\|_p)\} V_1(e(t)) \\
& + (m_k + q_k)(\|B^R + \Delta B^R\|_p + \|B^I + \Delta B^I\|_p)\} V_1(e(t - \tau(t))) \\
\leq & \{\mu_p(-C + \Omega) + (r_k + s_k)(\|A^R + \Delta A^R\|_p + \|A^I + \Delta A^I\|_p)\} V_1(e(t)) \\
& + (m_k + q_k)(\|B^R + \Delta B^R\|_p + \|B^I + \Delta B^I\|_p)\} sup_{t - \tau \leq s \leq t} V_1(e(s)).
\end{aligned} \tag{2.2.5}
$$

Let, $k_1 = -\{\mu_p(-C + \Omega) + (r_k + s_k)(\|A^R + \Delta A^R\|_p + \|A^I + \Delta A^I\|_p)\}$

and $\quad k_2 = (m_k + q_k)(\|B^R + \Delta B^R\|_p + \|B^I + \Delta B^I\|_p)$.

From the given condition (2.2.1), we have $0 < k_2 < k_1$. Using Lemma 2.1, it follows that

$$V_1(e(t)) \leq sup_{t_0-\tau\leq s\leq t_0} V_1(e(s))e^{-r(t-t_0)}, \tag{2.2.6}$$

where r is the unique positive solution of

$$\begin{aligned} r =& k_1 - k_2e^{r\tau} \\ =& -\{\mu_p(-C+\Omega) + (r_k+s_k)(\|A^R+\Delta A^R\|_p + \|A^I+\Delta A^I\|_p)\} \\ & - (m_k+q_k)(\|B^R+\Delta B^R\|_p + \|B^I+\Delta B^I\|_p)e^{r\tau}. \end{aligned} \tag{2.2.7}$$

Hence, it may be concluded from the definition 2.1.1 that $V_1(e(t))$ exponentially converges to zero with r convergence rate, which also implies that real part $e^R(t)$ and imaginary part $e^I(t)$ of the error system (2.1.7) converge globally and exponentially to zero. Hence every trajectory of $\widetilde{u}(t), \widetilde{v}(t)$ of equation (2.1.4) will be globally exponentially synchronized with the trajectory of $u(t), v(t)$ of equation (2.1.2), respectively. □

Note. The inequality (2.2.1) of the Theorem 2.2 can also be proved using the following corollary.

Corollary 2.3. *Under Assumption 1, the response CVRNN (2.1.3) with uncertain terms, and bounded and unbounded time-varying delay terms will be globally exponentially synchronized with considered master system (2.1.1) , if the coupling matrix Ω satisfies the following condition*

$$\begin{aligned} 0 <& (m_k+q_k)(\|B^R+\Delta B^R\|_p + \|B^I+\Delta B^I\|_p) \\ & < -\{\mu_p(-C) + \mu_p(\Omega) + (r_k+s_k)(\|A^R+\Delta A^R\|_p + \|A^I+\Delta A^I\|_p)\}, \end{aligned} \tag{2.2.8}$$

where $p = 1, 2, \infty$.

Now using the above corollary, we get

$$
\begin{aligned}
0 <& (m_k + q_k)(\|B^R + \Delta B^R\|_p + \|B^I + \Delta B^I\|_p) \\
&< -\{\mu_p(-C) + \mu_p(\Omega) + (r_k + s_k)(\|A^R + \Delta A^R\|_p + \|A^I + \Delta A^I\|_p)\} \\
&< -\{\mu_p(-C + \Omega) + (r_k + s_k)(\|A^R + \Delta A^R\|_p + \|A^I + \Delta A^I\|_p)\},
\end{aligned}
$$

[using inequality(iii) of Lemma 1.3]

which is the considered inequality (2.2.1) that helps to obtain global exponential synchronization of the CVRNNs (2.1.1) and (2.1.3) along with their conditions.

Remark 2.4. The sufficient condition of global exponential synchronization of CVRNNs (2.1.2) and (2.1.4) with uncertain parameters is independent of the delay terms but dependent on uncertain parameters, system parameters and also control gain matrix. In this chapter, construct of the control gain matrix is the main task. Here, we have chosen the control input vectors are the linear combination of differences between the state of the systems (2.1.2) and (2.1.4). The time delay term is considered as the bounded and unbounded time-varying delay terms. The researchers in [101] have considered the bounded and unbounded delay terms in non-autonomous RNNs with discrete and continuous time and other types of NNs ,[102, 103] etc. in real variables. In the present work the results are discussed in complex-valued RNNs.

Remark 2.5. The matrix measure has both negative and positive values as compared to matrix norm. In the literature, it is fact that the most of the results for synchronization and stability analysis are in the form of the norm and it has the limitation of values i.e., it can have only non-negative values. From [99], it is seen that the matrix measure method is more powerful method because it deals with the connection weight functions having positive and negative values and also it corresponds to the excitation and inhibition in neurons. So the proposed global exponential synchronization criteria of CVRNNs with uncertain parameters having bounded and unbounded time-varying delays by using matrix measure method are more precise compared to matrix norms.

2.3 Numerical Examples

In this section, two numerical examples are taken to validate the effectiveness and reliability of the proposed synchronization scheme.

Example 2.3.1. For n=2, let us consider CVRNNs as the master system (2.1.1) with the uncertain parameters and bounded time-varying delay terms as

$$C= \begin{pmatrix} 2.5 & 0 \\ 0 & 2 \end{pmatrix}, \ A= \begin{pmatrix} 1+2i & -2-i \\ -2.5-i & -1-0.5i \end{pmatrix}, \ \Delta A= \begin{pmatrix} 0.8+0.5i & 1-0.3i \\ 1+i & 0 \end{pmatrix},$$

$$B= \begin{pmatrix} 2-i & -1+i \\ -1+i & -1+2i \end{pmatrix}, \ \Delta B= \begin{pmatrix} 0.1+0.4i & 0.7+0.1i \\ 0.6-0.2i & 1 \end{pmatrix}, \ \Omega= \begin{pmatrix} -7.3 & 1.6 \\ 1.1 & -6.3 \end{pmatrix},$$

$$L(t) = \begin{pmatrix} 6sin(t+1) - 4cos(t-1)i \\ 5cos(t+1) - 6sin(t-1)i \end{pmatrix}, \ \tau(t) = (2 + cos(t)).$$

The activation functions are taken as

$$\begin{aligned}
f_j(w_j(t)) =& \frac{1-exp(-u_j)}{1+exp(-u_j)} + i\frac{1}{1+exp(-v_j)},\\
g_j(w_j(t)) =& \frac{1-exp(-v_j)}{1+exp(-v_j)} + i\frac{1}{1+exp(-u_j)}, (j=1,2),
\end{aligned}$$

which reduce the master system (2.1.2) as

$$\begin{aligned}
\dot{u}_1(t) =& -2.5u_1(t) + 1.8((1-exp(-u_1))/(1+exp(-u_1)))\\
& - 1((1-exp(-u_2))/(1+exp(-u_2))) - 2.5(1/(1+exp(-v_1)))\\
& + 1.3(1/(1+exp(-v_2))) + 2.1((1-exp(-v_1))/(1+exp(-v_1)))\\
& - 0.3((1-exp(-v_2))/(1+exp(-v_2))) + 0.6(1/(1+exp(-u_1)))\\
& - 1.1(1/(1+exp(-u_2))) + 6sin(t+1).\\
\dot{v}_1(t) =& -2.5v_1(t) + 2.5((1-exp(-u_1))/(1+exp(-u_1)))\\
& - 1.3((1-exp(-u_2))/(1+exp(-u_2))) + 1.8(1/(1+exp(-v_1)))\\
& - 1(1/(1+exp(-v_2))) - 0.6((1-exp(-v_1))/(1+exp(-v_1)))\\
& + 1.1((1-exp(-v_2))/(1+exp(-v_2))) + 2.1(1/(1+exp(-u_1)))\\
& - 0.3(1/(1+exp(-u_2))) - 4cos(t-1).\\
\dot{u}_2(t) =& -2u_2(t) - 1.5((1-exp(-u_1))/(1+exp(-u_1)))\\
& - 1((1-exp(-u_2))/(1+exp(-u_2))) + 0.5(1/(1+exp(-v_2)))\\
& - 0.4((1-exp(-v_1))/(1+exp(-v_1))) - 0.8(1/(1+exp(-u_1))\\
& - 2(1/(1+exp(-u_2))) + 5cos(t+1).\\
\dot{v}_2(t) =& -2v_2(t) - 0.5((1-exp(-u_2))/(1+exp(-u_2))) - 1.5(1/(1+exp(-v_1)))\\
& - 1(1/(1+exp(-v_2))) + 0.8((1-exp(-v_1))/(1+exp(-v_1)))\\
& + 2((1-exp(-v_2))/(1+exp(-v_2))) - 0.4(1/(1+exp(-u_1)))\\
& - 6sin(t-1).
\end{aligned} \tag{2.3.1}$$

The corresponding response system (2.1.4) with the same parameters and time delay terms can be written as

$$\begin{aligned}
\dot{\widetilde{u}}_1(t) =& -2.5\widetilde{u}_1(t) + 1.8((1-exp(-\widetilde{u}_1))/(1+exp(-\widetilde{u}_1)))\\
&-1((1-exp(-\widetilde{u}_2))/(1+exp(-\widetilde{u}_2))) - 2.5(1/(1+exp(-\widetilde{v}_1)))\\
&+1.3(1/(1+exp(-\widetilde{v}_2))) + 2.1((1-exp(-\widetilde{v}_1))/(1+exp(-\widetilde{v}_1)))\\
&-0.3((1-exp(-\widetilde{v}_2))/(1+exp(-\widetilde{v}_2))) + 0.6(1/(1+exp(-\widetilde{u}_1)))\\
&-1.1(1/(1+exp(-\widetilde{u}_2))) + 6sin(t+1).\\
\dot{\widetilde{v}}_1(t) =& -2.5\widetilde{v}_1(t) + 2.5((1-exp(-\widetilde{u}_1))/(1+exp(-\widetilde{u}_1)))\\
&-1.3((1-exp(-\widetilde{u}_2))/(1+exp(-\widetilde{u}_2))) + 1.8(1/(1+exp(-\widetilde{v}_1)))\\
&-1(1/(1+exp(-\widetilde{v}_2))) - 0.6((1-exp(-\widetilde{v}_1))/(1+exp(-\widetilde{v}_1)))\\
&+1.1((1-exp(-\widetilde{v}_2))/(1+exp(-\widetilde{v}_2))) + 2.1(1/(1+exp(-\widetilde{u}_1)))\\
&-0.3(1/(1+exp(-\widetilde{u}_2))) - 4cos(t-1).\\
\dot{\widetilde{u}}_2(t) =& -2\widetilde{u}_2(t) - 1.5((1-exp(-\widetilde{u}_1))/(1+exp(-\widetilde{u}_1)))\\
&-1((1-exp(-\widetilde{u}_2))/(1+exp(-\widetilde{u}_2))) + 0.5(1/(1+exp(-\widetilde{v}_2)))\\
&-0.4((1-exp(-\widetilde{v}_1))/(1+exp(-\widetilde{v}_1))) - 0.8(1/(1+exp(-\widetilde{u}_1)))\\
&-2(1/(1+exp(-\widetilde{u}_2))) + 5cos(t+1).\\
\dot{\widetilde{v}}_2(t) =& -2\widetilde{v}_2(t) - 0.5((1-exp(-\widetilde{u}_2))/(1+exp(-\widetilde{u}_2))) - 1.5(1/(1+exp(-\widetilde{v}_1)))\\
&-1(1/(1+exp(-\widetilde{v}_2))) + 0.8((1-exp(-\widetilde{v}_1))/(1+exp(-\widetilde{v}_1)))\\
&+2((1-exp(-\widetilde{v}_2))/(1+exp(-\widetilde{v}_2))) - 0.4((1/(1+exp(-\widetilde{u}_1))))\\
&-6sin(t-1).
\end{aligned} \tag{2.3.2}$$

Figure 2.1 depicts the 3-D plot of the trajectories $u_1(t), v_1(t), u_2(t)$ and $v_2(t)$ with time t of the system (2.3.1) without uncertain parameters and with bounded time-varying delay terms $\tau(t) = 2 + cos(t)$. Figure 2.2 depicts the 3-D plot of the trajectories of the system (2.3.1) with uncertain parameters and the same bounded time-varying delay terms.

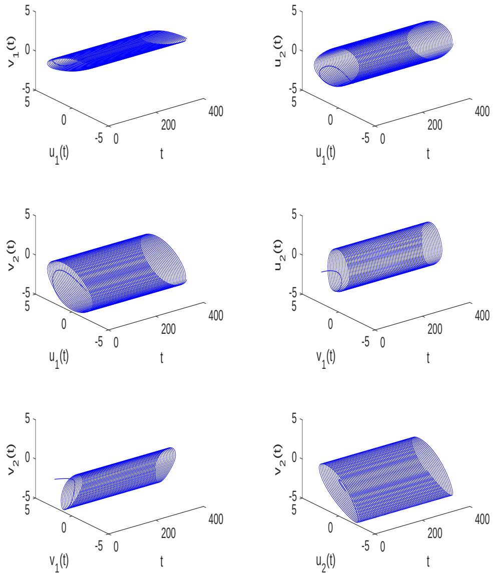

Figure 2.1: Plots of the state trajectories $u_1(t), v_1(t), u_2(t)$ and $v_2(t)$ with respect to time t in 3-dimensional space without uncertain parameters ΔA and ΔB of the system (2.3.1) and bounded time varying delay term $\tau(t) = (2 + cos(t))$.

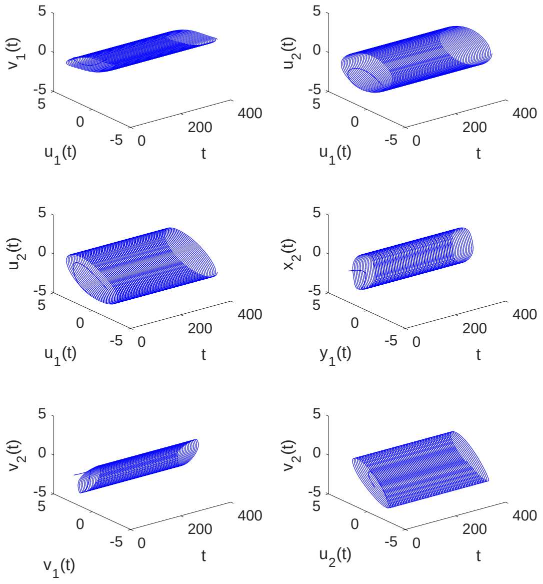

Figure 2.2: Plots of the state trajectories $u_1(t), v_1(t), u_2(t)$ and $v_2(t)$ with respect to time t in 3-dimensional space with uncertain parameters ΔA and ΔB and bounded time varying delay term $\tau(t) = (2 + cos(t))$.

$M(t) = \Omega(\widetilde{w}(t) - w(t))$ is the coupling control matrix.

Considering p=2, the system (2.3.1) satisfies the Assumption 1. The system parameters are considered as $r_k = 0.5, s_k = 0.25, m_k = 0.5$ and $q_k = 0.25$. The initial conditions of master and response systems (2.3.1) and (2.3.2) are taken as

$$w_1(s) = -2 + 2.4i, w_2(s) = -1 - 1.4i,$$

$$\widetilde{w}_1(s) = -2.3 - 1.8i,\ \widetilde{w}_2(s) = 1.2 - i,\ \ for\ s \in [-3, 0] \text{ respectively.}$$

It can be verified that

$$k_1{=}-\{\mu_p(-C + \Omega) + (r_k + s_k)(\|A^R + \Delta A^R\|_p + \|A^I + \Delta A^I\|_p)\} = 3.6236,$$

$$k_2{=}-(m_k + q_k)(\|B^R + \Delta B^R\|_p + \|B^I + \Delta B^I\|_p) = 3.3629,$$

i.e., $k_1 > k_2$. Thus from Theorem 2.2, we can conclude that the systems (2.3.1) and (2.3.2) will be globally exponential synchronized in the presence of uncertain parameters with bounded time-varying delay terms.

Figure 2.3 shows that the systems (2.3.1) and (2.3.2) are globally exponential synchronized with bounded time-varying delay terms and without uncertain parameters. Figure 2.4 also shows the global exponential synchronization of the considered systems with bounded time varying delay terms in the presence of uncertain parameters. Figures 2.5 and 2.6 depict that the error functions converge to zero after a small time duration with convergence rate 0.0228 which also confirms the global exponential synchronization of CVRNNs (2.3.1) and (2.3.2).

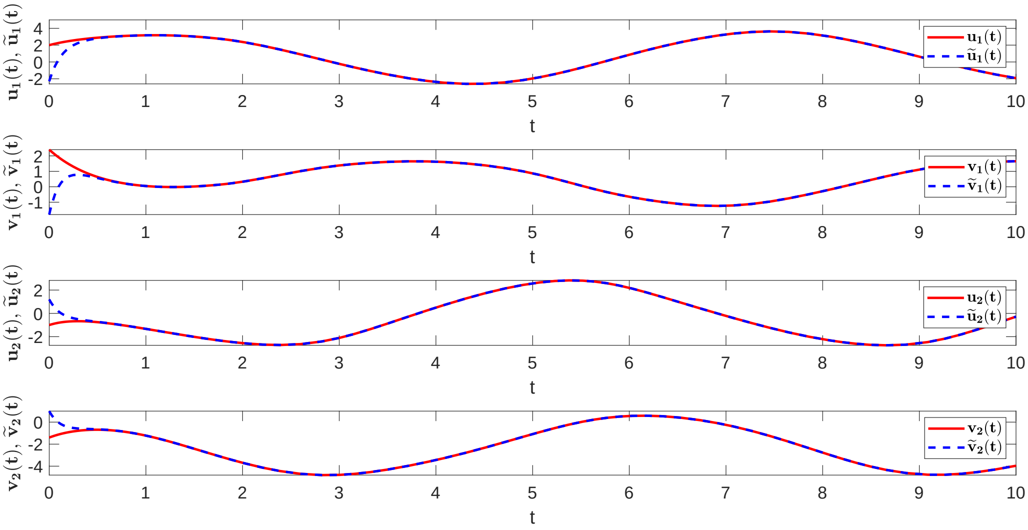

Figure 2.3: Plots of the state trajectories $u_1(t)$ and $\widetilde{u}_1(t), v_1(t)$ and $\widetilde{v}_1(t), u_2(t)$ and $\widetilde{u}_2(t), v_2(t)$ and $\widetilde{v}_2(t)$ with respect to time t, of master system (2.3.1) and response system (2.3.2) without uncertain parameters ΔA and ΔB and bounded time varying delay term $\tau(t) = (2 + cos(t))$.

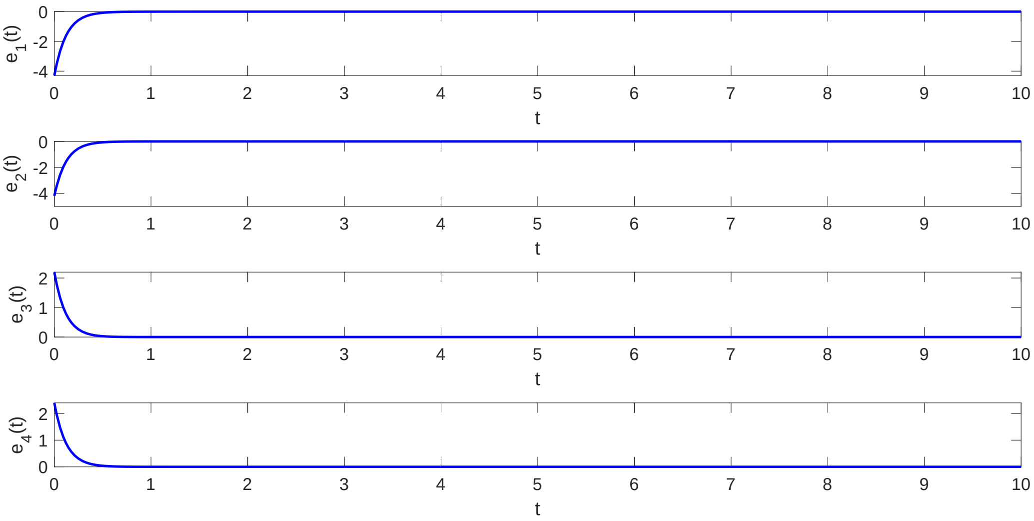

Figure 2.4: Plots of the global exponential synchronization errors $e_i(t), i = 1,2,3,4$ of the systems (2.3.1) and (2.3.2) without uncertain parameters ΔA and ΔB and bounded time varying delay term $\tau(t) = (2 + cos(t))$.

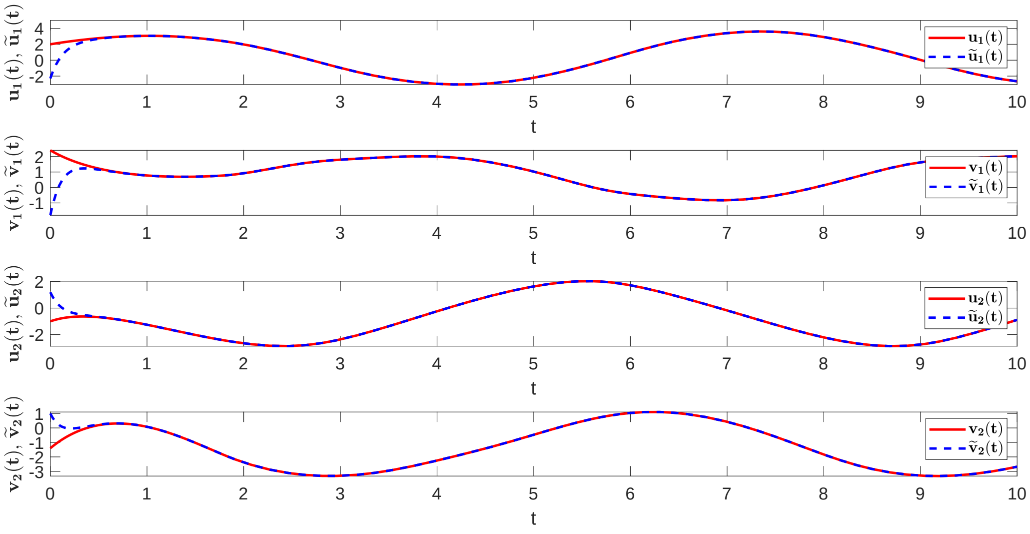

Figure 2.5: Plots of the state trajectories $u_1(t)$ and $\widetilde{u}_1(t), v_1(t)$ and $\widetilde{v}_1(t), u_2(t)$ and $\widetilde{u}_2(t), v_2(t)$ and $\widetilde{v}_2(t)$ with respect to time t, of master system (2.3.1) and response system (2.3.2) with uncertain parameters ΔA and ΔB and bounded time varying delay term $\tau(t) = (2 + cos(t))$.

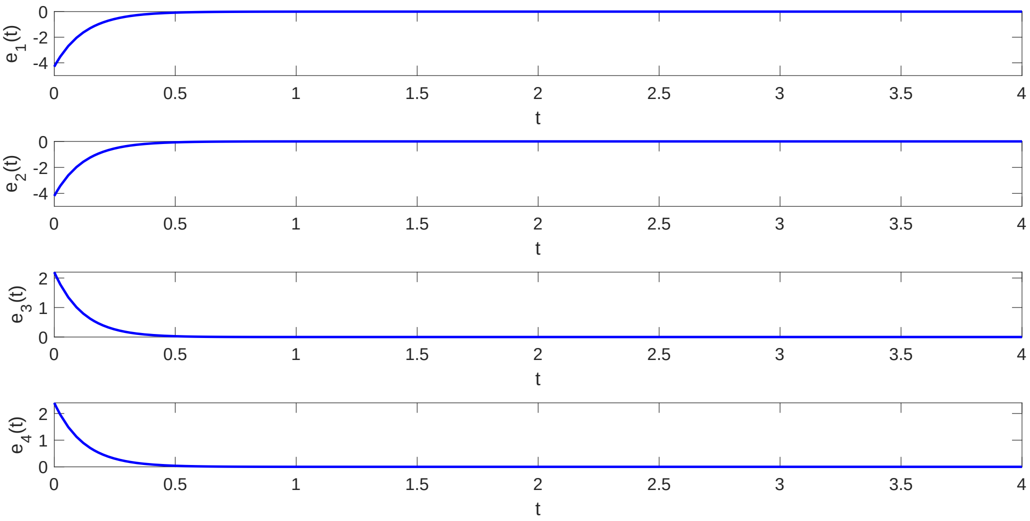

Figure 2.6: Plots of the global exponential synchronization errors $e_i(t), i = 1, 2, 3, 4$ of the systems (2.3.1) and (2.3.2) with uncertain parameters ΔA and ΔB and bounded time varying delay term $\tau(t) = (2 + cos(t))$.

Example 2.3.2. Let us consider CVRNN with uncertainty and unbounded time-varying delay terms as the master system (2.1.1) with the following parameters

$$C= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A= \begin{pmatrix} 2+8i & -0.1-0.2i \\ -8-1.2i & 8+4i \end{pmatrix}, \Delta A= \begin{pmatrix} 0.2+0.1i & 0.2-0.3i \\ 0.6+0.1i & -0.3+0.4i \end{pmatrix},$$

$$B = \begin{pmatrix} -1.5+1.2i & -0.1-0.2i \\ -0.2-0.2i & -4+i \end{pmatrix}, \Delta B= \begin{pmatrix} 0.2+0.7i & 0.3+05i \\ 0.5+0.1i & 0.4+0.5i \end{pmatrix},$$

$$\Omega = \begin{pmatrix} -64 & 10 \\ 15 & -60 \end{pmatrix}, L(t) = \begin{pmatrix} e^{-3t^2}+ie^{-5t^2} \\ e^{-2t^2}+ie^{-3t^2} \end{pmatrix},$$

$\tau(t) = ceil(3t/2)$ is the unbounded time-varying delay.

The activation functions are taken as

$$\begin{aligned}
f_j(w_j(t)) &= \frac{|u_j+1|-|u_j-1|}{2} + i\frac{|v_j+1|-|v_j-1|}{2},\\
g_j(w_j(t)) &= \frac{1}{1+exp(-u_j+2v_j)} + i\frac{1-exp(-2u_j-v_j)}{1+exp(-2u_j-v_j)}(j=1,2).
\end{aligned}$$

The master system (2.1.2) is written as

$$\begin{aligned}
\dot{u}_1(t) =& -u_1(t) + 2.2((|u_1+1|-|u_1-1|)/2) + 0.1((|u_2+1|-|u_2-1|)/2)\\
&- 8.1((|v_1+1|-|v_1-1|)/2) + 0.5((|v_2+1|-|v_2-1|)/2)\\
&- 1.3((1/(1+exp(-u_1+2v_1)) + 0.2((1/(1+exp(-u_2+2v_2))\\
&- 1.9((1-exp(-2u_1-v_1))/(1+exp(-2u_1-v_1)))\\
&- 0.3((1-exp(-2u_2-v_2))/(1+exp(-2u_2-v_2))) + exp(-3t^2).\\
\dot{v}_1(t) =& -v_1(t) + 8.1((|u_1+1|-|u_1-1|)/2) - 0.5((|u_2+1|-|u_2-1|)/2)\\
&+ 2.2((|v_1+1|-|v_1-1|)/2) + 0.1((|v_2+1|-|v_2-1|)/2)\\
&+ 1.9((1/(1+exp(-u_1+2v_1)))) + 0.3((1/(1+exp(-u_2+2v_2))))\\
&- 1.3((1-exp(-2u_1-v_1))/(1+exp(-2u_1-v_1)))\\
&+ 0.2((1-exp(-2u_2-v_2))/(1+exp(-2u_2-v_2))) + exp(-5t^2).\\
\dot{u}_2(t) =& -u_2(t) - 7.4((|u_1+1|-|u_1-1|)/2) + 7.7((|u_2+1|-|u_2-1|)/2)\\
&+ 1.1((|v_1+1|-|v_1-1|)/2) - 4.4((|v_2+1|-|v_2-1|)/2)\\
&+ 0.3((1/(1+exp(-u_1+2v_1)))) - 3.6((1/(1+exp(-u_2+2v_2))))\\
&+ 0.1((1-exp(-2u_1-v_1))/(1+exp(-2u_1-v_1)))\\
&- 1.5((1-exp(-2u_2-v_2))/(1+exp(-2u_2-v_2))) + exp(-2t^2).
\end{aligned}$$

$$\begin{aligned}
\dot{v_2}(t) =& -v_2(t) - 1.1((|u_1+1| - |u_1-1|)/2) + 4.4((|u_2+1| - |u_2-1|)/2)\\
&- 7.4((|v_1+1| - |v_1-1|)/2) + 7.7((|v_2+1| - |v_2-1|)/2)\\
&- 0.1((1/(1+exp(-u_1+2v_1)))) + 1.5((1/(1+exp(-u_2+2v_2))))\\
&+ 0.3((1-exp(-2u_1-v_1))/(1+exp(-2u_1-v_1)))\\
&- 3.6((1-exp(-2u_2-v_2))/(1+exp(-2u_2-v_2))) + exp(-3t^2). \qquad (2.3.3)
\end{aligned}$$

Figure 2.7 shows the 3-D plot of the trajectories $u_1(t), v_1(t), u_2(t), v_2(t)$ with respect to time and without uncertain parameters ΔA and ΔB, and having unbounded time-varying delay term $\tau(t) = ceil(3t/2)$.

The corresponding response system (2.1.4) with the same parameters is expressed as

$$\begin{aligned}
\dot{\tilde{u}}_1(t) =& -\tilde{u}_1(t) + 2.2((|\tilde{u}_1+1| - |\tilde{u}_1-1|)/2) + 0.1((|\tilde{u}_2+1| - |\tilde{u}_2-1|)/2)\\
&- 8.1((|\tilde{v}_1+1| - |\tilde{v}_1-1|)/2) + 0.5((|\tilde{v}_2+1| - |\tilde{v}_2-1|)/2)\\
&- 1.3((1/(1+exp(-\tilde{u}_1+2\tilde{v}_1)) + 0.2((1/(1+exp(-\tilde{u}_2+2\tilde{v}_2))\\
&- 1.9((1-exp(-2\tilde{u}_1-\tilde{v}_1))/(1+exp(-2\tilde{u}_1-\tilde{v}_1)))\\
&- 0.3((1-exp(-2\tilde{u}_2-\tilde{v}_2))/(1+exp(-2\tilde{u}_2-\tilde{v}_2))) + exp(-3t^2).\\
\dot{\tilde{v}}_1(t) =& -\widetilde{v_1}(t) + 8.1((|\tilde{u}_1+1| - |\tilde{u}_1-1|)/2) - 0.5((|\tilde{u}_2+1| - |\tilde{u}_2-1|)/2)\\
&+ 2.2((|\tilde{v}_1+1| - |\tilde{v}_1-1|)/2) + 0.1((|\tilde{v}_2+1| - |\tilde{v}_2-1|)/2)\\
&+ 1.9((1/(1+exp(-\tilde{u}_1+2\tilde{v}_1)))) + 0.3((1/(1+exp(-\tilde{u}_2+2\tilde{v}_2))))\\
&- 1.3((1-exp(-2\tilde{u}_1-\tilde{v}_1))/(1+exp(-2\tilde{u}_1-\tilde{v}_1)))\\
&+ 0.2((1-exp(-2\tilde{u}_2-\tilde{v}_2))/(1+exp(-2\tilde{u}_2-\tilde{v}_2))) + exp(-5t^2).\\
\dot{\tilde{u}}_2(t) =& -\tilde{u}_2(t) - 7.4((|\tilde{u}_1+1| - |\tilde{u}_1-1|)/2) - 7.7((|\tilde{u}_2+1| - |\tilde{u}_2-1|)/2)
\end{aligned}$$

$$
\begin{aligned}
&+1.1((|\widetilde{v}_1+1|-|\widetilde{v}_1-1|)/2)-4.4((|\widetilde{v}_2+1|-|\widetilde{v}_2-1|)/2)\\
&+0.3((1/(1+exp(-\widetilde{u}_1+2\widetilde{v}_1))))-3.6((1/(1+exp(-\widetilde{u}_2+2\widetilde{v}_2))))\\
&+0.1((1-exp(-2\widetilde{u}_1-\widetilde{v}_1))/(1+exp(-2\widetilde{u}_1-\widetilde{v}_1)))\\
&-1.5((1-exp(-2\widetilde{u}_2-\widetilde{v}_2))/(1+exp(-2\widetilde{u}_2-\widetilde{v}_2)))+exp(-2t^2).\\
\dot{\widetilde{v}}_2(t)=&-\widetilde{v}_2(t)-1.1((|\widetilde{u}_1+1|-|\widetilde{u}_1-1|)/2)+4.4((|\widetilde{u}_2+1|-|\widetilde{u}_2-1|)/2)\\
&-7.4((|\widetilde{v}_1+1|-|\widetilde{v}_1-1|)/2)+7.7((|\widetilde{v}_2+1|-|\widetilde{v}_2-1|)/2)\\
&-0.1((1/(1+exp(-\widetilde{u}_1+2\widetilde{v}_1))))+1.5((1/(1+exp(-\widetilde{u}_2+2\widetilde{v}_2))))\\
&+0.3((1-exp(-2\widetilde{u}_1-\widetilde{v}_1))/(1+exp(-2\widetilde{u}_1-\widetilde{v}_1)))\\
&-3.6((1-exp(-2\widetilde{u}_2-\widetilde{v}_2))/(1+exp(-2\widetilde{u}_2-\widetilde{v}_2)))+exp(-3t^2). \qquad (2.3.4)
\end{aligned}
$$

Figure 2.8 depicts the 3-D plot of the trajectory with uncertain parameters ΔA and ΔB with same unbounded time-varying delay term,

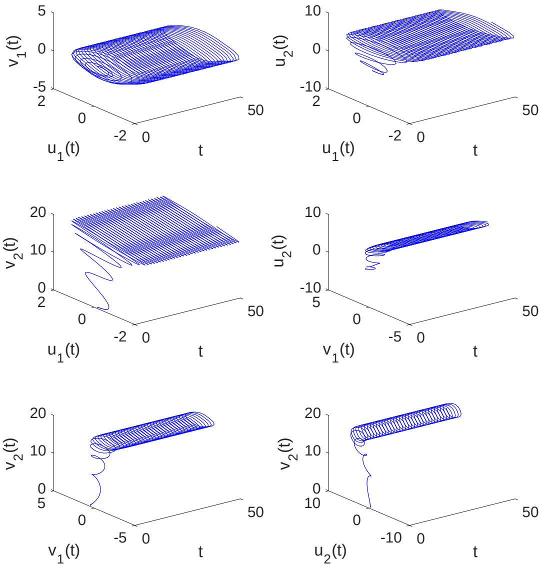

Figure 2.7: Plots of the state trajectories $u_1(t), v_1(t), u_2(t)$ and $v_2(t)$ with respect to time t in 3-dimensional space without uncertain parameters ΔA and ΔB of the system (2.3.3) and unbounded time varying delay term $\tau(t) = ceil(3t/2)$.

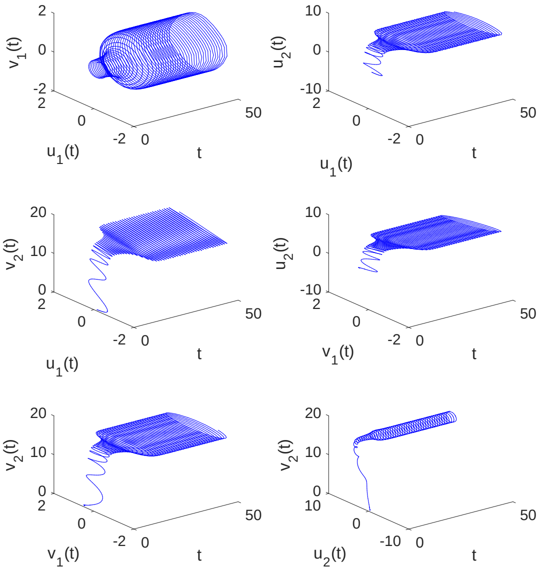

Figure 2.8: Plots of the state trajectories $u_1(t), v_1(t), u_2(t)$ and $v_2(t)$ with respect to time t in 3-dimensional space with uncertain parameters ΔA and ΔB and unbounded time varying delay term $\tau(t) = ceil(3t/2)$.

where $M(t) = \Omega(\widetilde{w}(t) - w(t))$ is the coupling control matrix. If we consider p=2, the system satisfies Assumption 1. The system parameters are taken as $r_k = 1, s_k = 1, m_k = \sqrt{2}/2$ and $q_k = \sqrt{2}$. Let us consider the initial conditions of master system (2.3.3) and response system (2.3.4) as

$w_1(s) = -0.15 + 0.45i, w_2(s) = -0.25 + 0.35i,$

$\widetilde{w}_1(s) = -0.30 - 0.30i,\ \widetilde{w}_2(s) = 0.30 + 0.30i,$

Then, we can verify that

$k_1{=}-\{\mu_p(-C + \Omega) + (r_k + s_k)(\|A^R + \Delta A^R\|_p + \|A^I + \Delta A^I\|_p)\} = 12.2356,$

$k_2{=}-(m_k + q_k)(\|B^R + \Delta B^R\|_p + \|B^I + \Delta B^I\|_p) = 11.7999,$

i.e., $k_1 > k_2$. Thus by Theorem 2.2, it can be concluded that the systems (2.3.3) and (2.3.4) will be globally exponentially synchronized.

Figure 2.9 shows that the systems (2.3.3) and (2.3.4) are globally exponential synchronized without uncertain parameters and with unbounded time-varying delay terms. Figure 2.10 shows that the global exponential synchronization of the systems (2.3.3) and (2.3.4) with uncertain parameters and unbounded time-varying delay. Figures 2.11 and 2.12 also confirm the global exponential synchronization from the error functions, which converge to zero after a small duration of time with the convergence rate 0.0283. From Figures 2.6 and 2.12, it is clearly observed that it takes less time for synchronization of the considered CVRNNs for unbounded time-varying delay as compared to bounded time-varying delay.

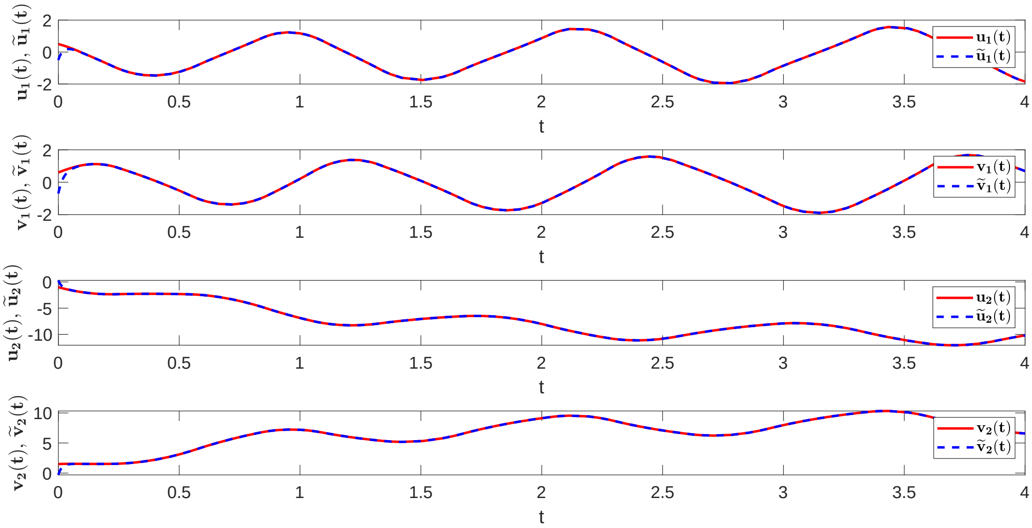

Figure 2.9: Plots of the state trajectories $u_1(t)$ and $\widetilde{u}_1(t), v_1(t)$ and $\widetilde{v}_1(t), u_2(t)$ and $\widetilde{u}_2(t), v_2(t)$ and $\widetilde{v}_2(t)$ with respect to time t, of master system (2.3.3) and response system (2.3.4) without uncertain parameters ΔA and ΔB and unbounded time varying delay term $\tau(t) = ceil(3t/2)$.

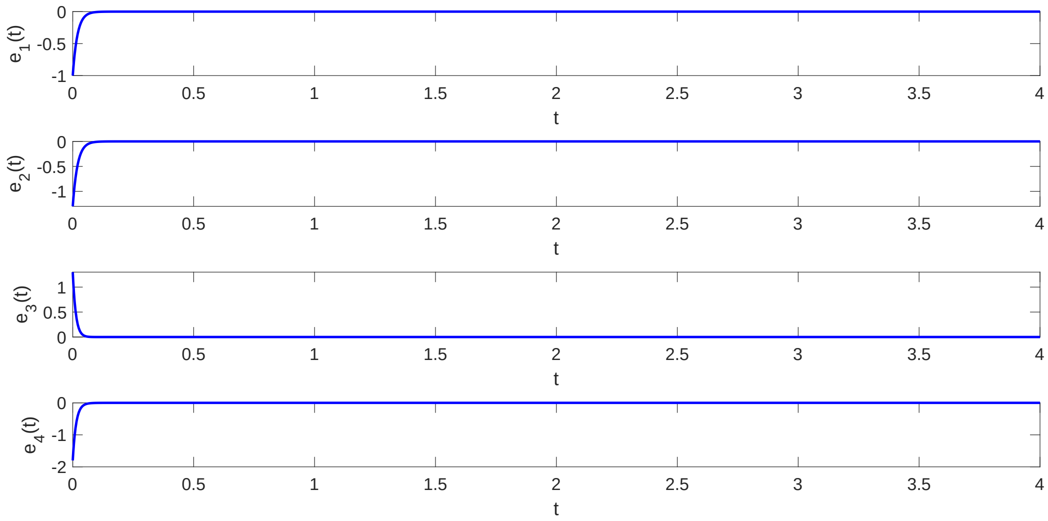

Figure 2.10: Plots of the global exponential synchronization errors $e_i(t), i = 1, 2, 3, 4$ of the systems (2.3.3) and (2.3.4) without uncertain parameters ΔA and ΔB and unbounded time varying delay term $\tau(t) = ceil(3t/2)$.

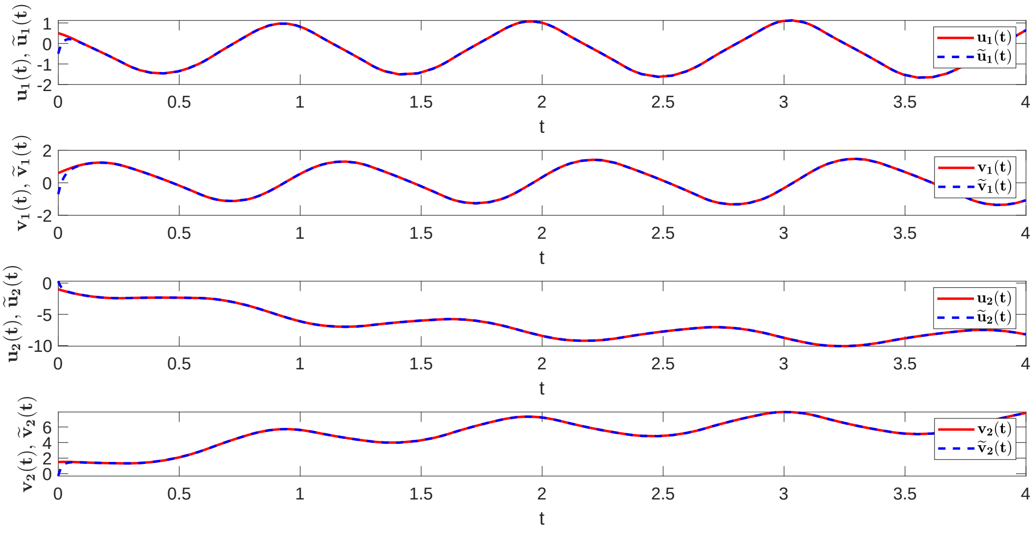

Figure 2.11: Plots of the state trajectories $u_1(t)$ and $\tilde{u}_1(t), v_1(t)$ and $\tilde{v}_1(t), u_2(t)$ and $\tilde{u}_2(t), v_2(t)$ and $\tilde{v}_2(t)$ with respect to time t, of master system (2.3.3) and response system (2.3.4) with uncertain parameters ΔA and ΔB and unbounded time varying delay term $\tau(t) = ceil(3t/2)$.

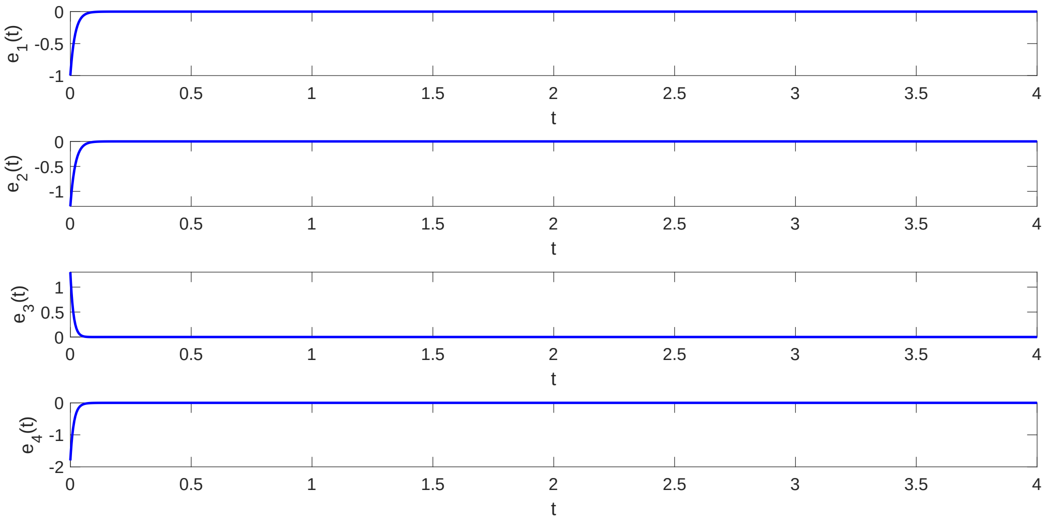

Figure 2.12: Plots of the global exponential synchronization errors $e_i(t), i = 1, 2, 3, 4$ of the systems (2.3.3) and (2.3.4) with uncertain parameters ΔA and ΔB and unbounded time varying delay term $\tau(t) = ceil(3t/2)$.

2.4 Conclusion

In this chapter, the global exponential synchronization criteria in bounded and unbounded time-varying CVRNNs with uncertain parameters have been discussed. The global exponential synchronization condition is obtained using matrix measure method with the help of Halanay inequality and Lyapunov stability theory. Two numerical examples are taken, one for bounded time-varying CVRNNs with uncertain parameters and another for unbounded time-varying CVRNNs with uncertain parameters to show the feasibility and effectiveness of the obtained global synchronization results. The salient feature of the chapter is the selection of the controller gain matrix in easier way to achieve the global exponential synchronization criteria by using the proposed method. The most important feature of the present scientific contribution is the graphical presentations of the global exponential synchronization of CVRNNs for different particular cases.

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