Chapter 4

Generalized-Hukuhara Dini Hadamard ϵ-Subdifferential and H$_\epsilon$-Subgradient and Their Applications on Interval Optimization

4.1 Introduction

Optimization theory for nonsmooth functions is known to have been hugely influenced by approximate subdifferentials or ϵ-subgradients. It has been observed that various numerical methods have been constructed with the help of ϵ-subgradients to minimize convex functions. The contribution of approximate subdifferentials on the calculus of convex subdifferentials helps to develop several generalized gradients of Clarke [16, 106, 107]. As Dini-Hadamard derivative preserves the linearity of the derivative with respect to the direction [16], ϵ-Dini-Hadamard plays a prominent role [108] in the advancement of nonsmooth analysis, especially nondifferentiable functions in the absence of convexity. In 1976, Mordukhovich [109] first obtained ϵ-subdifferentials as a byproduct of certain approximative techniques.

4.2 Motivation and Contribution

Mordukhovich [109] first obtained byproducts of certain finite dimensional approximative optimization techniques of nonconvex functions, which were named as approximate subdifferentials or ϵ-subdifferentials. It is known that in conventional optimization theory, the approximate subdifferentials and the Dini Hadamard ϵ-subdifferentials are found to be minimal among other conceivable subdifferentials. From the existing results on IOPs, it has been observed that the gH-subdifferential set may be empty and there is no theory to study the behaviour of such IVFs (see Example 4.1). However, with the help of the defined notion of gH-Dini Hadamard ϵ-subdifferentials, these IVFs can be studied. Moreover, the concept of gH-Dini Hadamard ϵ-subdifferential contains the set of gH-subdifferential and set of Fréchet derivatives; however, the converse is not true (see Theorem 4.1 and 4.4).

In this chapter, we have proposed the concept of gH-Dini Hadamard ϵ-subdifferentiability for IVFs and $\mathbf{H}_\epsilon$-subgradient, which is more general than all the existing subdifferentials on IOPs (see [3–8]) and also contains the set of gH-Dini Hadamard ϵ-subdifferential (see Theorem 4.6). A few relations between gH-Fréchet differentiability and gH-Dini Hadamard ϵ -subdifferentiability is given. Next, an important concept of $\mathbf{H}_\epsilon$-subgradient is given, which is based on the criterion of sponge of a set. Further, a variational interpretation of gH-Dini Hadamard ϵ-subdifferential based on the sponge of a set is discussed. Furthermore, the concept of ϵ-efficient solution followed by necessary and sufficient efficient conditions for finding an ϵ-efficient solution to an IOP with the gH-Dini Hadamard ϵ-subgradient of its objective function are given. An example to show the application of proposed results in sparsity regularizer for IOPs is given.

4.3 gH-Dini Hadamard ϵ-Subdifferentiability

We define the gH-Dini Hadamard ϵ-subdifferential set of an IVF using gH-Dini Hadamard derivative of an IVF followed by its several characterizations. Further, we prove that the concept of gH-Dini Hadamard ϵ-subdifferential is more general than the concept of gH-subdifferentiability. In the sequel, the relation of gH-subdifferentiability and gH-Fréchet differentiability with gH-Dini Hadamard ϵ-subdifferentiability is discussed.

Definition 4.1 (gH-Dini Hadamard derivative of an IVF). *Let $\boldsymbol{F}$ be an IVF on $\mathcal{Y}$. If for $\bar{y} \in \mathcal{Y}$ and $h \in \mathbb{R}^n$, the limit inferior*

$$\boldsymbol{F}_{\mathcal{DH}}(\bar{y})(h) = \liminf_{\substack{u \to h \\ \lambda \to 0+}} \frac{1}{\lambda} \odot \left(\boldsymbol{F}(\bar{y} + \lambda u) \ominus_{gH} \boldsymbol{F}(\bar{y}) \right),$$

exists and $\boldsymbol{F}_{\mathcal{DH}}(\bar{y})(h)$ is a linear IVF from $\mathcal{Y}$ to $I(\mathbb{R})$, then the limit value is called gH-Dini Hadamard derivative of $\boldsymbol{F}$ at $\bar{y}$ in the direction h.

Definition 4.2 (gH-Dini Hadamard ϵ-subdifferentiability of an IVF). *Let $\boldsymbol{F} : \mathcal{Y} \to \overline{I(\mathbb{R})}$ be an extended IVF that is finite at $\bar{y} \in \mathcal{Y}$. Then, for $\epsilon > 0$, the gH-Dini Hadamard ϵ-subdifferential of $\boldsymbol{F}$ at $\bar{y}$, denoted by $\partial_{\epsilon}^{\mathcal{DH}} \boldsymbol{F}(\bar{y})$, is defined as*

$$\partial_{\epsilon}^{\mathcal{DH}} \boldsymbol{F}(\bar{y}) = \left\{ \widehat{\boldsymbol{S}} \in I(\mathbb{R})^n : (w - \bar{y})^{\top} \odot \widehat{\boldsymbol{S}} \preceq \boldsymbol{F}_{\mathcal{DH}}(\bar{y})(w - \bar{y}) \oplus \epsilon \|w - \bar{y}\| \text{ for all } w \in \mathcal{Y} \right\}. \tag{4.1}$$

Then, $\widehat{\boldsymbol{S}}$ is called the gH-Dini Hadamard ϵ-subgradient of $\boldsymbol{F}$ at $\bar{y}$. Further, if $\partial_{\epsilon}^{\mathcal{DH}} \boldsymbol{F}(\bar{y}) \neq \emptyset$, we say IVF $\boldsymbol{F}$ is gH-Dini Hadamard ϵ-subdifferentiable at $\bar{y}$.

Example 4.1 *Let $\boldsymbol{F} : \mathbb{R} \to I(\mathbb{R})$ be an IVF given by $\boldsymbol{F}(y) = [1, 2] \odot |y|$.*

Let us check the gH-Dini Hadmard ϵ-subdifferentiability of $\boldsymbol{F}$ at 0.

Let us assume $\boldsymbol{S} = [\underline{s}, \overline{s}] \in \partial_{\epsilon}^{\mathcal{DH}} \boldsymbol{F}(0)$ *for* $\epsilon > 0$. *Therefore, for* $w \in \mathbb{R}$, *we have*

$$\begin{aligned} & [\underline{s}, \overline{s}] \odot (w - 0) \preceq \boldsymbol{F}_{\mathcal{DH}}(0)(w - 0) \oplus \epsilon |w - 0| \\ \implies & [\underline{s}, \overline{s}] \odot w \preceq \liminf_{\substack{u \to w \\ \lambda \to 0+}} \frac{1}{\lambda} \odot \big([1, 2] \odot |\lambda u|\big) \oplus \epsilon |w - 0| \\ \implies & [\underline{s}, \overline{s}] \odot w \preceq [1, 2] \odot |w| \oplus \epsilon |w|. \end{aligned}$$

We have the following two cases:

Case 1: For $w \geq 0$,

$$[\underline{s}, \overline{s}] \preceq [1, 2] \oplus \epsilon \implies \underline{s} \leq 1 + \epsilon \text{ and } \overline{s} \leq 2 + \epsilon.$$

Case 2: For $w < 0$,

$$\begin{aligned} [\underline{s}, \overline{s}] \odot w \preceq [1, 2] \odot (-w) \oplus \epsilon(-w) & \implies [1, 2] \odot (-1) \oplus \epsilon(-1) \preceq [\underline{s}, \overline{s}] \\ & \implies [-2 - \epsilon, -1 - \epsilon] \preceq [\underline{s}, \overline{s}] \\ & \implies -2 - \epsilon \leq \underline{s} \text{ and } -1 - \epsilon \leq \overline{s}. \end{aligned}$$

Hence, in view of Case 1: and Case 2:, we get

$$\partial_{\epsilon}^{\mathcal{DH}} \boldsymbol{F}(0) = \{\boldsymbol{S} \in I(\mathbb{R}) : -2 - \epsilon \leq \underline{s} \leq 1 + \epsilon \text{ and } -1 - \epsilon \leq \overline{s} \leq 2 + \epsilon\}.$$

A geometrical view of gH-Dini Hadamard ϵ*-subdifferentiability of* $\boldsymbol{F}$ *of Example 4.1 is given in Fig. 4.1. The IVF* $\boldsymbol{F}$ *is depicted by the pink region. For* $\epsilon = 1$, *the two possible gH-Dini Hadamard* ϵ*-subgradients of* $\boldsymbol{F}$ *at* 0 *are denoted by* $\boldsymbol{S}_1$ *and* $\boldsymbol{S}_2$ *and shown by green region.*

Example 4.2 *In this example, we check the gH-Dini Hadamard* ϵ*-subdifferentiability of an IVF* $\boldsymbol{F} : \mathbb{R} \to I(\mathbb{R})$ *which is given by* $\boldsymbol{F}(y) = [-2, -1] \odot |y|$.

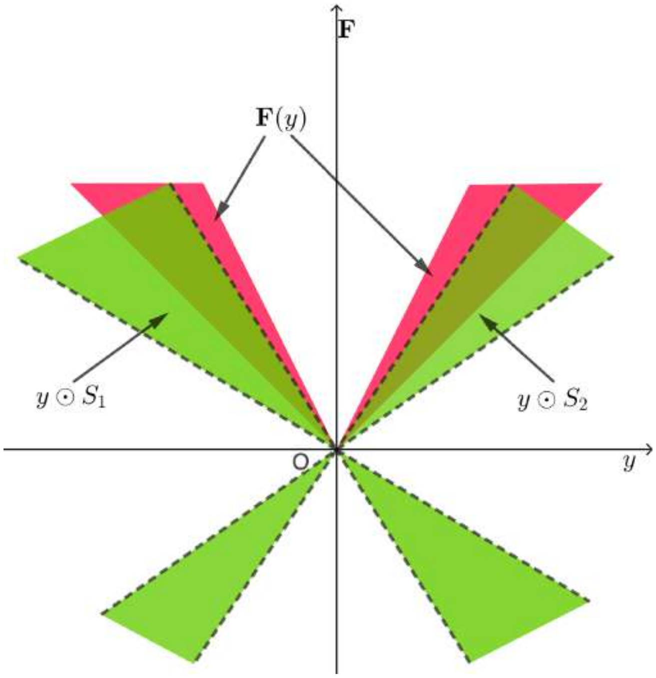


Figure 4.1: The geometrical representation of two possible gH-Dini Hadamard ϵ-subgradients of **F** of Example 4.1

Let us assume $\boldsymbol{S} \in \partial_{\epsilon}^{\mathcal{DH}} \boldsymbol{F}(0)$ *for* $0 < \epsilon < 2$. *Therefore, for* $w \in \mathbb{R}$, *we have*

$$\begin{aligned}
&(w-0) \odot \boldsymbol{S} \preceq \boldsymbol{F}_{\mathcal{DH}}(0)(w-0) \oplus \epsilon|w-0| \\
\implies\ & w \odot \boldsymbol{S} \preceq \liminf_{\substack{u \to w \\ \lambda \to 0+}} \frac{1}{\lambda} \odot (\boldsymbol{F}(\lambda u) \ominus_{gH} \boldsymbol{F}(0)) \oplus \epsilon|w| \\
\implies\ & w \odot \boldsymbol{S} \preceq \liminf_{\substack{u \to w \\ \lambda \to 0+}} \frac{1}{\lambda} \odot ([-2,-1] \odot |\lambda u|) \oplus \epsilon|w| \\
\implies\ & w \odot \boldsymbol{S} \preceq [-2,-1] \odot |w| \oplus \epsilon|w|.
\end{aligned}$$

We have the following two cases:

Case 1: For $w \geq 0$,

$$w \odot \boldsymbol{S} \preceq [-2,-1] \odot w \oplus \epsilon w \implies \boldsymbol{S} \preceq [-2,-1] \oplus \epsilon.$$

Case 2: For $w < 0$,

$$w \odot \boldsymbol{S} \preceq [-2,-1] \odot (-w) \oplus \epsilon(-w) \implies [1,2] \ominus_{gH} \epsilon \preceq \boldsymbol{S}.$$

From Case 1: and Case 2:, we can observe that for any $0 < \epsilon < 2$, there does not exist any $\boldsymbol{S} \in I(\mathbb{R})$ which satisfies both the cases simultaneously. Thus, $\partial_{\epsilon}^{\mathcal{DH}} \boldsymbol{F}(\bar{y})$ is empty.

Theorem 4.1 *Let $\boldsymbol{F}$ be a gH-subdifferentiable IVF at $\bar{y}$ in $\mathcal{Y}$. Then, $\boldsymbol{F}$ is gH-Dini Hadamard ϵ-subdifferentiable at $\bar{y}$ as well.*

Proof: If $\mathbf{F}$ is gH-subdifferentiable at $\bar{y}$, then there exists an $\widehat{\mathbf{S}} \in \partial \mathbf{F}(\bar{y})$ such that

$$(w-\bar{y})^{\top} \odot \widehat{\mathbf{S}} \preceq \mathbf{F}(w) \ominus_{gH} \mathbf{F}(\bar{y}) \text{ for all } w \in \mathcal{Y}. \tag{4.2}$$

Also, for any $\epsilon > 0$, the following relation holds:

$$\mathbf{F}(w) \ominus_{gH} \mathbf{F}(\bar{y}) \preceq \mathbf{F}(w) \ominus_{gH} \mathbf{F}(\bar{y}) \oplus \epsilon\|w-\bar{y}\| \text{ for all } w \in \mathcal{Y}. \tag{4.3}$$

Therefore, from (4.2) and (4.3), we conclude that for all $w \in \mathcal{Y}$,

$$\begin{aligned}
& (w-\bar{y})^{\top} \odot \widehat{\mathbf{S}} \preceq \mathbf{F}(w) \ominus_{gH} \mathbf{F}(\bar{y}) \oplus \epsilon\|w-\bar{y}\| \\
\implies\ & (w-\bar{y})^{\top} \odot \widehat{\mathbf{S}} \preceq \liminf_{\substack{u \to (w-\bar{y}) \\ \lambda \to 0+}} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{y}+\lambda u) \ominus_{gH} \mathbf{F}(\bar{y})) \oplus \epsilon\|w-\bar{y}\| \\
\implies\ & (w-\bar{y})^{\top} \odot \widehat{\mathbf{S}} \preceq \mathbf{F}_{\mathcal{DH}}(\bar{y})(w-\bar{y}) \oplus \epsilon\|w-\bar{y}\| \\
\implies\ & \widehat{\mathbf{S}} \in \partial_{\epsilon}^{\mathcal{DH}} \mathbf{F}(\bar{y}).
\end{aligned}$$

Thus, $\mathbf{F}$ is gH-Dini Hadamard ϵ-subdifferentiable at $\bar{y}$. □

Remark 4.1 *It is to be noted that the converse of the conclusion made in Theorem 4.1 need not be true. For instance, consider the IVF discussed in Example 4.2: $\boldsymbol{F}(y) =$*

$[-2, -1] \odot |y|$.

Let us check the gH-subdifferentiability and gH-Dini Hadamard ϵ-subdifferentiability of $\boldsymbol{F}$ at 0 for any $\epsilon \geq 2$. From Example 4.2, it can be observed that the gH-Dini Hadamard ϵ-subdifferentiability of $\boldsymbol{F}$ at 0 is given by

$$\partial_{\epsilon}^{\mathcal{DH}} \boldsymbol{F}(0) = \{\boldsymbol{S} \in I(\mathbb{R}) : [1, 2] \ominus_{gH} \epsilon \preceq \boldsymbol{S} \preceq [-2, -1] \oplus \epsilon\}.$$

Now the gH-subdifferential of $\boldsymbol{F}$ at 0 is

$$\begin{aligned} & w \odot \boldsymbol{S} \preceq \boldsymbol{F}(w) \ominus_{gH} \boldsymbol{F}(0) \qquad (4.4) \\ \implies\ & w \odot \boldsymbol{S} \preceq [-2, -1] \odot |w|. \end{aligned}$$

There arise the following two cases:

Case 1: For $w \geq 0$,

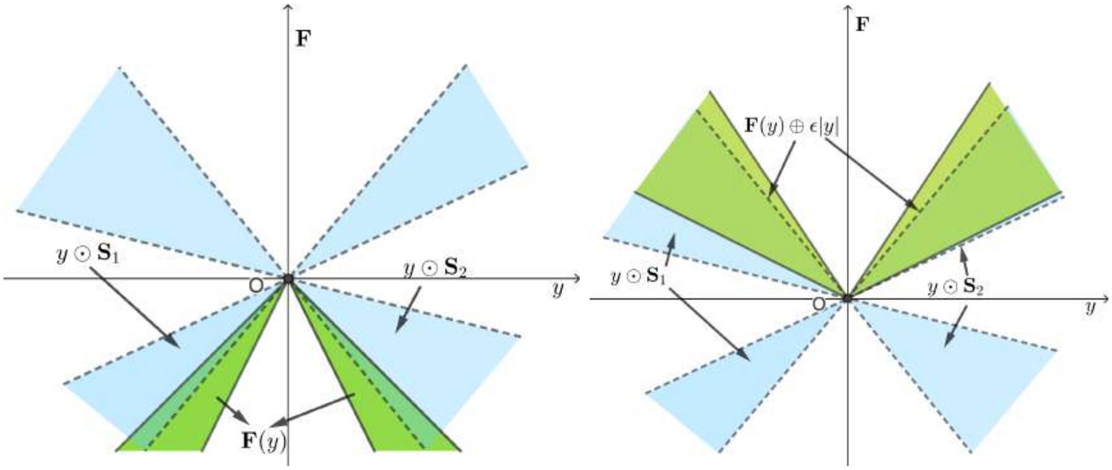
$$\boldsymbol{S} \odot w \preceq [-2, -1] \odot w \implies \boldsymbol{S} \preceq [-2, -1].$$

Case 2: For $w < 0$,

$$\boldsymbol{S} \odot w \preceq [-2, -1] \odot (-w) \implies [1, 2] \preceq \boldsymbol{S}.$$

From Case 1: and Case 2:, we observe that for any $\epsilon \geq 2$ there does not exist any $\boldsymbol{S} \in I(\mathbb{R})$ which satisfies both the cases simultaneously. Hence, $\partial \boldsymbol{F}(0) = \emptyset$. The graph of gH-Dini Hadamard ϵ-subdifferentiability of IVF $\boldsymbol{F}$ at $y = 0$ of Remark 4.1 is depicted in Figure 4.2. For $\epsilon = 2.5$, the two possible gH-Dini Hadamard ϵ-subgradients of $\boldsymbol{F}$ at 0 are denoted by $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$. It can be observed that with the help of gH-Dini Hadamard ϵ-subdifferentiability, the two subgradients $y \odot \boldsymbol{S}_1$ and $y \odot \boldsymbol{S}_2$ always supports the IVF $(\boldsymbol{F}(y) \oplus \epsilon|y|)$ from below as shown in Figure 4.2(b), which fails for IVF $\boldsymbol{F}(y)$ (see Figure

4.2(a)).



(a) The IVF $\mathbf{F}(y)$ and gH-Dini Hadamard ϵ-subgradients of $\mathbf{F}$

(b) The IVF $(\mathbf{F}(y) \oplus \epsilon|y|)$ and gH-Dini Hadamard ϵ-subgradients of $\mathbf{F}$

Figure 4.2: The geometrical representation of two possible gH-Dini Hadamard ϵ-subgradients of $\mathbf{F}$ of Remark 4.1

Lemma 4.1 (Monotonic property of gH-Dini Hadamard ϵ-subdifferential). *Let* $\boldsymbol{F}$ *be an IVF on* $\mathcal{Y}$. *Then, for* $0 < \epsilon_1 \leq \epsilon_2$, *and* $\bar{y} \in \mathcal{Y}$

$$\partial_{\epsilon_1}^{\mathcal{DH}} \boldsymbol{F}(\bar{y}) \subseteq \partial_{\epsilon_2}^{\mathcal{DH}} \boldsymbol{F}(\bar{y}).$$

Proof: Let $\widehat{\mathbf{S}} \in \partial_{\epsilon_1}^{\mathcal{DH}} \mathbf{F}(\bar{y})$. Then, for $\epsilon_1 > 0$ and for all $w \in \mathcal{Y}$, we have

$$(w - \bar{y})^\top \odot \widehat{\mathbf{S}} \ \preceq \ \mathbf{F}_{\mathcal{DH}}(\bar{y})(w - \bar{y}) \oplus \epsilon_1 \|w - \bar{y}\|.$$

Since $\epsilon_1 \leq \epsilon_2$, then from Lemma 1.1, we get

$$(w - \bar{y})^\top \odot \widehat{\mathbf{S}} \ \preceq \ \mathbf{F}_{\mathcal{DH}}(\bar{y})(w - \bar{y}) \oplus \epsilon_2 \|w - \bar{y}\| \text{ for all } w \in \mathcal{Y}.$$

Hence, $\partial_{\epsilon_1}^{\mathcal{DH}} \mathbf{F}(\bar{y}) \subseteq \partial_{\epsilon_2}^{\mathcal{DH}} \mathbf{F}(\bar{y})$. □

Theorem 4.2 *Let* $\boldsymbol{F}$ *be an IVF on* $\mathcal{Y}$. *Then, for any* $\epsilon > 0$ *and* $\bar{y} \in \mathcal{Y}$, $\partial_{\epsilon}^{\mathcal{DH}} \boldsymbol{F}(\bar{y})$ *is a convex set.*

Proof: Let us assume $\partial_{\epsilon}^{\mathcal{DH}}\mathbf{F}(\bar{y})$ is nonempty and there exist $\widehat{\mathbf{L}}$, $\widehat{\mathbf{K}} \in \partial_{\epsilon}^{\mathcal{DH}}\mathbf{F}(\bar{y})$ for any $\epsilon > 0$. Then, for all $w \in \mathcal{Y}$, we have

$$\begin{aligned}
&(w-\bar{y})^{\top} \odot \widehat{\mathbf{L}} \preceq \mathbf{F}_{\mathcal{DH}}(\bar{y})(w-\bar{y}) \oplus \epsilon\|w-\bar{y}\| \text{ and}\\
&(w-\bar{y})^{\top} \odot \widehat{\mathbf{K}} \preceq \mathbf{F}_{\mathcal{DH}}(\bar{y})(w-\bar{y}) \oplus \epsilon\|w-\bar{y}\|\\
\implies &\bigoplus_{i=1}^{n} \mathbf{L}_i \odot h_i \preceq \mathbf{F}_{\mathcal{DH}}(\bar{y})(h) \oplus \epsilon\|h\| \text{ and}\\
&\bigoplus_{i=1}^{n} \mathbf{K}_i \odot h_i \preceq \mathbf{F}_{\mathcal{DH}}(\bar{y})(h) \oplus \epsilon\|h\| \text{ taking } h=(w-\bar{y}) \in \mathcal{Y}.
\end{aligned}$$

Thus, for $h \in \mathcal{Y}$ and $\lambda_1, \lambda_2 > 0$ with $\lambda_1 + \lambda_2 = 1$, we have

$$\begin{aligned}
&\lambda_1 \odot \left(\bigoplus_{i=1}^{n} \mathbf{L}_i \odot h_i\right) \oplus \lambda_2 \odot \left(\bigoplus_{i=1}^{n} \mathbf{K}_i \odot h_i\right) \preceq \lambda_1 \odot \mathbf{F}_{\mathcal{DH}}(\bar{y})(h) \oplus\\
&\qquad\qquad\qquad\qquad \lambda_2 \odot \mathbf{F}_{\mathcal{DH}}(\bar{y})(h) \oplus \epsilon\|h\|\\
&\text{or, } \bigoplus_{i=1}^{n} (\lambda_1 \odot \mathbf{L}_i \oplus \lambda_2 \odot \mathbf{K}_i) \odot h_i \preceq \mathbf{F}_{\mathcal{DH}}(\bar{y})(h) \oplus \epsilon\|h\| \text{ from Lemma 1.3}\\
&\text{or, } h^{\top} \odot (\lambda_1 \odot \widehat{\mathbf{L}} \oplus \lambda_2 \odot \widehat{\mathbf{K}}) \preceq \mathbf{F}_{\mathcal{DH}}(\bar{y})(h) \oplus \epsilon\|h\|.
\end{aligned}$$

Therefore, $\lambda_1 \odot \widehat{\mathbf{L}} \oplus \lambda_2 \odot \widehat{\mathbf{K}} \in \partial_{\epsilon}^{\mathcal{DH}}\mathbf{F}(\bar{y})$. Hence, $\partial_{\epsilon}^{\mathcal{DH}}\mathbf{F}(\bar{y})$ is a convex set. □

Theorem 4.3 *Let $\boldsymbol{F}$ be an IVF on $\mathcal{Y}$. Then, for $\bar{y} \in \mathcal{Y}$ and $\epsilon > 0$, $\partial_{\epsilon}^{\mathcal{DH}}\boldsymbol{F}(\bar{y})$ is closed.*

Proof: Let $\{\widehat{\mathbf{S}}_k\}$ be a sequence in $\partial_{\epsilon}^{\mathcal{DH}}\mathbf{F}(\bar{y})$ which converges to $\widehat{\mathbf{S}} \in I(\mathbb{R})^n$, where $\widehat{\mathbf{S}}_k = (\mathbf{S}_{k1}, \mathbf{S}_{k2}, \ldots, \mathbf{S}_{kn})^{\top}$ and $\widehat{\mathbf{S}} = (\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_n)^{\top}$. Then, for all $w \in \mathcal{Y}$ and $\widehat{\mathbf{S}}_k \in \partial_{\epsilon}^{\mathcal{DH}}\mathbf{F}(\bar{y})$, we have

$$\begin{aligned}
&(w-\bar{y})^{\top} \odot \widehat{\mathbf{S}}_k \preceq \mathbf{F}_{\mathcal{DH}}(\bar{y})(w-\bar{y}) \oplus \epsilon\|w-\bar{y}\|\\
\implies &h^{\top} \odot \widehat{\mathbf{S}}_k \preceq \mathbf{F}_{\mathcal{DH}}(\bar{y})(h) \oplus \epsilon\|h\| \text{ taking } h=(w-\bar{y}) \in \mathcal{Y}\\
\implies &\bigoplus_{i=1}^{n} h_i^{\top} \odot \mathbf{S}_{ki} \preceq \mathbf{F}_{\mathcal{DH}}(\bar{y})(h) \oplus \epsilon\|h\|. \qquad (4.5)
\end{aligned}$$

Without loss of generality, let first m number of components of h in (4.5) be nonnegative and rest $(n-m)$ number of components be negative. Also, from Definition 4.1, assume $\mathbf{F}_{\mathcal{DH}}(\bar{y})(h) = \bigoplus_{i=1}^{n} \mathbf{R}_i \odot h_i$ and $\mathbf{R}_i \in I(\mathbb{R})$ for each $i = 1, 2, \ldots, n$. Therefore, we get

$$\bigoplus_{i'=1}^{m} h_{i'} \odot \widehat{\mathbf{S}}_{ki'} \oplus \bigoplus_{j=m+1}^{n} h_j \odot \widehat{\mathbf{S}}_{kj} \preceq \bigoplus_{i'=1}^{m} \mathbf{R}_{i'} \odot h_{i'} \oplus \bigoplus_{j=m+1}^{n} \mathbf{R}_j \odot h_j \oplus \epsilon\|h\|$$

$$\text{or, } \bigoplus_{i'=1}^{m} [\underline{s}_{ki'} h_{i'}, \overline{s}_{ki'} h_{i'}] \oplus \bigoplus_{j=m+1}^{n} [\overline{s}_{kj} h_j, \underline{s}_{kj} h_j] \preceq \bigoplus_{i'=1}^{m} [\underline{r}_{i'} h_i, \overline{r}_{i'} h_i] \oplus \bigoplus_{j=m+1}^{n} [\overline{r}_j h_j, \underline{r}_j h_j] \oplus \epsilon\|h\|.$$

Therefore, we have

$$\sum_{i'=1}^{m} \underline{s}_{ki'} h_{i'} + \sum_{j=m+1}^{n} \overline{s}_{kj} h_j \leq \sum_{i=1'}^{m} \underline{r}_{i'} h_{i'} + \sum_{j=m+1}^{n} \overline{r}_j h_j + \epsilon\|h\| \text{ and} \tag{4.6}$$

$$\sum_{i'=1}^{m} \overline{s}_{ki'} h_{i'} + \sum_{j=m+1}^{n} \underline{s}_{kj} h_j \leq \sum_{i'=1}^{m} \overline{r}_{i'} h_{i'} + \sum_{j=m+1}^{n} \underline{r}_j h_j + \epsilon\|h\|. \tag{4.7}$$

Since the sequence $\widehat{\mathbf{S}}_k$ converges to $\widehat{\mathbf{S}}$, from Remark 1.6, the sequences $\underline{s}_{ki'}$ and $\overline{s}_{ki'}$ converge to $\underline{s}_{i'}$, and $\overline{s}_{i'}$, respectively, for each i' and similarly for each j. Thus, we get

$$\left(\sum_{i'=1}^{m} \underline{s}_{ki'} h_{i'} + \sum_{j=m+1}^{n} \overline{s}_{kj} h_j \right) \to \left(\sum_{i'=1}^{m} \underline{s}_{i'} h_{i'} + \sum_{j=m+1}^{n} \overline{s}_j h_j \right) \text{ and} \tag{4.8}$$

$$\left(\sum_{i'=1}^{m} \overline{s}_{ki'} h_{i'} + \sum_{j=m+1}^{n} \underline{s}_{kj} h_j \right) \to \left(\sum_{i'=1}^{m} \overline{s}_{i'} h_{i'} + \sum_{j=m+1}^{n} \underline{s}_j h_j \right). \tag{4.9}$$

From (4.6), (4.7), (4.8), and (4.9), we obtain

$$\left(\sum_{i'=1}^{m} \underline{s}_{i'} h_{i'} + \sum_{j=m+1}^{n} \overline{s}_j h_j \right) \leq \sum_{i'=1}^{m} \underline{r}_{i'} h_{i'} + \sum_{j=m+1}^{n} \overline{r}_j h_j + \epsilon\|h\| \text{ and} \tag{4.10}$$

$$\left(\sum_{i'=1}^{m} \overline{s}_{i'} h_{i'} + \sum_{j=m+1}^{n} \underline{s}_j h_j \right) \leq \sum_{i'=1}^{m} \overline{r}_{i'} h_{i'} + \sum_{j=m+1}^{n} \underline{r}_j h_j + \epsilon\|h\|. \tag{4.11}$$

In view of (4.10) and (4.11), we obtain

$$\begin{aligned}
&\left[\sum_{i'=1}^{m} \underline{s}_{i'} h_{i'} + \sum_{j=m+1}^{n} \overline{s}_j h_j, \sum_{i'=1}^{m} \overline{s}_{i'} h_{i'} + \sum_{j=m+1}^{n} \underline{s}_j h_j\right] \preceq \mathbf{F}_{\mathcal{DH}}(\bar{y})(h) \oplus \epsilon\|h\| \\
\implies & \bigoplus_{i'=1}^{m} [\underline{s}_{i'} h_{i'}, \overline{s}_{i'} h_{i'}] \oplus \bigoplus_{j=m+1}^{n} [\overline{s}_j h_j, \underline{s}_j h_j] \preceq \mathbf{F}_{\mathcal{DH}}(\bar{y})(h) \oplus \epsilon\|h\| \\
\implies & \bigoplus_{i'=1}^{m} \mathbf{S}_{i'} \odot h_{i'} \oplus \bigoplus_{j=m+1}^{n} \mathbf{S}_j \odot h_j \preceq \mathbf{F}_{\mathcal{DH}}(\bar{y})(h) \oplus \epsilon\|h\| \\
\implies & h^\top \odot \widehat{\mathbf{S}} \preceq \mathbf{F}_{\mathcal{DH}}(\bar{y})(h) \oplus \epsilon\|h\|.
\end{aligned}$$

Thus, $\widehat{\mathbf{S}} \in \partial_\epsilon^{\mathcal{DH}} \mathbf{F}(\bar{y})$. Hence, $\partial_\epsilon^{\mathcal{DH}} \mathbf{F}(\bar{y})$ is closed. □

Theorem 4.4 *Let $\boldsymbol{F}$ be a gH-subdifferentiable and gH-Fréchet differentiable IVF at $\bar{y} \in \mathcal{Y}$. Then, for any $\epsilon > 0$,*

$$\{\boldsymbol{F}_{\mathscr{F}}(\bar{y})\} \subseteq \partial_\epsilon^{\mathcal{DH}} \boldsymbol{F}(\bar{y}),$$

where $\boldsymbol{F}_{\mathscr{F}}(\bar{y})$ is the Fréchet derivative of $\boldsymbol{F}$ at $\bar{y}$.

Proof: If $\mathbf{F}$ is gH-subdifferentiable at $\bar{y}$, then there exists $\widehat{\mathbf{S}} \in I(\mathbb{R})^n$ satisfying

$$(w - \bar{y})^\top \odot \widehat{\mathbf{S}} \preceq \mathbf{F}(w) \ominus_{gH} \mathbf{F}(\bar{y}) \text{ for any } w \in \mathcal{Y}. \tag{4.12}$$

Taking $w = \bar{y} + \lambda e,\ \lambda \geq 0$, and $e \in \mathcal{Y}$ in (4.12) such that $\|e\| = 1$, we get

$$\begin{aligned}
& (\lambda e)^\top \odot \widehat{\mathbf{S}} \preceq \mathbf{F}(\bar{y} + \lambda e) \ominus_{gH} \mathbf{F}(\bar{y}) \\
\implies \quad & e^\top \odot \widehat{\mathbf{S}} \preceq \frac{1}{\lambda} \odot (\mathbf{F}(\bar{y} + \lambda e) \ominus_{gH} \mathbf{F}(\bar{y})).
\end{aligned}$$

Also, $\mathbf{F}$ is gH-Fréchet differentiable at $\bar{y}$, then from Theorem 1.1 and Remark 1.7, for

$\lambda \to 0+$, we have

$$e^\top \odot \widehat{\mathbf{S}} \preceq \mathbf{F}_{\mathscr{F}}(\bar{y})(e). \tag{4.13}$$

Replacing e by $-e$ in the last relation, we get

$$\begin{aligned} & (-e)^\top \odot \widehat{\mathbf{S}} \preceq \mathbf{F}_{\mathscr{F}}(\bar{y})(-e) \\ \Longrightarrow \quad & \mathbf{F}_{\mathscr{F}}(\bar{y})(e) \preceq e^\top \odot \widehat{\mathbf{S}}. \end{aligned} \tag{4.14}$$

Hence, from (4.13), (4.14), and Theorem 4.1, we obtain

$$\mathbf{F}_{\mathscr{F}}(\bar{y}) \in \partial \mathbf{F}(\bar{y}) \implies \{\mathbf{F}_{\mathscr{F}}(\bar{y})\} \subseteq \partial_\epsilon^{\mathcal{DH}} \mathbf{F}(\bar{y}),$$

which is the required relation. □

Remark 4.2 *The conclusion in Theorem 4.4 can be strict. For instance, consider an IVF* $\boldsymbol{F} : \mathbb{R} \to I(\mathbb{R})$ *given by* $\boldsymbol{F}(y) = \boldsymbol{0} \odot y$.

Let us calculate the gH-Fréchet derivative and gH-Dini Hadamard ϵ*-subdifferential of* $\boldsymbol{F}$ *at* $\bar{y} = 0$. *Let us assume that there exists an* $\boldsymbol{S} \in \partial_\epsilon^{\mathcal{DH}} \boldsymbol{F}(0)$ *satisfying that for any* $w \in \mathbb{R}$,

$$\begin{aligned} & (w-0) \odot \boldsymbol{S} \preceq \boldsymbol{F}_{\mathcal{DH}}(0)(w-0) \oplus \epsilon|w-0| \\ \Longrightarrow \quad & w \odot \boldsymbol{S} \preceq \liminf_{\substack{u \to w \\ \lambda \to 0+}} \frac{1}{\lambda} \odot (\boldsymbol{F}(\lambda u) \ominus_{gH} \boldsymbol{F}(0)) \oplus \epsilon|w| \\ \Longrightarrow \quad & w \odot \boldsymbol{S} \preceq \boldsymbol{0} \oplus \epsilon|w| \\ \Longrightarrow \quad & w \odot \boldsymbol{S} \preceq \epsilon|w|. \end{aligned}$$

There arise the following two cases:

Case 1: For $w \geq 0$, $w \odot \boldsymbol{S} \preceq \epsilon w \implies \boldsymbol{S} \preceq \epsilon$.

Case 2: For $w < 0$, $\boldsymbol{S} \odot w \preceq \epsilon(-w) \implies -\epsilon \preceq \boldsymbol{S}$.

In view of Case 1: and Case 2:, we have $\partial_{\epsilon}^{\mathcal{DH}}\boldsymbol{F}(0) = \{\boldsymbol{S} : -\epsilon \preceq \boldsymbol{S} \preceq \epsilon\}$. *Now assume that* $\boldsymbol{F}_{\mathscr{F}}(\bar{y})$ *is the gH-Fréchet derivative of* $\boldsymbol{F}$ *at* $\bar{y} = 0$. *Then,*

$$\begin{aligned}
&\lim_{|h|\to 0} \frac{\|\boldsymbol{F}(\bar{y}+h) \ominus_{gH} \boldsymbol{F}(\bar{y}) \ominus_{gH} \boldsymbol{F}_{\mathscr{F}}(h)\|_{I(\mathbb{R})}}{|h|} = 0 \\
\implies &\lim_{|h|\to 0} \frac{\|\boldsymbol{F}_{\mathscr{F}}(h)\|_{I(\mathbb{R})}}{|h|} = 0 \implies \boldsymbol{F}_{\mathscr{F}}(0) = \boldsymbol{0}.
\end{aligned}$$

Thus, $\boldsymbol{F}_{\mathscr{F}}(0) \subset \partial_{\epsilon}^{\mathcal{DH}}\boldsymbol{F}(0)$.

Theorem 4.5 *Let* $\boldsymbol{F}$ *be gH-Dini Hadamard ϵ-subdifferentiable IVF at* $\bar{y}$ *in* $\mathcal{Y}$. *Then, for any* $\epsilon > 0$ *and* $\delta > 0$,

$$\partial_{\epsilon'}^{\mathcal{DH}}(\delta \odot \boldsymbol{F})(\bar{y}) = \delta \odot \partial_{\epsilon}^{\mathcal{DH}}(\boldsymbol{F})(\bar{y}), \ \textit{where } \epsilon' = \epsilon\delta.$$

Proof: Let $\widehat{\mathbf{S}} \in \delta \odot \partial_{\epsilon}^{\mathcal{DH}}(\mathbf{F})(\bar{y})$. Then, we can write $\widehat{\mathbf{S}} = \delta \odot \widehat{\mathbf{S}}'$ such that $\widehat{\mathbf{S}}' \in \partial_{\epsilon}^{\mathcal{DH}}(\mathbf{F})(\bar{y})$ for any $w \in \mathcal{Y}$ and

$$\begin{aligned}
&(w-\bar{y})^{\top} \odot \widehat{\mathbf{S}}' \preceq \mathbf{F}_{\mathcal{DH}}(\bar{y})(w-\bar{y}) \oplus \epsilon\|w-\bar{y}\| \\
\implies &(w-\bar{y})^{\top} \odot \left(\frac{1}{\delta} \odot \widehat{\mathbf{S}}\right) \preceq \mathbf{F}_{\mathcal{DH}}(\bar{y})(w-\bar{y}) \oplus \epsilon\|w-\bar{y}\| \\
\implies &(w-\bar{y})^{\top} \odot \widehat{\mathbf{S}} \preceq \delta \odot (\mathbf{F}_{\mathcal{DH}}(\bar{y})(w-\bar{y}) \oplus \epsilon\|w-\bar{y}\|) \\
\implies &(w-\bar{y})^{\top} \odot \widehat{\mathbf{S}} \preceq \delta \odot \left(\liminf_{\substack{u\to(w-\bar{y})\\ \lambda\to 0+}} \frac{1}{\lambda} \odot \big(\mathbf{F}(\bar{y}+\lambda u) \ominus_{gH} \mathbf{F}(\bar{y})\big) \right) \oplus \epsilon\delta\|w-\bar{y}\| \\
\implies &(w-\bar{y})^{\top} \odot \widehat{\mathbf{S}} \preceq \liminf_{\substack{u\to(w-\bar{y})\\ \lambda\to 0+}} \frac{1}{\lambda} \odot \big(\delta \odot \mathbf{F}(\bar{y}+\lambda u) \ominus_{gH} \delta \odot \mathbf{F}(\bar{y})\big) \oplus \epsilon\delta\|w-\bar{y}\| \\
\implies &(w-\bar{y})^{\top} \odot \widehat{\mathbf{S}} \preceq (\delta \odot \mathbf{F})_{\mathcal{DH}}(\bar{y})(w-\bar{y}) \oplus \epsilon'\|w-\bar{y}\|, \text{ where } \epsilon' = \epsilon\delta \\
\implies &\widehat{\mathbf{S}} \in \partial_{\epsilon'}^{\mathcal{DH}}(\delta \odot \mathbf{F})(\bar{y}).
\end{aligned}$$

Conversely, assume that $\widehat{\mathbf{S}} \in \partial_{\epsilon'}^{\mathcal{DH}}(\delta \odot \mathbf{F})(\bar{y})$. Then, for any $\epsilon > 0$ and $w \in \mathcal{Y}$, we have

$$
\begin{aligned}
&(w-\bar{y})^\top \odot \widehat{\mathbf{S}} \preceq (\delta \odot \mathbf{F})_{\mathcal{DH}}(\bar{y})(w-\bar{y}) \oplus \epsilon' \|w-\bar{y}\| \\
\implies &(w-\bar{y})^\top \odot \widehat{\mathbf{S}} \preceq \liminf_{\substack{u \to (w-\bar{y}) \\ \lambda \to 0+}} \frac{1}{\lambda} \odot \big(\delta \odot \mathbf{F}(\bar{y}+\lambda u) \ominus_{gH} \delta \odot \mathbf{F}(\bar{y})\big) \oplus \epsilon\delta \|w-\bar{y}\| \\
\implies &\frac{1}{\delta} \odot ((w-\bar{y})^\top \odot \widehat{\mathbf{S}}) \preceq \mathbf{F}_{\mathcal{DH}}(\bar{y})(w-\bar{y}) \oplus \epsilon \|w-\bar{y}\| \\
\implies &(w-\bar{y})^\top \odot \left(\frac{1}{\delta} \odot \widehat{\mathbf{S}}\right) \preceq \mathbf{F}_{\mathcal{DH}}(\bar{y})(w-\bar{y}) \oplus \epsilon \|w-\bar{y}\| \\
\implies &\frac{1}{\delta} \odot \widehat{\mathbf{S}} \in \partial_\epsilon^{\mathcal{DH}}(\mathbf{F})(\bar{y}) \\
\implies &\widehat{\mathbf{S}} \in \delta \odot \partial_\epsilon^{\mathcal{DH}}(\mathbf{F})(\bar{y}).
\end{aligned}
$$

Thus, we conclude that $\partial_{\epsilon'}^{\mathcal{DH}}(\delta \odot \mathbf{F})(\bar{y}) = \delta \odot \partial_\epsilon^{\mathcal{DH}}(\mathbf{F})(\bar{y})$, where $\epsilon' = \epsilon\delta$. □

4.4 $\mathbf{H}_\epsilon$-Subgradient

We define the notion of $\mathbf{H}_\epsilon$-subgradient for IVFs based on the notion of sponge of a set. Further, it has been proved that every gH-Dini Hadamard ϵ-subgradient at a point $\bar{y}$ is also an $\mathbf{H}_\epsilon$-subgradient of $\mathbf{F}$ at $\bar{y}$. However, the converse need not be true. Furthermore, an interpretation of gH-Dini Hadamard ϵ-subdifferential is discussed with the help of sponges.

Let $\mathbf{F}$ be an IVF on $\mathcal{Y}$. For all $\epsilon > 0$, define an IVF $\mathbf{F}_\epsilon : \mathcal{Y} \to I(\mathbb{R})$ by

$$\mathbf{F}_\epsilon(w) = \mathbf{F}(w) \oplus \epsilon \|w-\bar{y}\| \text{ for all } w \in \mathcal{Y}.$$

Lemma 4.2 *Let $\boldsymbol{F}$ be an IVF on $\mathcal{Y}$. Then, for $\bar{y} \in \mathcal{Y}$ and $\epsilon > 0$, we have*

$$\boldsymbol{F}_{\epsilon\mathcal{DH}}(\bar{y})(w-\bar{y}) = \boldsymbol{F}_{\mathcal{DH}}(\bar{y})(w-\bar{y}) \oplus \epsilon \|w-\bar{y}\|.$$

Proof: For $\epsilon > 0$ and $\bar{y} \in \mathcal{Y}$, we have

$$\begin{aligned}
\mathbf{F}_{\epsilon\mathcal{DH}}(\bar{y})(w-\bar{y}) &= \liminf_{\substack{u\to(w-\bar{y})\\ \lambda\to 0+}} \frac{1}{\lambda} \odot \big(\mathbf{F}_\epsilon(\bar{y}+\lambda u) \ominus_{gH} \mathbf{F}_\epsilon(\bar{y})\big)\\
&= \liminf_{\substack{u\to(w-\bar{y})\\ \lambda\to 0+}} \frac{1}{\lambda} \odot \big((\mathbf{F}(\bar{y}+\lambda u) \oplus \epsilon\|\lambda u\|) \ominus_{gH} \mathbf{F}(\bar{y})\big)\\
&= \liminf_{\substack{u\to(w-\bar{y})\\ \lambda\to 0+}} \frac{1}{\lambda} \odot \big(\mathbf{F}(\bar{y}+\lambda u) \ominus_{gH} (\mathbf{F}(\bar{y})\big) \oplus \lim_{u\to(w-\bar{y})} \epsilon\|u\|\\
&= \mathbf{F}_{\mathcal{DH}}(\bar{y})(w-\bar{y}) \oplus \epsilon\|w-\bar{y}\|.
\end{aligned}$$

Thus, we get the desired result. □

Remark 4.3 *From Lemma 4.2, we can observe that for all $w \in \mathcal{Y}$ and $\epsilon > 0$, an $\widehat{\boldsymbol{S}} \in \partial^{\mathcal{DH}}\boldsymbol{F}_\epsilon(\bar{y})$ if and only if*

$$\begin{aligned}
&(w-\bar{y})^\top \odot \widehat{\boldsymbol{S}} \preceq \boldsymbol{F}_{\epsilon\mathcal{DH}}(\bar{y})(w-\bar{y})\\
\iff\quad &(w-\bar{y})^\top \odot \widehat{\boldsymbol{S}} \preceq \boldsymbol{F}_{\mathcal{DH}}(\bar{y})(w-\bar{y}) \oplus \epsilon\|w-\bar{y}\|.
\end{aligned}$$

Therefore, for $\bar{y} \in \mathcal{Y}$ and $\epsilon > 0$, we have $\partial_\epsilon^{\mathcal{DH}}\boldsymbol{F}(\bar{y}) = \partial^{\mathcal{DH}}\boldsymbol{F}_\epsilon(\bar{y})$.

Definition 4.3 (Sponge of a set [110]). *Let $\mathcal{S} \subseteq \mathcal{Y}$. If for any $\bar{y} \in \mathcal{Y}$ and for all $h \in \mathcal{Y} \setminus \{0\}$, we can find a $\lambda > 0$ and $\delta > 0$ satisfying $\bar{y} + [0,\delta] \cdot \mathcal{B}(h,\delta) \subseteq \mathcal{S}$, then $\mathcal{S}$ is said to be a sponge set around $\bar{y}$.*

Definition 4.4 ($\mathbf{H}_\epsilon$-subgradient for IVF). *Let $\boldsymbol{F} : \mathcal{Y} \to \overline{I(\mathbb{R})}$ be an extended IVF on $\mathcal{Y}$. Then, an element $\widehat{\boldsymbol{S}} \in I(\mathbb{R})^n$ is an $\boldsymbol{H}_\epsilon$-subgradient of $\boldsymbol{F}$ at $\bar{y}$ if there exists a sponge $\mathcal{S}$ around $\bar{y} \in \mathcal{Y}$ such that*

$$(w-\bar{y})^\top \odot \widehat{\boldsymbol{S}} \ominus_{gH} \epsilon\|w-\bar{y}\| \preceq \boldsymbol{F}(w) \ominus_{gH} \boldsymbol{F}(\bar{y}) \text{ for all } w \in \mathcal{S}.$$

Theorem 4.6 *Let $\boldsymbol{F}$ be an IVF on $\mathcal{Y}$ and there exists an $\widehat{\boldsymbol{S}} \in \partial_\epsilon^{\mathcal{DH}}\boldsymbol{F}(\bar{y})$. Then, for all $\gamma > \epsilon$, $\widehat{\boldsymbol{S}}$ is an $\boldsymbol{H}_\gamma$-subgradient of $\boldsymbol{F}$ at $\bar{y}$.*

Proof: Let $\widehat{\mathbf{S}} \in \partial_\epsilon^{\mathcal{DH}}\mathbf{F}(\bar{y})$ and $\gamma > \epsilon$. Consider the set

$$\mathcal{S} = \{w \in \mathcal{Y} : (w - \bar{y})^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \gamma\|w - \bar{y}\| \preceq \mathbf{F}(w) \ominus_{gH} \mathbf{F}(\bar{y})\} \text{ for all } w \in \mathcal{Y}.$$

We show that $\mathcal{S}$ is a sponge around $\bar{y}$. Let $h \in \mathcal{Y} \setminus \{0\}$. Since $\widehat{\mathbf{S}} \in \partial_\epsilon^{\mathcal{DH}}\mathbf{F}(\bar{y})$, then

$$\begin{aligned} & h^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \epsilon\|h\| \preceq \mathbf{F}_{\mathcal{DH}}(\bar{y})(h) \\ \text{or, } & h^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \epsilon\|h\| \preceq \liminf_{\substack{u \to h \\ \lambda \to 0+}} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{y} + \lambda u) \ominus_{gH} \mathbf{F}(\bar{y})) \\ \text{or, } & h^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \left(\frac{\gamma + \epsilon}{2}\right) \|h\| \preceq \liminf_{\substack{u \to h \\ \lambda \to 0+}} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{y} + \lambda u) \ominus_{gH} \mathbf{F}(\bar{y})) \text{ from Lemma 1.1} \\ \text{or, } & h^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \gamma\|h\| \preceq \liminf_{\substack{u \to h \\ \lambda \to 0+}} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{y} + \lambda u) \ominus_{gH} \mathbf{F}(\bar{y})) \text{ from Lemma 1.1.} \end{aligned}$$

Therefore, we can find a $\delta_1 > 0$ such that for all $u \in \mathcal{B}(h, \delta_1)$

$$u^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \gamma\|u\| \preceq h^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \left(\frac{\gamma + \epsilon}{2}\right) \|h\|. \tag{4.15}$$

From definition of limit inferior, there exists $\delta_2 > 0$ satisfying that for all $\lambda \in (0, \delta_2)$ and $u \in \mathcal{B}(h, \delta_2)$, we have

$$h^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \left(\frac{\gamma + \epsilon}{2}\right) \|h\| \preceq \frac{1}{\lambda} \odot \big(\mathbf{F}(\bar{y} + \lambda u) \ominus_{gH} \mathbf{F}(\bar{y})\big). \tag{4.16}$$

Hence, from (4.15) and (4.16), there exists $\delta = \frac{1}{2}\min\{\delta_1, \delta_2\} > 0$ such that for all $\lambda \in (0, \delta]$ and $u \in \mathcal{B}(h, \delta)$,

$$(\lambda u)^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \gamma\|\lambda u\| \preceq \mathbf{F}(\bar{y} + \lambda u) \ominus_{gH} \mathbf{F}(\bar{y}).$$

Thus, for all $w \in \bar{y} + [0, \delta] \cdot \mathcal{B}(h, \delta)$, we have

$$(w - \bar{y})^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \gamma\|w - \bar{y}\| \preceq \mathbf{F}(w) \ominus_{gH} \mathbf{F}(\bar{y}).$$

Therefore, $\bar{y} + [0, \delta] \cdot \mathcal{B}(h, \delta) \in \mathcal{S}$. Thus, $\mathcal{S}$ is a sponge around $\bar{y}$. $\square$

Remark 4.4 *The converse of Theorem 4.6 need not be true. For example, consider an IVF* $\boldsymbol{F} : \mathbb{R} \to I(\mathbb{R})$ *such that*

$$\boldsymbol{F}(y) = \begin{cases} \boldsymbol{0}, & y \in \mathcal{S} \\ [-2, -1], & \textit{otherwise}, \end{cases}$$

where $\mathcal{S}$ *is a sponge set around some* $\bar{y} \in \mathbb{R}$. *We show that* $\boldsymbol{0}$ *is an* $\boldsymbol{H}_\epsilon$*-subgradient of* $\boldsymbol{F}$ *at* $\bar{y}$, *however* $\boldsymbol{0} \notin \partial_\epsilon^{\mathcal{DH}} \boldsymbol{F}(\bar{y})$.

Let $\boldsymbol{S} \in I(\mathbb{R})$ *be an* $\boldsymbol{H}_\epsilon$*-subgradient of* $\boldsymbol{F}$ *at* $\bar{y}$ *and* $\mathcal{S}$ *be a sponge set around* $\bar{y}$. *Then, for all* $w \in \mathcal{S}$,

$$\begin{aligned} & (w - \bar{y}) \odot \boldsymbol{S} \ominus_{gH} \epsilon |w - \bar{y}| \preceq \boldsymbol{F}(w) \ominus_{gH} \boldsymbol{F}(\bar{y}) \\ \implies \quad & (w - \bar{y}) \odot \boldsymbol{S} \ominus_{gH} \epsilon |w - \bar{y}| \preceq [0, 0] \ominus_{gH} \boldsymbol{F}(\bar{y}). \end{aligned}$$

We have the following cases:

Case 1: If $\bar{y} \in \mathcal{S}$, *then*

$$(w - \bar{y}) \odot \boldsymbol{S} \ominus_{gH} \epsilon |w - \bar{y}| \preceq \boldsymbol{0}.$$

There arise the following two subcases:

(a) If $(w - \bar{y}) \geq 0$, *then*

$$\boldsymbol{S} \ominus_{gH} \epsilon \preceq \boldsymbol{0} \implies \boldsymbol{S} \preceq \epsilon.$$

(b) If $(w - \bar{y}) < 0$, *then*

$$\boldsymbol{0} \preceq \boldsymbol{S} \ominus_{gH} (-\epsilon) \implies -\epsilon \preceq \boldsymbol{S}.$$

Since $\epsilon > 0$ *is arbitrary, therefore (a) and (b) of Case 1: occur simultaneously* *when* $\boldsymbol{S} = \boldsymbol{0}$.

Case 2: If $\bar{y} \notin \mathcal{S}$, *then*

$$(w - \bar{y}) \odot \boldsymbol{S} \ominus_{gH} \epsilon|w - \bar{y}| \preceq [1, 2].$$

There arise the following two sub cases:

(a) If $(w - \bar{y}) \geq 0$, *then*

$$\begin{aligned} & (w - \bar{y}) \odot \boldsymbol{S} \ominus_{gH} \epsilon(w - \bar{y}) \preceq [1, 2] \\ \implies \quad & (w - \bar{y})\underline{s} \leq 1 + \epsilon(w - \bar{y}) \text{ and } (w - \bar{y})\overline{s} \leq 2 + \epsilon(w - \bar{y}). \end{aligned}$$

(b) If $(w - \bar{y}) < 0$, *then*

$$\begin{aligned} & (w - \bar{y}) \odot \boldsymbol{S} \ominus_{gH} \epsilon(\bar{y} - w) \preceq [1, 2] \\ \implies \quad & (w - \bar{y})\overline{s} \leq 1 - \epsilon(w - \bar{y}) \text{ and } (w - \bar{y})\underline{s} \leq 2 - \epsilon(w - \bar{y}). \end{aligned}$$

Since $(w - \bar{y}) \in \mathbb{R}$ *and* $\epsilon > 0$ *are arbitrary, therefore (a) and (b) of Case 2: occur simultaneously when* $\boldsymbol{S} = \boldsymbol{0}$.

Thus, $\boldsymbol{0}$ *is an* $\boldsymbol{H}_\epsilon$*-subgradient of* $\boldsymbol{F}$ *at* $\bar{y}$. *Now, we show that* $\boldsymbol{0} \notin \partial_\epsilon^{\mathcal{DH}} \boldsymbol{F}(\bar{y})$. *Assume contrarily that* $\boldsymbol{0} \in \partial_\epsilon^{\mathcal{DH}} \boldsymbol{F}(\bar{y})$. *Therefore, for all* $w \in \mathbb{R}$, *we get*

$$\begin{aligned} & (w - \bar{y}) \odot \boldsymbol{S} \preceq \boldsymbol{F}_{\mathcal{DH}}(\bar{y})(w - \bar{y}) \oplus \epsilon|w - \bar{y}| \\ \implies \quad & \boldsymbol{0} \preceq \liminf_{\substack{u \to (w - \bar{y}) \\ \lambda \to 0+}} \frac{1}{\lambda} \odot \big(\boldsymbol{F}(\bar{y} + \lambda u) \ominus_{gH} \boldsymbol{F}(\bar{y})\big) \oplus \epsilon|w - \bar{y}|. \end{aligned}$$

We have the following cases:

Case 1: If $\bar{y} \in \mathcal{S}$, *then*

$$\boldsymbol{0} \preceq \liminf_{\substack{u \to (w - \bar{y}) \\ \lambda \to 0+}} \frac{1}{\lambda} \odot \boldsymbol{F}(\bar{y} + \lambda u) \oplus \epsilon|w - \bar{y}|.$$

Therefore, we observe that for $(\bar{y}+\lambda u) \notin \mathcal{S}$, *we get*

$$\boldsymbol{0} \preceq \liminf_{\lambda \to 0+} \frac{1}{\lambda} \odot [-2,-1] \oplus \epsilon|w-\bar{y}|,$$

which does not exist for any $w \in \mathbb{R}$.

Case 2: If $\bar{y} \notin \mathcal{S}$, *then*

$$\boldsymbol{0} \preceq \liminf_{\substack{u \to (w-\bar{y}) \\ \lambda \to 0+}} \frac{1}{\lambda} \odot \big(\boldsymbol{F}(\bar{y}+\lambda u) \ominus_{gH} [-2,-1]\big) \oplus \epsilon|w-\bar{y}|.$$

Therefore, we observe that for $(\bar{y}+\lambda u) \in \mathcal{S}$, *we get*

$$\boldsymbol{0} \preceq \liminf_{\lambda \to 0+} \frac{1}{\lambda} \odot [1,2] \oplus \epsilon|w-\bar{y}|,$$

which does not exist for any $w \in \mathbb{R}$. *In view of Case 1: and Case 2:, we can conclude that* $\boldsymbol{0} \notin \partial_\epsilon^{\mathcal{DH}} \boldsymbol{F}$ *at* $\bar{y}$.

Definition 4.5 (gH-calm IVF). *Let* $\boldsymbol{F}$ *be an IVF on* $\mathcal{Y}$. *Then,* $\boldsymbol{F}$ *is said to be a* gH*-calm IVF at* $\bar{y} \in \mathcal{Y}$ *if there exist* $c \geq 0$ *and* $\delta > 0$ *such that*

$$-c\|w-\bar{y}\| \preceq \boldsymbol{F}(w) \ominus_{gH} \boldsymbol{F}(\bar{y}) \text{ for all } w \in \mathcal{B}(\bar{y},\delta).$$

Remark 4.5 *(i) In view of Definition 4.5, it can be observed that there exist some constant* $c \geq 0$ *such that for every* h *in* $\mathcal{Y}$, *we have*

$$-c\|h\| \preceq \boldsymbol{F}_{\mathcal{DH}}(\bar{y})(h).$$

(ii) From Definition 4.5, it can be noted that if $\boldsymbol{F}$ *is* gH*-calm IVF at* $\bar{y} \in \mathcal{Y}$. *Then,* $\underline{f}, \overline{f}$ *are calm at* $\bar{y} \in \mathcal{Y}$ *and vice-versa.*

Lemma 4.3 *Let $\boldsymbol{F}$ be a gH-calm IVF at $\bar{y} \in \mathcal{Y}$, and $\widehat{\boldsymbol{S}}$ be an $\boldsymbol{H}_\epsilon$-subgradient of $\boldsymbol{F}$ at $\bar{y} \in \mathcal{Y}$. Then, $\widehat{\boldsymbol{S}} \in \partial_\epsilon^{\mathcal{DH}} \boldsymbol{F}(\bar{y})$.*

Proof: If $\widehat{\mathbf{S}}$ is an $\mathbf{H}_\epsilon$-subgradient of $\mathbf{F}$ at $\bar{y} \in \mathcal{Y}$, then there exists a sponge $\mathcal{S}$ around $\bar{y}$ satisfying

$$(w - \bar{y})^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \epsilon \|w - \bar{y}\| \preceq \mathbf{F}(w) \ominus_{gH} \mathbf{F}(\bar{y}) \text{ for all } w \in \mathcal{S}.$$

Let $h \in \mathcal{Y} \setminus \{0\}$, we can find a $\beta > 0$, $\delta > 0$ such that $\lambda \in (0, \beta]$ and $u \in \mathcal{B}(h, \delta)$, $\bar{y} + \lambda u \in \mathcal{S}$ and

$$\begin{aligned} & u^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \epsilon \|u\| \preceq \frac{1}{\lambda} \odot \big(\mathbf{F}(\bar{y} + \lambda u) \ominus_{gH} \mathbf{F}(\bar{y})\big) \\ \implies \quad & \liminf_{\substack{u \to h \\ \lambda \to 0+}} u^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \epsilon \|u\| \preceq \liminf_{\substack{u \to h \\ \lambda \to 0+}} \frac{1}{\lambda} \odot \big(\mathbf{F}(\bar{y} + \lambda u) \ominus_{gH} \mathbf{F}(\bar{y})\big) \\ \implies \quad & h^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \epsilon \|h\| \preceq \mathbf{F}_{\mathcal{DH}}(\bar{y})(h). \end{aligned}$$

Therefore, $\widehat{\mathbf{S}} \in \partial_\epsilon^{\mathcal{DH}} \mathbf{F}(\bar{y})$. □

We next provide an interpretation of gH-Dini Hadamard ϵ-subdifferential by replacing neighborhood with sponges.

Theorem 4.7 *Let $\boldsymbol{F}$ be an extended IVF on $\mathcal{Y}$. For $\epsilon > 0$ and $\bar{y} \in \mathcal{Y}$, $\widehat{\boldsymbol{S}} \in \partial_\epsilon^{\mathcal{DH}} \boldsymbol{F}(\bar{y})$ if and only if $\boldsymbol{F}$ is gH-calm IVF at $\bar{y}$ and for every $\alpha > 0$, there exists a sponge $\mathcal{S}$ around $\bar{y}$ such that*

$$(w - \bar{y})^\top \odot \widehat{\boldsymbol{S}} \ominus_{gH} (\alpha + \epsilon) \|w - \bar{y}\| \preceq \boldsymbol{F}(w) \ominus_{gH} \boldsymbol{F}(\bar{y}). \tag{4.17}$$

Proof: Let $\widehat{\mathbf{S}} \in \partial_\epsilon^{\mathcal{DH}} \mathbf{F}(\bar{y})$. Then, from Theorem 4.6, there exists a sponge $\mathcal{S}$ around $\bar{y}$ satisfying

$$(w - \bar{y})^\top \odot \widehat{\mathbf{S}} \ominus_{gH} (\alpha + \epsilon) \|w - \bar{y}\| \preceq \mathbf{F}(w) \ominus_{gH} \mathbf{F}(\bar{y}) \text{ for all } w \in \mathcal{S}.$$

To prove the converse, assume that $\mathbf{F}$ is gH-calm IVF at $\bar{y}$ and there exists an $\widehat{\mathbf{S}} \in I(\mathbb{R})^n$ such that (4.17) holds. We show that $\widehat{\mathbf{S}} \in \partial_{\epsilon}^{\mathcal{DH}}\mathbf{F}(\bar{y})$, i.e.,

$$h^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \epsilon\|h\| \preceq \mathbf{F}_{\mathcal{DH}}(\bar{y})(h) \text{ for all } h \in \mathcal{Y}.$$

Now, for all $p \in \mathbb{N}$, take $\alpha_p = \frac{1}{p}$. By the hypothesis, there exists a sponge $\mathcal{S}_p$ around $\bar{y}$ such that

$$(w - \bar{y})^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \left(\tfrac{1}{p} + \epsilon\right) \|w - \bar{y}\| \preceq \mathbf{F}(w) \ominus_{gH} \mathbf{F}(\bar{y}) \text{ for all } w \in \mathcal{S}_p.$$

Thus, for $h \in \mathcal{Y}$ and for every $p \in \mathbb{N}$, there exist $t_p > 0$ and $\delta_p > 0$ such that for all $\lambda \in (0, t_p)$ and $u \in \mathcal{B}(h, \delta_p)$, we have $\bar{y} + \lambda u \in \mathcal{S}_p$ and

$$\begin{aligned}
&(\lambda u)^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \left(\tfrac{1}{p} + \epsilon\right) \|\lambda u\| \preceq \mathbf{F}(\bar{y} + \lambda u) \ominus_{gH} \mathbf{F}(\bar{y})\\
\text{or,}\quad &\liminf_{\substack{u \to h\\ \lambda \to 0+}} \; u^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \left(\tfrac{1}{p} + \epsilon\right) \|u\| \preceq \liminf_{\substack{u \to h\\ \lambda \to 0+}} \; \tfrac{1}{\lambda} \odot \big(\mathbf{F}(\bar{y} + \lambda u) \ominus_{gH} \mathbf{F}(\bar{y})\big)\\
\text{or,}\quad &h^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \left(\tfrac{1}{p} + \epsilon\right) \|h\| \preceq \mathbf{F}_{\mathcal{DH}}(\bar{y})(h).
\end{aligned}$$

Therefore, as $p \to \infty$, $h^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \epsilon\|h\| \preceq \mathbf{F}_{\mathcal{DH}}(\bar{y})(h)$. Thus, $\widehat{\mathbf{S}} \in \partial_{\epsilon}^{\mathcal{DH}}\mathbf{F}(\bar{y})$. $\square$

4.5 Optimality Conditions on Nonsmooth Interval Optimization

In this section, we define a concept of an ϵ-efficient solution for IOPs and characterization of ϵ-efficient solutions with the help of the derived results on gH-Dini Hadamard ϵ-subdifferentiability of IVFs.

Definition 4.6 (ϵ-efficient solution of an IOP). *Let* $\mathbf{F}$ *be an IVF on* $\mathcal{Y}$*. Then, for*

$\epsilon > 0$, a point $\bar{y} \in \mathcal{Y}$ is called an ϵ-efficient solution of the IOP

$$\min_{y \in \mathcal{Y}} \boldsymbol{F}(y) \tag{4.18}$$

if for all $w \in \mathcal{Y}$, we have $\boldsymbol{F}(\bar{y}) \preceq \boldsymbol{F}(w) \oplus \epsilon\|w - \bar{y}\|$.

Theorem 4.8 *Let $\boldsymbol{F}$ be an IVF on $\mathcal{Y}$. If $\widehat{\boldsymbol{0}} \in \partial_{\epsilon}^{\mathcal{DH}} \boldsymbol{F}(\bar{y})$ for some $\bar{y} \in \mathcal{Y}$, then $\bar{y}$ is an ϵ-efficient solution of the IOP (4.18).*

Proof: Let $\widehat{\mathbf{0}} \in \partial_{\epsilon}^{\mathcal{DH}} \mathbf{F}(\bar{y})$ for some $\bar{y} \in \mathcal{Y}$. Then, for $\epsilon > 0$ and for all $w \in \mathcal{Y}$, we have

$$\begin{aligned}
&(w - \bar{y})^\top \odot \widehat{\mathbf{0}} \preceq \mathbf{F}_{\mathcal{DH}}(\bar{y})(w - \bar{y}) \oplus \epsilon\|w - \bar{y}\| \\
\text{or, } &\mathbf{0} \preceq \liminf_{\substack{u \to (w - \bar{y}) \\ \lambda \to 0+}} \frac{1}{\lambda} \odot \big(\mathbf{F}(\bar{y} + \lambda u) \ominus_{gH} \mathbf{F}(\bar{y})\big) \oplus \epsilon\|w - \bar{y}\| \\
\text{or, } &\mathbf{0} \preceq \left[\min\left\{\liminf_{\substack{u \to (w - \bar{y}) \\ \lambda \to 0+}} \frac{\underline{f}(\bar{y} + \lambda u) - \underline{f}(\bar{y})}{\lambda}, \liminf_{\substack{u \to (w - \bar{y}) \\ \lambda \to 0+}} \frac{\overline{f}(\bar{y} + \lambda u) - \overline{f}(\bar{y})}{\lambda}\right\},\right. \\
&\left.\max\left\{\liminf_{\substack{u \to (w - \bar{y}) \\ \lambda \to 0+}} \frac{\underline{f}(\bar{y} + \lambda u) - \underline{f}(\bar{y})}{\lambda}, \liminf_{\substack{u \to (w - \bar{y}) \\ \lambda \to 0+}} \frac{\overline{f}(\bar{y} + \lambda u) - \overline{f}(\bar{y})}{\lambda}\right\}\right] \oplus \epsilon\|w - \bar{y}\|.
\end{aligned} \tag{4.19}$$

Now, there arises the following two cases.

Case 1: Let

$$\begin{aligned}
&\left[\min\left\{\liminf_{\substack{u \to (w - \bar{y}) \\ \lambda \to 0+}} \frac{\underline{f}(\bar{y} + \lambda u) - \underline{f}(\bar{y})}{\lambda}, \liminf_{\substack{u \to (w - \bar{y}) \\ \lambda \to 0+}} \frac{\overline{f}(\bar{y} + \lambda u) - \overline{f}(\bar{y})}{\lambda}\right\}\right] \\
= &\liminf_{\substack{u \to (w - \bar{y}) \\ \lambda \to 0+}} \frac{\underline{f}(\bar{y} + \lambda u) - \underline{f}(\bar{y})}{\lambda}.
\end{aligned}$$

In this case, from (4.19), we have

$$\mathbf{0} \preceq \left[\liminf_{\substack{u \to (w-\bar{y}) \\ \lambda \to 0+}} \frac{\underline{f}(\bar{y}+\lambda u) - \underline{f}(\bar{y})}{\lambda}, \liminf_{\substack{u \to (w-\bar{y}) \\ \lambda \to 0+}} \frac{\overline{f}(\bar{y}+\lambda u) - \overline{f}(\bar{y})}{\lambda} \right] \oplus \epsilon \|w - \bar{y}\|.$$

Thus, there exists a $\delta_1 > 0$ such that for all $\lambda \in (0, \delta_1)$, we obtain

$$0 \leq \frac{\underline{f}(\bar{y}+\lambda(w-\bar{y})) - \underline{f}(\bar{y})}{\lambda} \text{ and } 0 \leq \frac{\overline{f}(\bar{y}+\lambda(w-\bar{y})) - \overline{f}(\bar{y})}{\lambda}$$
$$\text{or, } 0 \leq \underline{f}(\bar{y}+\lambda(w-\bar{y})) - \underline{f}(\bar{y}) \;\text{ and }\; 0 \leq \overline{f}(\bar{y}+\lambda(w-\bar{y})) - \overline{f}(\bar{y}).$$

Therefore, for all $w \in \mathcal{Y}$, we have

$$\begin{aligned} & \mathbf{0} \preceq \mathbf{F}(\bar{y}+\lambda(w-\bar{y})) \ominus_{gH} \mathbf{F}(\bar{y}) \oplus \epsilon \|w-\bar{y}\| \\ \implies & \mathbf{0} \preceq \mathbf{F}(w) \ominus_{gH} \mathbf{F}(\bar{y}) \oplus \epsilon \|w-\bar{y}\| \\ \implies & \mathbf{F}(\bar{y}) \preceq \mathbf{F}(w) \oplus \epsilon \|w-\bar{y}\|. \end{aligned}$$

Case 2: Let

$$\begin{aligned} & \left[\min \left\{ \liminf_{\substack{u \to (w-\bar{y}) \\ \lambda \to 0+}} \frac{\underline{f}(\bar{y}+\lambda u) - \underline{f}(\bar{y})}{\lambda}, \liminf_{\substack{u \to (w-\bar{y}) \\ \lambda \to 0+}} \frac{\overline{f}(\bar{y}+\lambda u) - \overline{f}(\bar{y})}{\lambda} \right\} \right] \\ = & \liminf_{\substack{u \to (w-\bar{y}) \\ \lambda \to 0+}} \frac{\overline{f}(\bar{y}+\lambda u) - \overline{f}(\bar{y})}{\lambda}. \end{aligned}$$

Proceeding in a similar manner as in Case 1:, we get the desired result. Thus,

$$\mathbf{F}(\bar{y}) \preceq \mathbf{F}(w) \oplus \epsilon \|w-\bar{y}\| \text{ for all } w \in \mathcal{Y}.$$

Thus, in view of Case 1: and Case 2:, $\bar{y}$ is an ϵ-efficient solution of IOP (4.18). $\square$

Theorem 4.9 *Let $\boldsymbol{F} : \mathcal{Y} \to I(\mathbb{R})$ be an IVF on $\mathcal{Y}$. If $\bar{y}$ is an ϵ-efficient solution of the*

IOP (4.18), then for each $\epsilon > 0$

$$\widehat{\boldsymbol{0}} \in \partial_{\epsilon}^{\mathcal{DH}} \boldsymbol{F}(\bar{y}).$$

Proof: Let $\bar{y}$ be an ϵ-efficient solution of the IOP (4.18). Then, for all $w \in \mathcal{Y}$,

$$\begin{aligned} & \mathbf{F}(\bar{y}) \preceq \mathbf{F}(w) \oplus \epsilon \|w - \bar{y}\| \\ \Longrightarrow\ & \mathbf{0} \preceq (\mathbf{F}(w) \ominus_{gH} \mathbf{F}(\bar{y})) \oplus \epsilon \|w - \bar{y}\| \\ \Longrightarrow\ & \mathbf{0} \preceq \liminf_{\substack{u \to (w - \bar{y}) \\ \lambda \to 0+}} \frac{1}{\lambda} \odot \big(\mathbf{F}(\bar{y} + \lambda u) \ominus_{gH} \mathbf{F}(\bar{y})\big) \oplus \epsilon \|w - \bar{y}\| \\ \Longrightarrow\ & (w - \bar{y})^{\top} \odot \widehat{\mathbf{0}} \preceq \mathbf{F}_{\mathcal{DH}}(\bar{y})(w - \bar{y}) \oplus \epsilon \|w - \bar{y}\|. \end{aligned}$$

Therefore, $\widehat{\mathbf{0}} \in \partial_{\epsilon}^{\mathcal{DH}} \mathbf{F}(\bar{y})$. □

Theorem 4.10 (Necessary condition for efficient points to an IOP). *Let* $\boldsymbol{F} : \mathcal{Y} \to I(\mathbb{R})$ *be an IVF and* $\bar{y}$ *be an* ϵ*-efficient solution of the IOP (4.18). If there exists an* $\widehat{\boldsymbol{S}} \in \partial_{\epsilon}^{\mathcal{DH}} \boldsymbol{F}(\bar{y})$*, then*

$$\boldsymbol{0} \preceq (w - \bar{y})^{\top} \odot \widehat{\boldsymbol{S}} \text{ for all } w \in \mathcal{Y}.$$

Proof: Let $\bar{y}$ be an ϵ-efficient solution of IOP (4.18) and there does not exist any $w \in \mathcal{Y}$ such that

$$\mathbf{0} \preceq (w - \bar{y})^{\top} \odot \widehat{\mathbf{S}}.$$

Since $\widehat{\mathbf{S}} \in \partial_{\epsilon}^{\mathcal{DH}} \mathbf{F}(\bar{y})$, there does not exist any $w \in \mathcal{Y}$ such that

$$\begin{aligned} & \mathbf{0} \preceq (w - \bar{y})^{\top} \odot \widehat{\mathbf{S}} \preceq \mathbf{F}_{\mathcal{DH}}(\bar{y})(w - \bar{y}) \oplus \epsilon \|w - \bar{y}\| \\ \text{or, } & \mathbf{0} \preceq \liminf_{\substack{u \to (w - \bar{y}) \\ \lambda \to 0+}} \frac{1}{\lambda} \odot \big(\mathbf{F}(\bar{y} + \lambda u) \ominus_{gH} \mathbf{F}(\bar{y})\big) \oplus \epsilon \|w - \bar{y}\|. \end{aligned}$$

Proceeding in a similar manner as in Theorem 4.8, we can conclude that there does not exist any $w \in \mathcal{Y}$ such that

$$\mathbf{F}(\bar{y}) \preceq \mathbf{F}(w) \oplus \epsilon \|w - \bar{y}\|,$$

which is a contradiction. Therefore, we obtain

$$\mathbf{0} \preceq (w-\bar{y})^\top \odot \widehat{\mathbf{S}} \text{ for all } w \in \mathcal{Y},$$

which is the required relation. □

Theorem 4.11 (Sufficient condition for efficient points to an IOP). *Let $\boldsymbol{F}$ be a convex IVF on $\mathcal{Y}$. If there exists an $\widehat{\boldsymbol{S}} \in \partial_\epsilon^{\mathcal{DH}} \boldsymbol{F}(\bar{y})$ for some $\bar{y} \in \mathcal{Y}$ and $\epsilon > 0$ such that*

$$\boldsymbol{0} \preceq (w-\bar{y})^\top \odot \widehat{\boldsymbol{S}} \textit{ for all } w \in \mathcal{Y}, \tag{4.20}$$

then $\bar{y}$ is an ϵ-efficient solution of the IOP (4.18).

Proof: Let $\widehat{\mathbf{S}} \in \partial_\epsilon^{\mathcal{DH}} \mathbf{F}(\bar{y})$ be such that the relation (4.20) is true. Therefore, for all $w \in \mathcal{Y}$, we have

$$\begin{aligned}
& (w-\bar{y})^\top \odot \widehat{\mathbf{S}} \preceq \mathbf{F}_{\mathcal{DH}}(\bar{y}) \oplus \epsilon \|w-\bar{y}\| \\
\text{or, } & \mathbf{0} \preceq \mathbf{F}_{\mathcal{DH}}(\bar{y}) \oplus \epsilon \|w-\bar{y}\| \\
\text{or, } & \mathbf{0} \preceq \liminf_{\substack{u \to (w-\bar{y}) \\ \lambda \to 0+}} \frac{1}{\lambda} \odot \big(\mathbf{F}(\bar{y}+\lambda u) \ominus_{gH} \mathbf{F}(\bar{y})\big) \oplus \epsilon \|w-\bar{y}\| \\
\text{or, } & \mathbf{0} \preceq \left[\min\left\{ \liminf_{\substack{u \to (w-\bar{y}) \\ \lambda \to 0+}} \frac{\underline{f}(\bar{y}+\lambda u) - \underline{f}(\bar{y}))}{\lambda}, \liminf_{\substack{u \to (w-\bar{y}) \\ \lambda \to 0+}} \frac{\overline{f}(\bar{y}+\lambda u) - \overline{f}(\bar{y}))}{\lambda} \right\},\right. \\
& \quad \left. \max\left\{ \liminf_{\substack{u \to (w-\bar{y}) \\ \lambda \to 0+}} \frac{\underline{f}(\bar{y}+\lambda u) - \underline{f}(\bar{y}))}{\lambda}, \liminf_{\substack{u \to (w-\bar{y}) \\ \lambda \to 0+}} \frac{\overline{f}(\bar{y}+\lambda u) - \overline{f}(\bar{y}))}{\lambda} \right\}\right] \oplus \epsilon \|w-\bar{y}\|.
\end{aligned} \tag{4.21}$$

Proceeding in a similar manner as in Theorem 4.8, we can conclude that $\bar{y}$ is an ϵ-efficient solution of the IOP (4.18). □

Theorem 4.12 *Let $(-\boldsymbol{F}_1)$ be gH-subdifferentiable and gH-Fréchet differentiable IVF at $\bar{y}$ with Fréchet derivative $(-\boldsymbol{F}_{1\mathcal{F}})$ on $\mathcal{Y}$. Let $\boldsymbol{F}_2$ be an IVF on $\mathcal{Y}$. If for any $\epsilon > 0$, $\bar{y}$ is an ϵ-efficient solution of $\boldsymbol{F}_1 \oplus \boldsymbol{F}_2$, then*

$$(-\boldsymbol{F}_{1\mathcal{F}})(\bar{y}) \in \partial_{\epsilon'}^{\mathcal{DH}} \boldsymbol{F}_2(\bar{y}), \ \textit{where } \epsilon' = 2\epsilon. \tag{4.22}$$

Proof: Let $(-\mathbf{F}_1)$ is a gH-subdifferentiable and gH-Fréchet differentiable IVF at $\bar{y}$ with Fréchet derivative $(-\mathbf{F}_{1\mathcal{F}})$ on $\mathcal{Y}$, then from Theorem 4.4, we have

$$\{(-\mathbf{F}_{1\mathcal{F}})(\bar{y})\} \subseteq \partial_{\epsilon}^{\mathcal{DH}}(-\mathbf{F}_1)(\bar{y}) \text{ for } \epsilon > 0. \tag{4.23}$$

Also, as $\bar{y}$ is an ϵ-efficient solution of $\mathbf{F}_1 \oplus \mathbf{F}_2$, we have

$$(\mathbf{F}_1 \oplus \mathbf{F}_2)(\bar{y}) \preceq (\mathbf{F}_1 \oplus \mathbf{F}_2)(w) \oplus \epsilon\|w - \bar{y}\|$$

or, $\mathbf{F}_1(\bar{y}) \ominus_{gH} \mathbf{F}_1(w) \preceq \mathbf{F}_2(w) \ominus_{gH} \mathbf{F}_2(\bar{y}) \oplus \epsilon\|w - \bar{y}\|$ from (i) of Lemma 1.4

or, $\liminf\limits_{\substack{u \to (w-\bar{y}) \\ \lambda \to 0+}} \dfrac{1}{\lambda} \odot \big(\mathbf{F}_1(\bar{y}) \ominus_{gH} \mathbf{F}_1(\bar{y} + \lambda u)\big) \preceq \liminf\limits_{\substack{u \to (w-\bar{y}) \\ \lambda \to 0+}} \dfrac{1}{\lambda} \odot \big(\mathbf{F}_2(\bar{y} + \lambda u) \ominus_{gH} \mathbf{F}_2(\bar{y})\big) \oplus \epsilon\|w - \bar{y}\|$

or, $(-\mathbf{F}_{1\mathcal{DH}}(\bar{y})) \preceq \mathbf{F}_{2\mathcal{DH}}(\bar{y}) \oplus \epsilon\|w - \bar{y}\|$

or, $(-\mathbf{F}_{1\mathcal{DH}}(\bar{y})) \oplus \epsilon\|w - \bar{y}\| \preceq \mathbf{F}_{2\mathcal{DH}}(\bar{y}) \oplus (\epsilon + \epsilon)\|w - \bar{y}\|$

or, $(-\mathbf{F}_{1\mathcal{DH}}(\bar{y})) \oplus \epsilon\|w - \bar{y}\| \preceq \mathbf{F}_{2\mathcal{DH}}(\bar{y}) \oplus \epsilon'\|w - \bar{y}\|, \ 2\epsilon = \epsilon'.$ (4.24)

In view of (4.23) and (4.24), we have

$$(w - \bar{y})^\top \odot (-\mathbf{F}_{1\mathcal{F}})(\bar{y}) \preceq (-\mathbf{F}_{1\mathcal{DH}}(\bar{y})) \oplus \epsilon\|w - \bar{y}\| \preceq \mathbf{F}_{2\mathcal{DH}}(\bar{y}) \oplus \epsilon'\|w - \bar{y}\|$$

$$(w - \bar{y})^\top \odot (-\mathbf{F}_{1\mathcal{F}})(\bar{y}) \preceq \mathbf{F}_{2\mathcal{DH}}(\bar{y}) \oplus \epsilon'\|w - \bar{y}\|.$$

Thus, $(-\mathbf{F}_{1\mathcal{F}})(\bar{y}) \in \partial_{\epsilon'}^{\mathcal{DH}} \mathbf{F}_2(\bar{y})$. □

Theorem 4.13 *Let $\boldsymbol{F}$ be an IVF on $\mathcal{Y}$ and $\bar{y}$ be an ϵ-efficient solution of the IOP*

(4.18). Then, for any $\epsilon > 0$*, we have*

$$\partial_{\epsilon}^{\mathcal{DH}} \boldsymbol{0}(\bar{y}) \subseteq \partial_{\epsilon'}^{\mathcal{DH}} \boldsymbol{F}(\bar{y}), \text{ where } \epsilon' = 2\epsilon.$$

Proof: Let $\widehat{\mathbf{S}} \in \partial_{\epsilon}^{\mathcal{DH}} \mathbf{0}(\bar{y})$. Then, for all $w \in \mathcal{Y}$ and $\epsilon > 0$,

$$(w - \bar{y})^\top \odot \widehat{\mathbf{S}} \preceq \mathbf{0}_{\mathcal{DH}}(\bar{y}) \oplus \epsilon \|w - \bar{y}\|. \tag{4.25}$$

Also, as $\bar{y}$ is an ϵ-efficient solution of (4.18), for each $w \in \mathcal{Y}$ and $\epsilon > 0$, we have

$$\begin{aligned} & \mathbf{F}(\bar{y}) \preceq \mathbf{F}(w) \oplus \epsilon \|w - \bar{y}\| \\ \implies & \mathbf{0} \preceq \mathbf{F}(w) \ominus_{gH} \mathbf{F}(\bar{y}) \oplus \epsilon \|w - \bar{y}\| \\ \implies & \epsilon \|w - \bar{y}\| \preceq \mathbf{F}(w) \ominus_{gH} \mathbf{F}(\bar{y}) \oplus (\epsilon + \epsilon) \|w - \bar{y}\| \\ \implies & \epsilon \|w - \bar{y}\| \preceq \liminf_{\substack{u \to (w - \bar{y}) \\ \lambda \to 0+}} \frac{1}{\lambda} \odot \big(\mathbf{F}(\bar{y} + \lambda u) \ominus_{gH} \mathbf{F}(\bar{y})\big) \oplus 2\epsilon \|w - \bar{y}\| \\ \implies & \epsilon \|w - \bar{y}\| \preceq \mathbf{F}_{\mathcal{DH}}(\bar{y}) \oplus \epsilon' \|w - \bar{y}\|, \ \epsilon' = 2\epsilon. \end{aligned} \tag{4.26}$$

From (4.25), (4.26), and Lemma 1.1, we have

$$(w - \bar{y})^\top \odot \widehat{\mathbf{S}} \preceq \mathbf{F}_{\mathcal{DH}}(\bar{y}) \oplus \epsilon' \|w - \bar{y}\|.$$

Therefore, $\widehat{\mathbf{S}} \in \partial_{\epsilon'}^{\mathcal{DH}} \mathbf{F}(\bar{y})$. Thus, $\partial_{\epsilon}^{\mathcal{DH}} \mathbf{0}(\bar{y}) \subseteq \partial_{\epsilon'}^{\mathcal{DH}} \mathbf{F}(\bar{y})$. □

Remark 4.6 *In Theorem 4.13, the condition on* $\bar{y}$ *to be an* ϵ*-efficient solution is necessary. For instance, consider the IVF that was discussed in Remark 4.1:* $\boldsymbol{F}(\bar{y}) = [-2, -1] \odot |y|$*. It can be observed that* $\boldsymbol{0}$ *is not an* ϵ*-efficient point of* $\boldsymbol{F}$ *and the gH-Dini Hadamard* ϵ*-subdifferentials of IVFs* $\boldsymbol{F}$ *and* $\boldsymbol{0}$ *at* $\bar{y} = 0$*, are given by*

$$\partial_{\epsilon'}^{\mathcal{DH}} \boldsymbol{F}(0) = \{\boldsymbol{S} : [1, 2] \ominus_{gH} \epsilon' \preceq \boldsymbol{S} \preceq [-2, -1] \oplus \epsilon'\} \text{ and} \tag{4.27}$$

$$\partial_{\epsilon}^{\mathcal{DH}} \boldsymbol{O}(0) = \{\boldsymbol{S} : -\epsilon \preceq \boldsymbol{S} \preceq \epsilon\}, \text{ respectively.} \tag{4.28}$$

Therefore, for $\epsilon = 1$, we have

$$\partial_{\epsilon'}^{\mathcal{DH}} \boldsymbol{F}(0) = \{\boldsymbol{S} : [-1, 0] \preceq \boldsymbol{S} \preceq [0, 1]\} \text{ and } \partial_{\epsilon}^{\mathcal{DH}} \boldsymbol{O}(0) = \{\boldsymbol{S} : [-1, -1] \preceq \boldsymbol{S} \preceq [1, 1]\}.$$

It can be observed that $[-1, -1] \in \partial_{\epsilon}^{\mathcal{DH}} \boldsymbol{O}(\bar{y})$ but $[-1, -1] \notin \partial_{\epsilon'}^{\mathcal{DH}} \boldsymbol{F}(\bar{y})$.
Thus, $\partial_{\epsilon}^{\mathcal{DH}} \boldsymbol{O}(\bar{y}) \not\subset \partial_{\epsilon'}^{\mathcal{DH}} \boldsymbol{F}(\bar{y})$.

Example 4.3 (An application example: sparsity regularizer for IOPs). *In many classification problems, the data set may not be precise and thus involves uncertainty. This may be due to errors in measurement, implementation, etc. We know that overfitting in a model is a common problem which one faces; to remove this, we induce sparsity in our model. Let us consider the following interval-valued regression problem:*

$$\min_{w \in \mathbb{R}^n} \tfrac{1}{2} \odot \|y - w\|_2^2 \odot \boldsymbol{P}, \tag{4.29}$$

where $y \in \mathbb{R}^n$ and $\boldsymbol{O} \prec \boldsymbol{P}$. Let us assume that $w^* = (w_1, w_2, \ldots, w_n)^\top$ be an efficient solution to the IOP (4.29), and our aim is to constrain the efficient solution w^* to be zero for some range of y. To achieve our aim, we consider the following approximated IOP:

$$\min_{w \in \mathbb{R}^n} (\boldsymbol{F}_1(w, y) \oplus \boldsymbol{F}_2(w, y)) = \min_{w \in \mathbb{R}^n} \left(\tfrac{1}{2} \odot \|y - w\|_2^2 \odot \boldsymbol{P} \oplus \lambda \odot \|w\|_1 \odot \boldsymbol{Q}\right),$$

where $\boldsymbol{F}_1(y, w) = \frac{1}{2} \odot \|y - w\|_2^2 \odot \boldsymbol{P}$, $\boldsymbol{F}_2(y, w) = \lambda \odot \|w\|_1 \odot \boldsymbol{Q}$, $\lambda > 0$, and $\boldsymbol{O} \preceq \boldsymbol{Q}$. From Theorem 4.12, we can see that if w^* is an ϵ-efficient solution of $(\boldsymbol{F}_1(w, b) \oplus \boldsymbol{F}_2(w, b))$, then relation (4.22) holds. In the below, we characterize w^* with the help of (4.22).

From Definition 1.26, we observe that $\boldsymbol{F}_1$ is gH-Fréchet differentiable at $w^* \in \mathbb{R}^n$, *and we have*

$$\begin{aligned}\boldsymbol{F}_{1\mathscr{F}}(w^*) = \nabla \boldsymbol{F}(w^*) =& (D_1\boldsymbol{F}(w^*), D_2\boldsymbol{F}(w^*), \ldots, D_n\boldsymbol{F}(w^*))^\top \\ =& ((w_1^* - y_1) \odot \boldsymbol{P}, (w_2^* - y_2) \odot \boldsymbol{P}, \ldots, (w_n^* - y_n) \odot \boldsymbol{P})^\top.\end{aligned}$$

Also, by using Example 4.1, the gH-Dini Hadamard ϵ*-subgradient of* $\boldsymbol{F}_2$ *at* w^* *is given by*

$$\partial_\epsilon^{\mathcal{DH}} \boldsymbol{F}_2(w^*) \in \begin{cases} \lambda \odot \boldsymbol{Q}, & \text{if } w_i^* > 0 \\ (-1) \odot \lambda \odot \boldsymbol{Q}, & \text{if } w_i^* < 0 \\ \boldsymbol{G}_i \in I(\mathbb{R}) : (-1) \odot \lambda \odot \boldsymbol{Q} \oplus (-\epsilon) \preceq \boldsymbol{G}_i \preceq \lambda \odot \boldsymbol{Q} \oplus \epsilon, & \text{if } w_i^* = 0. \end{cases}$$

Therefore, in view of Theorem 4.12, w^* *is an* ϵ*-efficient solution of* $(\boldsymbol{F}_1 \oplus \boldsymbol{F}_2)$ *if*

$$(w_i^* - y_i) \odot \boldsymbol{P} \in \begin{cases} \lambda \odot \boldsymbol{Q}, & \text{if } w_i^* > 0 \\ (-1) \odot \lambda \odot \boldsymbol{Q}, & \text{if } w_i^* < 0 \\ \boldsymbol{G}_i \in I(\mathbb{R}) : (-1) \odot \lambda \odot \boldsymbol{Q} \oplus (-\epsilon) \preceq \boldsymbol{G}_i \preceq \lambda \odot \boldsymbol{Q} \oplus \epsilon, & \text{if } w_i^* = 0. \end{cases}$$

In view of the above relation, we can observe that for $w^* = 0$*, we have*

$$(w_i^* - y_i) \odot \boldsymbol{P} \in \boldsymbol{G}_i \implies y_i^* \in [w_i^*, w_i^*] \ominus_{gH} (\boldsymbol{G}_i \oslash \boldsymbol{P}). \tag{4.30}$$

With the help of the above relation, we can obtain a range of y_i *for which* w^* *is zero, and this will help in achieving* w^* *to be an optimal solution to the problem.*

4.6 Conclusion

In this chapter, the concept of gH-Dini Hadamard ϵ-subdifferentiabilty of IVF (Definition 4.1) with its several characterizations have been studied. It has been observed that the gH-subdifferentiability implies the gH-Dini Hadamard ϵ-subdifferentiability (Theorem 4.1). However, the converse need not be true (Remark 4.1). Further, a relation of gH-Dini Hadamard ϵ-subdifferentiability with the Fréchet derivative of an IVF (Theorem 4.4) is discussed. Next, we have proposed the notion of $\mathbf{H}_\epsilon$-subgradient (Definition 4.4) of IVF with the help of sponge of a set. A variational interpretation of gH-Dini Hadamard ϵ-subdifferentiabilty of an IVF with sponges and gH-calm IVF (Theorem 4.7) has been given. To develop this relation, we have derived two important results (Theorem 4.6 and Lemma 4.3) based on sponges and gH-calm IVF. We have further defined the notion of ϵ-efficient solution of an IOP. Thereafter, we have discussed several necessary and sufficient conditions for an ϵ-efficient solution of an IOP (Theorems 4.8, 4.9, 4.10, and 4.11). Finally, an example to show the application of the proposed results in sparsity regularizer is given (Example 4.3).

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