Chapter 5

Computational Approach Based on Operational Matrix/Finite Difference for Multi-term Fractional Wave Model Arise in Dielectric Medium

5.1 Introduction

The history of fractional calculus is almost as old as the classical calculus. But, Leibniz was the first who talked about half order derivative with some remark and about its possibility [144]. The utility of half order derivative and integral leads to formulations of certain electrochemical problems which are more economical and useful in comparison to the classical calculus. The main advantage of fractional derivative in comparison to classical integer-order models is that it provides an excellent tool for the description of memory and hereditary properties of various materials and processes. However, over the most recent couple of decades, different authors reported that the derivative and integration of fractional order are suitable for the description of the properties of diverse real material [145]. Fundamental physical considerations are in favor of derivatives of non-integer order based models [146, 147]. Fractional order derivatives are more interesting because they do not have an obvious interpretation. Moreover, unlike the classical derivative, the fractional derivative has more than one definition. Out of these definitions most frequently used definitions are, the Riemann-Liouville and the Caputo derivative due to their background in the research [30, 61, 62, 148].

In recent years, researchers have given attention to the appearance and applications of FDEs in electromagnetic theory [149–151]. In [152, 153], authors show that the electromagnetic fields and waves in a vast class of dielectric medium can be represented by FDEs. Tarasov [154] have shown that, the electromagnetic field in a dielectric medium whose susceptibility follows a fractional power-law dependence can be described by FDEs with non-integer order time derivative.

In this chapter, let us consider the following electromagnetic wave equation ([154])

$$ {}_0D_t^{\alpha}u(t,x) - \lambda_1\ {}_0D_t^{\beta}u(t,x) - \lambda_2\nabla^2 u(t,x) = f(t,x), \tag{5.1} $$

with the initial conditions as

$$ \begin{cases} u(0,x) = \phi(x), \\ \left.\dfrac{\partial u(t,x)}{\partial t}\right|_{t=0} = \psi(x), \end{cases} \tag{5.2} $$

and the Dirichlet boundary conditions as

$$ u(t,a) = g_1(t), \quad u(t,b) = g_2(t), \tag{5.3} $$

where $u(t,x)$ denotes the electromagnetic field, $t \in (0,T]$, $x \in [a,b]$ and $1 < \beta < \alpha < 2$. The constants λ_1 and λ_2 depend on dielectric medium. Moreover, the fractional terms are defined in the Caputo-derivative sense with respect to t.

Due to the wide use of fractional calculus in the areas of science and engineering, it is necessary to develop the tools to solve the differential equations which involve fractional derivative. However, there are some analytic methods valid for solving some linear and special FDEs, e.g., the Laplace transform method, the Fourier transform method, the Green function method and the Mellin transform method [155]. But, there are a wide range of FDEs, on which these analytical methods do not work. In this progress, we need to develop some numerical techniques to solve FDEs. Some well-known methods have been applied by various authors. In [62, 156, 157], authors applied finite difference schemes to solve FDEs. In [158–160], the authors used finite element methods. Moreover, spectral methods, collocation methods, finite volume methods have also been applied to solve FDEs numerically [26–28, 30–32]. The operational matrix methods have also been used to solve FDEs numerically [56, 161, 162]. Moreover, Dehghan et al. [163] have used the homotopy method to solve FDEs. Multi fractional order terms have also been handled by various authors [164–166].

Some research articles have motivated us to construct a numerical method for problems (5.1)-(5.3), which are given as follows:

- H. Zhang et al. [167] have proposed a discrete implicit numerical scheme for time fractional Black-Scholes equation.
- S. Singh et al. [36] have established a numerical scheme based on operational matrices for nonlinear weakly singular partial integro-differenial equation.
- Z. Liu et al. [168] have proposed a finite difference scheme for solving time fractional diffusion-wave equation.
- F. Mirzaee et al. [169] have solved fractional stochastic advection-diffusion equation by developing a method based on a combination of the difference method and the meshless method.
- N. Samadyar et al. [170] have solved the two-dimensional weakly singular stochastic integral equation using radial basis functions.
- F. Mirzaee et al. [171] have developed a numerical scheme based on hybrid block-pulse and parabolic functions to solve the nonlinear stochastic integral equation.

Our objective is to design an efficient and stable method to solve the proposed problem (5.1)-(5.3). The main points of this chapter are as follows:

- The fractional order terms of order α and β are approximated by finite difference scheme of order $\mathcal{O}(\tau^{3-\alpha})$ and $\mathcal{O}(\tau^{3-\beta})$, $1<\beta<\alpha<2$, respectively.
- After using the approximation, the proposed problem (5.1)-(5.3) is converted into a system of second order ODEs with independent variable x and the operational matrix method is used to solve the system of ODEs.
- Convergences in the time and space directions are derived. It is proven that the convergence rates in the time and spatial direction are $\mathcal{O}\left(\tau^{min\{3-\alpha,3-\beta\}}\right)$ and $\mathcal{O}\left(\frac{1}{2^{N+1}(N+1)!}\right)$,

respectively.

- The stability of the semi-discrete scheme is also verified.

- Numerical experiments are performed to support the theoretical results.

Moreover, the emphasis is given to solve the system of ODEs with the existing scheme which is reliable and easy to implement. However, there are many software (Mathematica, Matlab) available those having built-in ODE solvers. Various authors have used in built ODE solver to solve the system of ODEs ([172–174]). But, one can not trust them blindly. In [175], an example has been quoted to show how a mathematician might misunderstand the numerical solution to be correct. In articles [36, 39, 171, 176], one can see that the methods based on operational matrices are efficient and convergent. Moreover, operational matrix is a technique that converts the differential equation, integral equation into a system of algebraic equations, which can be solved easily. This motivates for using the operational matrix methods to solve the system of ODEs, rather than the ODE solvers. Proceeding this, the operational matrix approach is applied to solve the system of ODEs.

5.2 Approximation for the fractional order terms

Let, M be a positive integer. Let us discretized co-ordinate t with M uniformly space grid points $t_k = t_{k-1} + \tau,\ t_0 = 0,\ t_M = 1,\ k = 1, 2, \cdots, M$. At point (t_k, x), the functional value and approximated value of $u(t, x)$ are denoted by $u^k = u(t_k, x)$ and $\bar{u}^k$, respectively. The fractional order term ${}_0D_t^{\alpha} u(t, x)$ is discretized as [168]

$$\begin{aligned} {}_0D_t^{\alpha} u(t_k, x) = &\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)} \left[\sum_{m=1}^{k-1} (\mathrm{G}_{k-m+1} - \mathrm{G}_{k-m}) \delta_t u^m + \mathrm{G}_1 \delta_t u^k - \mathrm{G}_k \psi(x) \right] \\ &+ \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)} \frac{\delta_t u^k - \delta_t u^{k-1}}{2^{2-\alpha}} + R_1 + \mathcal{O}(\tau^{3-\alpha}), \end{aligned} \tag{5.4}$$

where $\mathrm{G}_i = (i + \frac{1}{2})^{2-\alpha} - (i - \frac{1}{2})^{2-\alpha}$, $v(t, x) = \dfrac{\partial u(t, x)}{\partial t}$, $\delta_t u^k = \dfrac{u^k - u^{k-1}}{\tau}$ and
$\mathrm{R}_1 = \dfrac{1}{\Gamma(2-\alpha)} \sum\limits_{m=0}^{k-1} \int\limits_{t_{m-\frac{1}{2}}}^{t_{m+\frac{1}{2}}} \Big(v_t(t, x) - \dfrac{v^{m+\frac{1}{2}} - v^{m-\frac{1}{2}}}{\tau} (t_k - t^{1-\alpha}) \Big) dt$, $|\mathrm{R}_1| = \mathcal{O}(\tau^{3-\alpha})$ (see [168]).

Similarly,

$$
\begin{aligned}
{}_0D_t^{\beta}u(t_k,x) = & \frac{\tau^{1-\beta}}{\Gamma(3-\beta)}\left[\sum_{m=1}^{k-1}(\mathrm{H}_{k-m+1}-\mathrm{H}_{k-m})\delta_t u^m + \mathrm{H}_1\delta_t u^k - \mathrm{H}_k\psi(x)\right] \\
& + \frac{\tau^{1-\beta}}{\Gamma(3-\beta)}\frac{\delta_t u^k - \delta_t u^{k-1}}{2^{2-\beta}} + \mathrm{R}_2 + \mathcal{O}(\tau^{3-\beta}),
\end{aligned} \tag{5.5}
$$

where $\mathrm{H}_i = (i+\frac{1}{2})^{2-\beta} - (i-\frac{1}{2})^{2-\beta}$, $\mathrm{R}_2 = \frac{1}{\Gamma(2-\beta)}\sum_{m=0}^{k-1}\int_{t_{m-\frac{1}{2}}}^{t_{m+\frac{1}{2}}}\Big(v_t(t,x) - \frac{v^{m+\frac{1}{2}} - v^{m-\frac{1}{2}}}{\tau}(t_k - t^{1-\beta})\Big)dt$, and $|\mathrm{R}_2| = \mathcal{O}(\tau^{3-\beta})$.

Substituting the approximated value of fractional order derivative from equations (5.4) and (5.5) into equation (5.1), we get

$$
\begin{aligned}
\lambda_2\frac{d^2u^k}{dx^2} = & \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\left[\sum_{m=1}^{k-1}(\mathrm{G}_{k-m+1}-\mathrm{G}_{k-m})\delta_t u^m + \mathrm{G}_1\delta_t u^k - \mathrm{G}_k\psi(x)\right] \\
& + \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\frac{\delta_t u^k - \delta_t u^{k-1}}{2^{2-\alpha}} \\
& - \lambda_1\frac{\tau^{1-\beta}}{\Gamma(3-\beta)}\left[\sum_{m=1}^{k-1}(\mathrm{H}_{k-m+1}-\mathrm{H}_{k-m})\delta_t u^m + \mathrm{H}_1\delta_t u^k - \mathrm{H}_k\psi(x)\right] \\
& - \lambda_1\frac{\tau^{1-\beta}}{\Gamma(3-\beta)}\frac{\delta_t u^k - \delta_t u^{k-1}}{2^{2-\beta}} \\
& - f(t_k,x) + \mathrm{R}_1 + \mathrm{R}_2 + \mathcal{O}(\tau^{3-\alpha} + \tau^{3-\beta}).
\end{aligned} \tag{5.6}
$$

Omitting the small term $\mathrm{R}_1 + \mathrm{R}_2 + \mathcal{O}(\tau^{3-\alpha} + \tau^{3-\beta})$, we get the following system of ordinary differential equations (ODEs) with independent variable x as

$$
\begin{aligned}
\lambda_2\frac{d^2\bar{u}^k}{dx^2} = & \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\left[\sum_{m=1}^{k-1}(\mathrm{G}_{k-m+1}-\mathrm{G}_{k-m})\delta_t \bar{u}^m + \mathrm{G}_1\delta_t \bar{u}^k - \mathrm{G}_k\psi(x)\right] \\
& + \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\frac{\delta_t \bar{u}^k - \delta_t \bar{u}^{k-1}}{2^{2-\alpha}} \\
& - \lambda_1\frac{\tau^{1-\beta}}{\Gamma(3-\beta)}\left[\sum_{m=1}^{k-1}(\mathrm{H}_{k-m+1}-\mathrm{H}_{k-m})\delta_t \bar{u}^m + \mathrm{H}_1\delta_t \bar{u}^k - \mathrm{H}_k\psi(x)\right] \\
& - \lambda_1\frac{\tau^{1-\beta}}{\Gamma(3-\beta)}\frac{\delta_t \bar{u}^k - \delta_t \bar{u}^{k-1}}{2^{2-\beta}} - f(t_k,x).
\end{aligned} \tag{5.7}
$$

The above equation can be written in more compact form as

$$\begin{aligned}
\lambda_2 \frac{d^2\bar{u}^k}{dx^2} &= a_1 \left[\sum_{m=1}^{k-1} (\mathrm{G}_{k-m+1} - \mathrm{G}_{k-m}) \delta_t \bar{u}^m + \mathrm{G}_1 \delta_t \bar{u}^k - \mathrm{G}_k \psi(x) \right] \\
&+ a_2 \left(\bar{u}^k - 2\bar{u}^{k-1} + \bar{u}^{k-2} \right) \\
&- b_1 \left[\sum_{m=1}^{k-1} (\mathrm{H}_{k-m+1} - \mathrm{H}_{k-m}) \delta_t \bar{u}^m + \mathrm{H}_1 \delta_t \bar{u}^k - \mathrm{H}_k \psi(x) \right] \\
&- b_2 \left(\bar{u}^k - 2\bar{u}^{k-1} + \bar{u}^{k-2} \right) - f(t_k, x),
\end{aligned} \tag{5.8}$$

where $a_1 = \dfrac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}$, $b_1 = \lambda_1 \dfrac{\tau^{1-\beta}}{\Gamma(3-\beta)}$, $a_2 = \dfrac{a_1}{\tau.2^{2-\alpha}}$ and $b_2 = \lambda_1 \dfrac{b_1}{\tau.2^{2-\beta}}$,
with boundary conditions given by

$$\bar{u}^k(a) = g_1(t_k), \ \bar{u}^k(b) = g_2(t_k). \tag{5.9}$$

The equation (5.8) is the system of second order ODEs with the boundary conditions given in equation (5.9). Our next task is to solve this system of ODEs for unknowns $\bar{u}^1(x), \bar{u}^2(x), ..., \bar{u}^M(x)$.

5.3 Basis functions and operational matrices

If P_{N+1} is the Legendre polynomial of degree $(N+1)$ with roots x_k for $k = 0, 1, 2, ..., N$, then the Lagrange polynomials at distinct node points $x_0, x_1, ..., x_N$ are defined as [97]

$$l_j(x) = \prod_{\substack{i=0 \\ i \neq j}}^{N} \frac{(x - x_i)}{(x_j - x_i)}, \tag{5.10}$$

with Kronecker property $l_i(x_j) = \delta_{ij}$. The interpolating scaling functions (ISF) defined in [98], are given below:

$$\phi_k(x) = \begin{cases} \sqrt{\dfrac{2}{w_k}} l_k(2x-1), & x \in [0, 1], \\ 0, & otherwise, \end{cases} \tag{5.11}$$

where $w_k = \dfrac{1}{(N+1)P'_{N+1}(x_k)P_N(x_k)}$, $k = 0, 1, ..., N$ are the Gauss-Legendre quadrature weights and $\Phi(x) = [\phi_0(x), \phi_1(x), \phi_2(x), ..., \phi_N(x)]^T$. Therefore, any function $f(x) \in L^2[0, 1]$

can be expanded in terms of basis functions ISF as

$$f(x) = \sum_{i=0}^{\infty} c_i \phi_i(x),$$

or

$$f(x) \simeq \sum_{i=0}^{N} c_i \phi_i(x),$$

where the unknown coefficients are given by

$$c_i = \sqrt{\frac{w_i}{2}} f(\hat{x}_i), \qquad i = 0, 1, ..., N,$$

and

$$\hat{x}_i = \frac{x_i + 1}{2}.$$

Let, $\Phi(x) = [\phi_0(x), \phi_1(x), \cdots, \phi_N(x)]$ be the basis of $L^2[0,1]$, then

$$\int_0^{x'} \begin{bmatrix} \phi_0(x') \\ \phi_1(x') \\ \vdots \\ \phi_N(x') \end{bmatrix} dx' \approx \begin{bmatrix} i_{00} & i_{01} & i_{02} & \cdots & i_{0N} \\ i_{10} & i_{11} & i_{12} & \cdots & i_{1N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ i_{N0} & i_{N1} & i_{N2} & \cdots & i_{NN} \end{bmatrix} \begin{bmatrix} \phi_0(x) \\ \phi_1(x) \\ \vdots \\ \phi_N(x) \end{bmatrix} = P\Phi^T(x),$$

and

$$\frac{x}{b}\int_a^b\int_a^{x'} \Phi(t)dtdx' = P\frac{x}{b}\int_a^b \Phi(x)dx = \frac{x}{b}P\int_a^b \Phi(x)dx = P\Big(\frac{x}{b}\int_a^b \Phi(x)dx\Big) = PQ\Phi(x),$$

where the matrix Q is calculated in the following manner

$$\frac{x}{b}\int_a^b \Phi(x)dx = \frac{x}{b}\int_a^b \begin{bmatrix} \phi_0(x) \\ \phi_1(x) \\ \vdots \\ \phi_N(x) \end{bmatrix} dx = Q\Phi(x).$$

In the above discussion, P is the matrix of integration and the coefficients of matrix Q are calculated as

$$Q_{ij} = \int_0^1 \Big(\frac{x}{b}\int_a^b \phi_i(x)dx\Big)\phi_j(x)dx.$$

Remark 5.3.1. It is to be noted that the domain is taken as $[0, 1]$, so $a = 0$ and $b = 1$. Thus the above formula takes the following form

$$Q_{ij} = \int_0^1 \Big(x\int_0^1 \phi_i(x)dx\Big)\phi_j(x)dx.$$

5.4 Method of solution for system of differential equations

In this section, the operational matrix based method is discussed for solving the system of ODEs. In order to solve the system of ODEs (equation (5.8)), we approximate $\frac{d^2\bar{u}(x)}{dx^2}$ at k^th level of time t in terms of basis functions as

$$\frac{d^2\bar{u}^k(x)}{dx^2} = \sum_{i=0}^{\infty} a_i\phi_i(x) \approx \sum_{i=0}^{N} a_i\phi_i(x) = A^T\Phi(x). \tag{5.12}$$

Integrating above equation from a to x, we get

$$\frac{d\bar{u}^k(x)}{dx} - \frac{d\bar{u}^k(a)}{dx} \approx A_k^T \int_a^x \Phi(t)dt. \tag{5.13}$$

Again integrating from a to x, we obtain

$$\bar{u}^k(x) - \bar{u}^k(a) - x\frac{d\bar{u}^k(a)}{dx} \approx A_k^T \int_a^x\int_a^{x'} \Phi(t)dtdx'. \tag{5.14}$$

Now using boundary condition $\bar{u}^k(a) = g_1(t_k)$ from equation (5.9), one can obtain

$$\bar{u}^k(x) \approx g_1(t_k) + x\frac{d\bar{u}^k(a)}{dx} + A_k^T \int_a^x\int_a^{x'} \Phi(t)dtdx'. \tag{5.15}$$

In order to use second boundary condition $\bar{u}^k(b) = g_2(t_k)$, putting $x = b$ in equation (5.15), we get

$$\bar{u}^k(b) \approx b\frac{d\bar{u}^k(a)}{dx} + g_1(t_k) + A_k^T \int_a^b \int_a^{x'} \Phi(t)dtdx', \tag{5.16}$$

or

$$\frac{d\bar{u}^k(a)}{dx} \approx \frac{1}{b}\Big(\bar{u}^k(b) - g_1(t_k) - A_k^T \int_a^b \int_a^{x'} \Phi(t)dtdx'\Big). \tag{5.17}$$

Substituting the value of $\dfrac{\partial \bar{u}^k(a)}{\partial x}$ from equation (5.17) into equation (5.15), the following relation is obtained as

$$\bar{u}^k(x) \approx g_1(t_k)(1 - \frac{x}{b}) + \frac{x}{b}g_2(t_k) - \frac{x}{b}A_k^T \int_a^b \int_a^{x'} \Phi(t)dtdx' + A_k^T \int_a^x \int_a^{x'} \Phi(t)dtdx'. \tag{5.18}$$

Now, approximating the integration terms with the help of operational matrices as

$$\begin{cases} \int_a^x \int_a^{x'} \Phi(t)dtdx' \approx P_1\Phi(x), \\ \frac{x}{b}\int_a^b \int_a^{x'} \Phi(t)dtdx' \approx P_2\Phi(x), \end{cases} \tag{5.19}$$

in which $P_1 = P^2$ and $P_2 = PQ$, where the matrices P and Q are defined in section 3.

Also, $g_1(t_k)(1 - \frac{x}{b}) + \frac{x}{b}g_2(t_k)$ is a function of x, hence it can be written as the linear combination of basis functions as

$$g_1(t_k)(1 - \frac{x}{b}) + \frac{x}{b}g_2(t_k) \approx B_k^T\Phi(x), \tag{5.20}$$

where B_k^T is a $1 \times (N+1)$ vector.

Using equations (5.19) and (5.20) into equation (5.18), we get the solution at the k^th level of t, which is given as

$$\bar{u}^k(x) \approx B_k^T\Phi(x) + A_k^T(P_1 - P_2)\Phi(x), \qquad \forall\ k = 1, 2, \cdots, M. \tag{5.21}$$

Now, in order to convert the system of ODEs into system of algebraic equations, the approximations of $\bar{u}^k(x)$ and its derivative $\dfrac{\partial \bar{u}^k(x)}{\partial x^2}$ from equations (5.21) and (5.12) are used in equation (5.8) for each k.

For $k=1$, the equation (5.8) gives

$$(a_1\mathrm{G}_1 - b_1\mathrm{H}_1)\Big(\frac{\bar{u}^1-\bar{u}^0}{\tau}\Big) - (a_1\mathrm{G}_1 - b_1\mathrm{H}_1)\psi(x) + (a_2-b_2)(\bar{u}^1 - 2\bar{u}^0 + \bar{u}^{-1}) \\ = \lambda_2\frac{d^2\bar{u}^1}{dx^2} + f(t_1,x). \tag{5.22}$$

To remove the dummy point $\bar{u}^{-1}$, we use the initial condition $\left.\dfrac{\partial u(t,x)}{\partial t}\right|_{t=0} = \psi(x)$. Therefore,

$$\frac{\partial\bar{u}^0}{\partial t} = \frac{\bar{u}^0-\bar{u}^{-1}}{\tau} \Rightarrow \bar{u}^{-1} = \phi(x) - \tau\psi(x). \tag{5.23}$$

Using equation (5.23) in equation (5.22), we get

$$c_1\bar{u}^1 - \lambda_2\frac{d^2\bar{u}^1}{dx^2} = c_2\phi(x) + (a_1\mathrm{G}_1 - b_1\mathrm{H}_1)\,\psi(x) + (a_2-b_2)\,(\phi(x)+\tau\psi(x)) + f(t_k,x), \tag{5.24}$$

where $c_1 = \left(\frac{a_1\mathrm{G}_1-b_1\mathrm{H}_1}{\tau} + (a_2-b_2)\right)$ and $c_2 = \left(\frac{a_1\mathrm{G}_1-b_1\mathrm{H}_1}{\tau}\right)$.
In general, we can write equation (5.8) for $k \geq 2$ as

$$c_1\bar{u}^k - \lambda_2\frac{d^2\bar{u}^k}{dx^2} = f(t_k,x) + (a_1\mathrm{G}_k - b_1\mathrm{H}_k)\psi(x) - a_1\sum_{m=1}^{k-1}(\mathrm{G}_{k-m+1}-\mathrm{G}_{k-m})\frac{(\bar{u}^m-\bar{u}^{m-1})}{\tau} \\ + b_1\sum_{m=1}^{k-1}(\mathrm{H}_{k-m+1}-\mathrm{H}_{k-m})\frac{(\bar{u}^m-\bar{u}^{m-1})}{\tau} + \frac{a_1\mathrm{G}_1-b_1\mathrm{H}_1}{\tau}\bar{u}^{k-1} \\ - (a_2-b_2)(-2\bar{u}^{k-1}+\bar{u}^{k-2}). \tag{5.25}$$

It is apparent that the right hand side of equation (5.24) is a function of x alone, say $r(x)$, so it can be approximated in terms of basis functions as

$$r(x) \approx R^T\Phi(x), \tag{5.26}$$

and

$$f(t_k,x) \approx F_k^T\Phi(x), \\ \psi(x) \approx S^T\Phi(x), \tag{5.27}$$

where $R,\ S$ and F_k are $(N+1)\times 1$ vectors.

After using the approximations of $\bar{u}^k(x)$, $\dfrac{d^2\bar{u}^k(x)}{dx^2}$ and $r(x)$ from equations (5.12), (5.21) and equation (5.26), respectively into the equation (5.24), we obtain

$$c_1\left[B_1^T\Phi(x)+A_1^T(P_1-P_2)\Phi(x)\right]-\lambda_2A_1^T\Phi(x)=R^T\Phi(x), \tag{5.28}$$

or

$$A_1^T\left[c_1(P_1-P_2)-\lambda_2I\right]=\left[R^T-c_1B_1^T\right], \tag{5.29}$$

where I is the identity matrix.

Now, using the approximations for $\bar{u}(x)$ and its second order derivative $\dfrac{d^2\bar{u}(x)}{dx^2}$ at each level which is present in equation (5.25), we get

$$\begin{aligned}
&c_1\left[B_k^T\Phi(x)+A_k^T(P_1-P_2)\Phi(x)\right]-\lambda_2A_k^T\Phi(x)=F_k\Phi(x)+(a_1\mathrm{G}_k-b_1\mathrm{H}_k)S^T\Phi(x)\\
&-a_1\sum_{m=1}^{k-1}(\mathrm{G}_{k-m+1}-\mathrm{G}_{k-m})\left(\frac{\left[B_m^T\Phi(x)+A_m^T(P_1-P_2)\Phi(x)\right]-\left[B_{m-1}^T\Phi(x)+A_{m-1}^T(P_1-P_2)\Phi(x)\right]}{\tau}\right)\\
&+b_1\sum_{m=1}^{k-1}(\mathrm{H}_{k-m+1}-\mathrm{H}_{k-m})\left(\frac{\left[B_m^T\Phi(x)+A_m^T(P_1-P_2)\Phi(x)\right]-\left[B_{m-1}^T\Phi(x)+A_{m-1}^T(P_1-P_2)\Phi(x)\right]}{\tau}\right)\\
&\qquad+(a_2-b_2)\left(-2\left[B_{k-1}^T\Phi(x)+A_{k-1}^T(P_1-P_2)\Phi(x)\right]+\left[B_{k-2}^T\Phi(x)+A_{k-2}^T(P_1-P_2)\Phi(x)\right]\right)\\
&\qquad+c_2\left(\frac{\left[B_{k-1}^T\Phi(x)+A_{k-1}^T(P_1-P_2)\Phi(x)\right]}{\tau}\right),
\end{aligned} \tag{5.30}$$

or

$$\begin{aligned}
&A_k^T\left[c_1\tilde{P}-\lambda_2I\right]\Phi(x)=\left[F_k^T+(a_1\mathrm{G}_k-b_1\mathrm{H}_k)S^T-c_1B_k^T\right]\Phi(x)\\
&-\left(a_1\sum_{m=1}^{k-1}(\mathrm{G}_{k-m+1}-\mathrm{G}_{k-m})-b_1\sum_{m=1}^{k-1}(\mathrm{H}_{k-m+1}-\mathrm{H}_{k-m})\right)\left(\frac{\left[B_m^T+A_m^T\tilde{P}\right]-\left[B_{m-1}^T+A_{m-1}^T\tilde{P}\right]\Phi(x)}{\tau}\right)\\
&\quad+(c_2-2\tau(a_2-b_2))\left[B_{k-1}^T+A_{k-1}^T\tilde{P}\right]\Phi(x)+(a_2-b_2)\left[B_{k-2}^T+A_{k-2}^T\tilde{P}\right]\Phi(x),
\end{aligned} \tag{5.31}$$

or

$$
\begin{aligned}
A_k^T\left[c_1\tilde{P}-\lambda_2 I\right] &= \left[F_k^T+(a_1\mathrm{G}_k-b_1\mathrm{H}_k)S^T-c_1B_k^T\right]\\
&-\left(a_1\sum_{m=1}^{k-1}(\mathrm{G}_{k-m+1}-\mathrm{G}_{k-m})-b_1\sum_{m=1}^{k-1}(\mathrm{H}_{k-m+1}-\mathrm{H}_{k-m})\right)\left(\frac{\left[B_m^T+A_m^T\tilde{P}\right]-\left[B_{m-1}^T+A_{m-1}^T\tilde{P}\right]}{\tau}\right)\\
&+(c_2-2\tau(a_2-b_2))\left[B_{k-1}^T+A_{k-1}^T\tilde{P}\right]+(a_2-b_2)\left[B_{k-2}^T+A_{k-2}^T\tilde{P}\right].
\end{aligned}
\tag{5.32}
$$

One can see that the equations (5.29) and (5.32) are the system of algebraic equations with A_1^T and A_k^T as unknown vectors, respectively. After solving the system of equations at k^th level, we get the unknown vector A_k^T. Using the value A_k^T into equation (5.21), we get the numerical solution at $k-th$ level of t. On combining the equations (5.29) and (5.32), we can write

$$
A_k^T=\begin{cases}
\left[R_1^T-c_1B_1^T\right]\tilde{Z}^{-1}, & if\ k=1,\\
\\
[F_k^T+(a_1\mathrm{G}_k-b_1\mathrm{H}_k)S-c_1B_k^T & \\
-(a_1\sum\limits_{m=1}^{k-1}\tilde{g}_m-b_1\sum\limits_{m=1}^{k-1}\tilde{h}_m)\left(\frac{\mathcal{Y}_m^T-\mathcal{Y}_{m-1}^T}{\tau}\right) & \\
+\tilde{\gamma}\mathcal{Y}_{k-1}^T+(a_2-b_2)\mathcal{Y}_{k-2}^T]\tilde{Z}^{-1}, & if\ k\geq 2,
\end{cases}
\tag{5.33}
$$

where

$$
\begin{aligned}
&\mathcal{Y}_k^T=\left[B_k^T+A_k^T\tilde{P}\right],\tilde{\gamma}=\Big(c_2-2\tau(a_2-b_2)\Big),\\
&\tilde{h}_m=\Big(\mathrm{H}_{k-m+1}-\mathrm{H}_{k-m}\Big),\tilde{g}_m=\Big(\mathrm{G}_{k-m+1}-\mathrm{G}_{k-m}\Big),\\
&\tilde{P}=P_1-P_2,\ \tilde{Z}=c_1\tilde{P}-\lambda_2 I,
\end{aligned}
$$

in which $\mathcal{Y}_k^T$ is $1\times(N+1)$ vector and $\tilde{Z}$ is of order $(N+1)\times(N+1)$ matrix.

5.5 *Convergence analysis in the spatial direction*

Lemma 5.5.1. Let $L^2[0,1]$ be the Hilbert space and $S_N=\sum\limits_{i=0}^{N}a_i\phi_i(x)$ be the N^th partial sum sequence of $u_{xx}^k=\sum\limits_{i=0}^{\infty}a_i\phi_i(x)$, then as $N\to\infty$, S_N and u_{xx}^k converges to the same limit, where

$$a_i = \left\langle \frac{d^2u^k(x)}{dx^2}, \phi_i(x) \right\rangle = \int_0^1 \frac{d^2u^k(x)}{dx^2} \phi_i(x) dx.$$

Proof. Consider the series representation of $\frac{d^2u(x)}{dx^2}$ at k^{th} level i.e., $\frac{d^2u^k(x)}{dx^2} = \sum_{i=0}^{\infty} a_i\phi_i(x)$, where

$$a_i = \left\langle \frac{d^2u^k(x)}{dx^2}, \phi_i(x) \right\rangle = \int_0^1 \frac{d^2u^k(x)}{dx^2} \phi_i(x) dx,$$

and the norm is defined as $\|.\|_2^2 = \int_0^1 (.)^2 dx$.

Let $S_N = \sum_{i=0}^{N} a_i\phi_i(x)$ be the N^{th} partial sum sequence of u_{xx}^k. To show that the S_N is a Cauchy sequence, let $N_1 < N$, then

$$\|S_N - S_{N_1}\|^2 = \|\sum_{i=N_1+1}^{N} a_i\phi_i(x)\|^2.$$

Using the definition of inner product, we get

$$\begin{aligned} \|S_N - S_{N_1}\|^2 = & \int_0^1 [\sum_{i=N_1+1}^{N} a_i\phi_i(x)]^2 dx \\ = & \int_0^1 \left(\sum_{i=N_1+1}^{N} a_i\phi_i(x)\right)\left(\sum_{j=N_1+1}^{N} a_j\phi_j(x)\right) dx \\ = & \sum_{i=N_1+1}^{N} \sum_{j=N_1+1}^{N} a_i a_j \int_0^1 \phi_i(x)\phi_j(x) dx. \end{aligned} \tag{5.34}$$

Using the orthogonality condition, we obtain from (5.34)

$$\|S_N - S_{N_1}\|^2 = \sum_{i=N_1+1}^{N} a_i^2. \tag{5.35}$$

By making use of Bessel's inequality, series $\sum_{i=N_1+1}^{N} a_i^2$ is convergent as $N \to \infty$. Hence S_N

is a Cauchy sequence, so convergent. Now, let $S_N \to \bar{u}^k_{xx}$, then

$$
\begin{aligned}
\langle \bar{u}^k_{xx} - u^k_{xx}, \phi_i(x)\rangle &= \int_0^1 \bar{u}^k_{xx}\phi_i(x)dx - \int_0^1 u^k_{xx}\phi_i(x)dx \\
&= \int_0^1 \lim_{N\to\infty} S_N\phi_i(x)dx - \int_0^1 u^k_{xx}\phi_i(x)dx \\
&= \int_0^1 \lim_{N\to\infty} S_N\phi_i(x)dx - a_i \\
&= \lim_{N\to\infty} \int_0^1 S_N\phi_i(x)dx - a_i \\
&= \lim_{N\to\infty} \int_0^1 \Big(\sum_{i=1}^{N} a_i\phi_i(x)\Big)\phi_i(x)dx - a_i \\
&= a_i - a_i = 0, \quad \forall i.
\end{aligned}
$$

From the above, we can conclude that $\langle\, \bar{u}^k_{xx} - u^k_{xx}, \phi_i(x)\rangle = 0,\ \forall i$, which is possible only when $\bar{u}^k_{xx} = u^k_{xx}$. Hence $\bar{u}^k_{xx}$ and u^k_{xx} converge to the same value.

Lemma 5.5.2. *[177] Suppose $x_0, x_1, ..., x_N$ are $(N+1)$ distinct roots of Legendre's polynomial, where N be a non-negative integer. Also, assume that the smooth function F(x) is approximated by ISF, then*

$$F(x) - F_N(x) = \tfrac{F^{N+1}(\mu_x)}{2^{N+1}(N+1)!}[(2x-1) - x_0] + ... + [(2x-1) - x_N], \quad \mu_x \in (x_0, x_N), \tag{5.36}$$

where $F_N(x)$ is the numerical approximation of $F(x)$ by ISF. So,

$$|F(x) - F_N(x)| \leq \tfrac{K_1 K_2}{2^{N+1}(N+1)!}, \tag{5.37}$$

where $K_1 = \max_{0\leq\mu_x\leq 1} |F^{N+1}(\mu_x)|, \quad K_2 = \max_{0\leq x\leq 1} |[(2x-1) - x_0] + ... + [(2x-1) - x_N]|.$

Theorem 5.5.1. Let $\{\bar{u}^k_N\}_{k=1}^{M}$ is the solution of equation (5.7) in terms of ISF, then

$$\|u^k - \bar{u}^k_N\| \approx \mathcal{O}\bigg(\frac{1}{2^{N+1}(N+1)!}\bigg),$$

where $\bar{u}^k_N(x) \approx B_k^T\Phi(x) + A_k^T(P_1 - P_2)\Phi(x)$.

Proof. After applying Lemma 5.5.2 for the function $\dfrac{d^2u^k}{dx^2}$, we get

$$u_{xx}^k - A_k^T\Phi(x) = \frac{u_{xx}^k(\mu_x)}{2^{N+1}(N+1)!}[(2x-1)-x_0] + ... + [(2x-1)-x_N], \quad \mu_x \in (x_0, x_N). \tag{5.38}$$

Integrating both sides of the above equation from a to x, we get

$$u_x^k(x) - u_x(a) - A_k^T\int_0^x \Phi(x)dx = \int_0^x \frac{u_{xx}^k(\mu_x)}{2^{N+1}(N+1)!}[(2x-1)-x_0] + ... + [(2x-1)-x_N]dx. \tag{5.39}$$

Again integrating above equation from a to x, we get

$$u^k(x) - g_1(t_k) - xu_x(a) - A_k^T\int_0^x\int_0^{x'} \Phi(t)dtdx' = \frac{u_{xx}^k(\mu_x)}{2^{N+1}(N+1)!}\mathcal{F}(x). \tag{5.40}$$

In order to use the second boundary condition $u^k(b) = g_2(t_k)$, putting $x = b$ in equation (5.40), we get

$$g_2(t_k) - g_1(t_k) - bu_x(a) - A_k^T\int_0^b\int_0^{x'} \Phi(t)dtdx' = \frac{u_{xx}^k(\mu_x)}{2^{N+1}(N+1)!}\mathcal{F}(b), \tag{5.41}$$

$$\text{or,} \quad u_x(a) = \tfrac{1}{b}\left(g_2(t_k) - g_1(t_k) - A_k^T\int_0^b\int_0^{x'} \Phi(t)dtdx' - \frac{u_{xx}^k(\mu_x)}{2^{N+1}(N+1)!}\mathcal{F}(b)\right). \tag{5.42}$$

Now, putting the value of $u_x(a)$ from equation (5.42) into equation (5.40), we get

$$\begin{aligned} u^k(x) - g_1(t_k) - \tfrac{x}{b}\left(g_2(t_k) - g_1(t_k) - A_k^T\int_0^b\int_0^{x'} \Phi(t)dtdx' - \tfrac{u_{xx}^k(\mu_x)}{2^{N+1}(N+1)!}\mathcal{F}(b)\right) \\ -A_k^T\int_0^x\int_0^{x'} \Phi(t)dtdx' = \tfrac{u_{xx}^k(\mu_x)}{2^{N+1}(N+1)!}\mathcal{F}(x). \end{aligned} \tag{5.43}$$

Rearranging the terms of above equation, we get

$$\begin{aligned} u^k(x) - \left(\frac{x}{b} - 1\right) g_1(t_k) - \frac{x}{b}g_2(t_k) + \frac{x}{b}A_k^T\int_0^b\int_0^{x'} \Phi(t)dtdx' - A_k^T\int_0^x\int_0^{x'} \Phi(t)dtdx' \\ = \frac{u_{xx}^k(\mu_x)}{2^{N+1}(N+1)!}\left(\mathcal{F}(x) - \frac{x}{b}\mathcal{F}(b)\right). \end{aligned} \tag{5.44}$$

Now using equations (5.19) and (5.20), the above equation reduces to the following relation

$$u^k(x) - \left[B_k^T\Phi(x) + A_k^T(P_1 - P_2)\Phi(x)\right] = \frac{u_{xx}^k(\mu_x)}{2^{N+1}(N+1)!}\left(\mathcal{F}(x) - \tfrac{x}{b}\mathcal{F}(b)\right). \tag{5.45}$$

Also, we know from equation (5.21) that, $B_k^T\Phi(x) + A_k^T(P_1 - P_2)\Phi(x)$ is the ISF approximation of $u^k(x)$, denoting this by $\bar{u}_N^k(x)$, we get

$$u^k(x) - \bar{u}_N^k(x) = \frac{u_{xx}^k(\mu_x)}{2^{N+1}(N+1)!}\left(\mathcal{F}(x) - \frac{x}{b}\mathcal{F}(b)\right) \leq \frac{\mathcal{L}_1\mathcal{L}_2}{2^{N+1}(N+1)!}. \tag{5.46}$$

From above equation we can conclude that

$$\|u^k(x) - \bar{u}_N^k(x)\| \approx \mathcal{O}\left(\frac{1}{2^{N+1}(N+1)!}\right), \tag{5.47}$$

where $\mathcal{L}_1 = \max\limits_{a\leq\mu_x\leq b} u_{xx}^k(\mu_x)$, $\mathcal{L}_2 = \max\limits_{a\leq\mu_x\leq b}\left(\mathcal{F}(x) - \frac{x}{b}\mathcal{F}(b)\right)$.

Remark 5.5.1. Stability and convergence analysis in time direction are done by taking λ_1 negative and λ_2 positive (for more details see [154]).

5.6 Stability analysis

Lemma 5.6.1. *[168]* For the definition $\mathrm{G}_i = \left(i+\frac{1}{2}\right)^{2-\alpha} - \left(i-\frac{1}{2}\right)^{2-\alpha}$, and $\mathrm{H}_i = \left(i+\frac{1}{2}\right)^{2-\beta} - \left(i-\frac{1}{2}\right)^{2-\beta}$, $i = 1,\ldots,\mathrm{M}-1$, we have $\mathrm{G}_i > 0$, $\mathrm{H}_i > 0$ and $\mathrm{G}_{i+1} \leq \mathrm{G}_i$, $\mathrm{H}_{i+1} \leq \mathrm{H}_i$.

Theorem 5.6.1. The semi-discrete form of equation (5.1) is unconditionally stable with the inequality

$$\|\bar{u}^n\|^2 \leq 2\|\bar{u}^0\|^2 + C_1\left(O(\bar{u}^0) + \mathrm{T}\mathcal{M} + \sum_{m=1}^{n}\left(\tfrac{a_1\mathrm{G}_m\tau + b_1\mathrm{H}_m\tau}{2}\right)\|\psi(x)\|^2 + \tfrac{\tau}{2\epsilon}\sum_{m=1}^{n}\|f_m\|^2\right),$$

provided that $\dfrac{d^2\bar{u}^k}{dx^2}$ is bounded by positive quantity $\mathcal{M}$, for $k = 0, 1, ..., M$.

Proof. Equation (5.8) can be written in the following form

$$\begin{aligned}
&\tfrac{a_1}{\lambda_2}\left[\sum_{m=1}^{k-1}(\mathrm{G}_{k-m+1} - \mathrm{G}_{k-m})\delta_t\bar{u}^m + \mathrm{G}_1\delta_t\bar{u}^k - \mathrm{G}_k\psi(x)\right] + \tfrac{a_2}{\lambda_2}\left(\delta_t\bar{u}^k - \delta_t\bar{u}^{k-1}\right)\\
&-\tfrac{b_1}{\lambda_2}\left[\sum_{m=1}^{k-1}(\mathrm{H}_{k-m+1} - \mathrm{H}_{k-m})\delta_t\bar{u}^m + \mathrm{H}_1\delta_t\bar{u}^k - \mathrm{H}_k\psi(x)\right] - \tfrac{b_2}{\lambda_2}\left(\delta_t\bar{u}^k - \delta_t\bar{u}^{k-1}\right)\\
&\qquad\qquad\qquad\qquad = \frac{d^2\bar{u}^k}{dx^2} + \tfrac{f(t_k,x)}{\lambda_2},
\end{aligned} \tag{5.48}$$

where $a_1 = \dfrac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}$, $a_2 = \dfrac{a_1}{2^{2-\alpha}}$, $b_1 = \lambda_1 \dfrac{\tau^{1-\beta}}{\Gamma(3-\beta)}$ and $b_2 = \dfrac{b_1}{2^{2-\beta}}$.

Since b_1 and b_2 are negative quantities (as λ_1 is negative quantity). So, we can write the above equation in the following form

$$\begin{aligned}
&\tfrac{a_1}{\lambda_2}\left[\sum_{m=1}^{k-1}(\mathrm{G}_{k-m+1}-\mathrm{G}_{k-m})\delta_t\bar{u}^m + \mathrm{G}_1\delta_t\bar{u}^k - \mathrm{G}_k\psi(x)\right] + \tfrac{a_2}{\lambda_2}\left(\delta_t\bar{u}^k - \delta_t\bar{u}^{k-1}\right)\\
&+\tfrac{\hat{b_1}}{\lambda_2}\left[\sum_{m=1}^{k-1}(\mathrm{H}_{k-m+1}-\mathrm{H}_{k-m})\delta_t\bar{u}^m + \mathrm{H}_1\delta_t\bar{u}^k - \mathrm{H}_k\psi(x)\right] + \tfrac{\hat{b_2}}{\lambda_2}\left(\delta_t\bar{u}^k - \delta_t\bar{u}^{k-1}\right)\\
&\qquad = \frac{d^2\bar{u}^k}{dx^2} + \tfrac{f(t_k,x)}{\lambda_2}.
\end{aligned} \tag{5.49}$$

The above equation can be written in the form

$$\begin{aligned}
&\tfrac{(a_2+\hat{b_2})}{\lambda_2}\left(\delta_t\bar{u}^k - \delta_t\bar{u}^{k-1}\right) + \tfrac{a_1}{\lambda_2}\mathrm{G}_1\delta_t\bar{u}^k + \tfrac{\hat{b_1}}{\lambda_2}\mathrm{H}_1\delta_t\bar{u}^k = \tfrac{a_1}{\lambda_2}\left[\sum_{m=1}^{k-1}(\mathrm{G}_{k-m}-\mathrm{G}_{k-m+1})\delta_t\bar{u}^m + \mathrm{G}_k\psi(x)\right] +\\
&\qquad +\tfrac{\hat{b_1}}{\lambda_2}\left[\sum_{m=1}^{k-1}(\mathrm{H}_{k-m}-\mathrm{H}_{k-m+1})\delta_t\bar{u}^m + \mathrm{H}_k\psi(x)\right] + \frac{d^2\bar{u}^k}{dx^2} + \tfrac{f(t_k,x)}{\lambda_2},
\end{aligned} \tag{5.50}$$

where $\hat{b_1}$ and $\hat{b_2}$ are positive.

Further, taking inner product of every term of equation (5.50) with $\delta_t\bar{u}^k$, we get

$$\begin{aligned}
&\tfrac{(a_2+\hat{b_2})}{\lambda_2}\langle\delta_t\bar{u}^k - \delta_t\bar{u}^{k-1}, \delta_t\bar{u}^k\rangle + \tfrac{a_1}{\lambda_2}\mathrm{G}_1\|\delta_t\bar{u}^k\|^2 + \tfrac{\hat{b_1}}{\lambda_2}\|\mathrm{H}_1\delta_t\bar{u}^k\|^2 =\\
&\qquad \tfrac{a_1}{\lambda_2}\left[\sum_{m=1}^{k-1}(\mathrm{G}_{k-m}-\mathrm{G}_{k-m+1})\langle\delta_t\bar{u}^m, \delta_t\bar{u}^k\rangle + \mathrm{G}_k\langle\psi(x), \delta_t\bar{u}^k\rangle\right]\\
&+\tfrac{\hat{b_1}}{\lambda_2}\left[\sum_{m=1}^{k-1}(\mathrm{H}_{k-m}-\mathrm{H}_{k-m+1})\langle\delta_t\bar{u}^m, \delta_t\bar{u}^k\rangle + \mathrm{H}_k\langle\psi(x), \delta_t\bar{u}^k\rangle\right] + \langle\frac{d^2\bar{u}^k}{dx^2}, \delta_t\bar{u}^k\rangle + \tfrac{1}{\lambda_2}\langle f(t_k,x), \delta_t\bar{u}^k\rangle.
\end{aligned} \tag{5.51}$$

Now, first we evaluate the term in the right hand side of (5.51) as

$$\begin{aligned}
\tfrac{(a_2+\hat{b_2})}{\lambda_2}\langle\delta_t\bar{u}^k - \delta_t\bar{u}^{k-1}, \delta_t\bar{u}^k\rangle &= (a_2+\hat{b_2})\left(\delta_t\|\bar{u}^k\|^2 - \langle\delta_t\bar{u}^{k-1}, \delta_t\bar{u}^k\rangle\right)\\
&\geq \tfrac{(a_2+\hat{b_2})}{\lambda_2}\left(\|\delta_t\bar{u}^k\|^2 - \tfrac{(\|\delta_t\bar{u}^k\|^2+\|\delta_t\bar{u}^{k-1}\|^2)}{2}\right)\\
&= \tfrac{(a_2+\hat{b_2})}{\lambda_2}\tfrac{(\|\delta_t\bar{u}^k\|^2-\|\delta_t\bar{u}^{k-1}\|^2)}{2}.
\end{aligned} \tag{5.52}$$

Also, by Lemma 5.5.2, we have $\mathrm{G}_{k-m}-\mathrm{G}_{k-m+1}>0$. So, we have

$$\begin{aligned}\sum_{m=1}^{k-1}(\mathrm{G}_{k-m}-\mathrm{G}_{k-m+1})\langle\delta_t\bar{u}^m,\delta_t\bar{u}^k\rangle &\leq \tfrac{1}{2}\sum_{m=1}^{k-1}(\mathrm{G}_{k-m}-\mathrm{G}_{k-m+1})(\|\delta_t\bar{u}^m\|^2+\|\delta_t\bar{u}^k\|^2)\\ &=\tfrac{1}{2}\sum_{m=1}^{k-1}\mathrm{G}_m\|\delta_t\bar{u}^{k-m}\|^2-\tfrac{1}{2}\sum_{m=1}^{k-1}\mathrm{G}_m\|\delta_t\bar{u}^{k-m+1}\|^2\\ &\quad+\tfrac{1}{2}\mathrm{G}_1\|\bar{u}^k\|^2+\tfrac{1}{2}(\mathrm{G}_1-\mathrm{G}_k)\|\bar{u}^k\|^2.\end{aligned}\tag{5.53}$$

Similarly,

$$\begin{aligned}\sum_{m=1}^{k-1}(\mathrm{H}_{k-m}-\mathrm{H}_{k-m+1})\langle\delta_t\bar{u}^m,\delta_t\bar{u}^k\rangle &\leq \tfrac{1}{2}\sum_{m=1}^{k-1}\mathrm{H}_m\|\delta_t\bar{u}^{k-m}\|^2-\tfrac{1}{2}\sum_{m=1}^{k-1}\mathrm{H}_m\|\delta_t\bar{u}^{k-m+1}\|^2\\ &\quad+\tfrac{1}{2}\mathrm{H}_1\|\bar{u}^k\|^2+\tfrac{1}{2}(\mathrm{H}_1-\mathrm{H}_k)\|\bar{u}^k\|^2.\end{aligned}\tag{5.54}$$

Using Cauchy-Schwarz inequality there exists a constant $\epsilon=\frac{(2-\alpha)2^{\alpha-2}}{\Gamma(3-\alpha)}$ such that

$$\langle f(t_kx),\delta_t\bar{u}^k\rangle=\langle f_k,\delta_t\bar{u}^k\rangle\leq\tfrac{\epsilon}{2}\|\delta_t\bar{u}^k\|^2+\tfrac{1}{2\epsilon}\|f_k\|^2.\tag{5.55}$$

Now, multiplying equation (5.51) with τ and using equations from (5.52)-(5.55), we get

$$\begin{aligned}&\tfrac{\tau}{\lambda_2}\tfrac{a_2+\hat{b_2}}{2}\|\delta_t\bar{u}^k\|^2+\tfrac{a_1\tau}{2\lambda_2}\sum_{m=1}^{k-1}\mathrm{G}_m\|\delta_t\bar{u}^{k-m+1}\|^2+\tfrac{b_1\tau}{2\lambda_2}\sum_{m=1}^{k-1}\mathrm{H}_m\|\delta_t\bar{u}^{k-m+1}\|^2\\ &\leq\tfrac{\tau}{\lambda_2}\tfrac{a_2+\hat{b_2}}{2}\|\delta_t\bar{u}^{k-1}\|^2+\tfrac{a_1\tau}{2\lambda_2}\sum_{m=1}^{k-1}\mathrm{G}_m\|\delta_t\bar{u}^{k-m}\|^2+\tfrac{b_1\tau}{2\lambda_2}\sum_{m=1}^{k-1}\mathrm{H}_m\|\delta_t\bar{u}^{k-m}\|^2\\ &\quad+\tfrac{\tau}{2}\left(\|\tfrac{d^2\bar{u}^k}{dx^2}\|^2+\|\delta_t\bar{u}^k\|^2\right)+\tfrac{\tau}{\lambda_2}\left(\tfrac{1}{2\epsilon}\|f_k\|^2+\tfrac{\epsilon}{2}\|\delta_t\bar{u}^k\|^2\right)\\ &\quad+\tfrac{a_1\mathrm{G}_k\tau}{2\lambda_2}\|\psi(x)\|^2+\tfrac{b_1\mathrm{H}_k\tau}{2\lambda_2}\|\psi(x)\|^2.\end{aligned}\tag{5.56}$$

Let,

$$O(\bar{u}^k)=\tfrac{\tau}{\lambda_2}\tfrac{a_2+\hat{b_2}}{2}\|\delta_t\bar{u}^k\|^2+\tfrac{a_1\tau}{2\lambda_2}\sum_{m=1}^{k-1}\mathrm{G}_m\|\delta_t\bar{u}^{k-m+1}\|^2+\tfrac{b_1\tau}{2\lambda_2}\sum_{m=1}^{k-1}\mathrm{H}_m\|\delta_t\bar{u}^{k-m+1}\|^2.\tag{5.57}$$

Then equation (5.56) reduces to

$$O(\bar{u}^k)\leq O(\bar{u}^{k-1})+\tfrac{\tau}{2}\left(\|\tfrac{d^2\bar{u}^k}{dx^2}\|^2\right)+\left(\tfrac{\tau}{2}+\tfrac{\tau\epsilon}{2\lambda_2}\right)\|\delta_t\bar{u}^k\|^2+\left(\tfrac{a_1\mathrm{G}_k\tau+b_1\mathrm{H}_k\tau}{2\lambda_2}\right)\|\psi(x)\|^2+\tfrac{\tau}{2\epsilon\lambda_2}\|f_k\|^2.\tag{5.58}$$

Summing with respect to k from 1 to K, we get

$$\begin{aligned} O(\bar{u}^K) \leq O(\bar{u}^0) + \tfrac{\tau}{2} \sum_{m=1}^{K} \left(\| \tfrac{d^2\bar{u}^m}{dx^2} \|^2 \right) + \left(\tfrac{\tau}{2} + \tfrac{\tau\epsilon}{2\lambda_2} \right) \sum_{m=1}^{K} \|\delta_t \bar{u}^m\|^2 \\ + \sum_{m=1}^{K} \left(\tfrac{a_1 \mathrm{G}_m \tau + b_1 \mathrm{H}_m \tau}{2\lambda_2} \right) \|\psi(x)\|^2 + \tfrac{\tau}{2\epsilon\lambda_2} \sum_{m=1}^{K} \|f_m\|^2. \end{aligned} \tag{5.59}$$

Using Lemma 5.6.1, we have

$$\tfrac{(2-\alpha)2^{\alpha-2}}{\Gamma(3-\alpha)} \tfrac{\tau}{\lambda_2} \sum_{m=1}^{K} \|\delta_t \bar{u}^m\|^2 \leq \left(\tfrac{1}{2} \tfrac{a_1\tau}{2\lambda_2} \right) \sum_{m=1}^{K} \mathrm{G}_N \|\delta_t \bar{u}^{k-m+1}\|^2 \leq \left(\tfrac{1}{2} \tfrac{a_1\tau}{2\lambda_2} \right) \sum_{m=1}^{K} \mathrm{G}_m \|\delta_t \bar{u}^{k-m+1}\|^2 \leq \tfrac{1}{2} O(\bar{u}^K). \tag{5.60}$$

The above relation implies that

$$\left(\tfrac{\tau}{2} + \tfrac{\tau\epsilon}{2\lambda_2} \right) \sum_{m=1}^{K} \|\delta_t \bar{u}^m\|^2 \leq \left(\tfrac{1}{2} \tfrac{a_1\tau}{2\lambda_2} + \tfrac{\tau}{2} \right) \sum_{m=1}^{K} \mathrm{G}_m \|\delta_t \bar{u}^{k-m+1}\|^2 \leq \left(\tfrac{1}{2} + \tfrac{\tau}{2} \right) O(\bar{u}^K). \tag{5.61}$$

Using equation (5.61) into equation (5.59), we get the following relation

$$\begin{aligned} O(\bar{u}^K) \leq O(\bar{u}^0) + \tfrac{\tau}{2} \sum_{m=1}^{K} \left(\| \tfrac{d^2\bar{u}^m}{dx^2} \|^2 \right) + \sum_{m=1}^{K} \left(\tfrac{a_1 \mathrm{G}_m \tau + b_1 \mathrm{H}_m \tau}{2\lambda_2} \right) \|\psi(x)\|^2 \\ + \tfrac{\tau}{2\epsilon\lambda_2} \sum_{m=1}^{K} \|f_m\|^2 + \left(\tfrac{1}{2} + \tfrac{\tau}{2} \right) O(\bar{u}^K), \end{aligned} \tag{5.62}$$

$$\begin{aligned} \text{or,} \quad \left(\tfrac{1}{2} - \tfrac{\tau}{2} \right) O(\bar{u}^K) \leq O(\bar{u}^0) + \tfrac{\tau}{2} \sum_{m=1}^{K} \left(\| \tfrac{d^2\bar{u}^m}{dx^2} \|^2 \right) + \sum_{m=1}^{K} \left(\tfrac{a_1 \mathrm{G}_m \tau + b_1 \mathrm{H}_m \tau}{2\lambda_2} \right) \|\psi(x)\|^2 \\ + \tfrac{\tau}{2\epsilon\lambda_2} \sum_{m=1}^{K} \|f_m\|^2. \end{aligned} \tag{5.63}$$

Also, note that $1 - \tau > 0$.

$$\text{Now,} \quad \bar{u}^n = \bar{u}^0 + \tau \sum_{k=1}^{n} \delta_t \bar{u}^k, \qquad 1 \leq n \leq M. \tag{5.64}$$

Using equation (27) from [168], we get

$$
\begin{aligned}
\|\bar{u}^n\|^2 &\leq 2\|\bar{u}^0\|^2 + 2T\tau \sum_{m=1}^{n} \|\delta_t \bar{u}^k\|^2 \\
&\leq 2\|\bar{u}^0\|^2 + CO(\bar{u}^n) \\
&\leq 2\|\bar{u}^0\|^2 + C_1\Bigg(O(\bar{u}^0) + \tfrac{\tau}{2} \sum_{m=1}^{n} \|\tfrac{d^2\bar{u}^m}{dx^2}\|^2 \\
&+ \sum_{m=1}^{n} \left(\tfrac{a_1 \mathrm{G}_m \tau + b_1 \mathrm{H}_m \tau}{2\lambda_2}\right) \|\psi(x)\|^2 + \tfrac{\tau}{2\epsilon\lambda_2} \sum_{m=1}^{n} \|f_m\|^2 \Bigg),
\end{aligned} \tag{5.65}
$$

$$
\text{or,} \quad \|\bar{u}^n\|^2 \leq 2\|\bar{u}^0\|^2 + C_1\Bigg(O(\bar{u}^0) + \mathrm{T}\mathcal{M} + \sum_{m=1}^{n} \left(\tfrac{a_1 \mathrm{G}_m \tau + b_1 \mathrm{H}_m \tau}{2}\right) \|\psi(x)\|^2 + \tfrac{\tau}{2\epsilon} \sum_{m=1}^{n} \|f_m\|^2 \Bigg), \tag{5.66}
$$

where $C = \frac{T\lambda_2}{\epsilon}$, $C_1 = \left(\frac{2T\lambda_2}{\epsilon(1-\tau)}\right)$ and $O(\bar{u}^0) = \tau\left(\frac{a_2 + \hat{b_2}}{2\lambda_2}\right)\|\psi(x)\|^2$.

5.7 Convergence analysis in time direction

Theorem 5.7.1. If $\{\bar{u}^k\}_{k=1}^{M}$ be the time-discrete solution of the problem equation (5.1), then

$$
\|u^k(x) - \bar{u}^k(x)\| \leq \mathcal{C}\mathcal{O}(t^{3-\alpha}).
$$

Proof. Subtracting equation (5.7) from (5.6) and rearranging the terms, we get

$$
\begin{aligned}
&\tfrac{a_1}{\lambda_2}\left[\sum_{m=1}^{k-1}(\mathrm{G}_{k-m+1} - \mathrm{G}_{k-m})\delta_t e^m + \mathrm{G}_1 \delta_t e^k\right] + \tfrac{a_2}{\lambda_2}\left(\delta_t e^k - \delta_t e^{k-1}\right) \\
&+ \tfrac{\hat{b_1}}{\lambda_2}\left[\sum_{m=1}^{k-1}(\mathrm{H}_{k-m+1} - \mathrm{H}_{k-m})\delta_t e^m + \mathrm{H}_1 \delta_t e^k\right] + \tfrac{\hat{b_2}}{\lambda_2}\left(\delta_t e^k - \delta_t e^{k-1}\right) \\
&\qquad = \frac{d^2 e^k}{dx^2} + \tfrac{\mathrm{R}^k}{\lambda_2},
\end{aligned} \tag{5.67}
$$

where $e_k = u^k(x) - \bar{u}^k(x)$, $\mathrm{R}^k = \mathcal{O}(\tau^{3-\alpha} + \tau^{3-\beta})$.

Now, proceeding in similar manner as in Theorem 5.6.1, we get

$$\begin{aligned}
&\tfrac{\tau}{\lambda_2}\tfrac{a_2+\hat{b_2}}{2}\|\delta_t e^k\|^2+\tfrac{a_1\tau}{2\lambda_2}\sum_{m=1}^{k-1}\mathrm{G}_m\|\delta_t e^{k-m+1}\|^2+\tfrac{b_1\tau}{2\lambda_2}\sum_{m=1}^{k-1}\mathrm{H}_m\|\delta_t e^{k-m+1}\|^2\\
&\leq \tfrac{\tau}{\lambda_2}\tfrac{a_2+\hat{b_2}}{2}\|\delta_t e^{k-1}\|^2+\tfrac{a_1\tau}{2\lambda_2}\sum_{m=1}^{k-1}\mathrm{G}_m\|\delta_t e^{k-m}\|^2+\tfrac{b_1\tau}{2\lambda_2}\sum_{m=1}^{k-1}\mathrm{H}_m\|\delta_t e^{k-m}\|^2\\
&\qquad+\tfrac{\tau}{2}\left(\|\tfrac{d^2e^k}{dx^2}\|^2+\|\delta_t e^k\|^2\right)+\tfrac{\tau}{2\epsilon\lambda_2}\|\mathrm{R}^k\|^2+\tfrac{\tau\epsilon}{2\lambda_2}\|\delta_t e^k\|^2.
\end{aligned} \tag{5.68}$$

Above equation reduces to the following relation after using the value of $O(e^k)$ from equation (5.57).

$$O(e^k)\leq O(e^{k-1})+\tfrac{\tau}{2}\|\tfrac{d^2e^k}{dx^2}\|^2+\left(\tfrac{\tau}{2}+\tfrac{\tau\epsilon}{2\lambda_2}\right)\|\delta_t e^k\|^2+\tfrac{\tau}{2\epsilon\lambda_2}\|\mathrm{R}^k\|^2. \tag{5.69}$$

Summing up from 1 to K, we obtain

$$O(e^K)\leq O(e^0)+\tfrac{\tau}{2}\sum_{m=1}^{K}\|\tfrac{d^2e^m}{dx^2}\|^2+\left(\tfrac{\tau}{2}+\tfrac{\tau\epsilon}{2\lambda_2}\right)\sum_{m=1}^{K}\|\delta_t e^m\|^2+\tfrac{\tau}{2\epsilon\lambda_2}\sum_{m=1}^{K}\|\mathrm{R}^m\|^2. \tag{5.70}$$

Also, we have the following relation similar to equation (5.60)

$$\tfrac{\tau\epsilon}{2\lambda_2}\sum_{m=1}^{K}\|\delta_t e^m\|^2\leq\tfrac{1}{2}\tfrac{a_1\tau}{2\lambda_2}\sum_{m=1}^{K}\mathrm{G}_m\|\delta_t e^{k-m+1}\|^2\leq\tfrac{1}{2}O(e^K). \tag{5.71}$$

Above equation implies that,

$$\left(\tfrac{\tau}{2}+\tfrac{\tau\epsilon}{2\lambda_2}\right)\sum_{m=1}^{K}\|\delta_t e^m\|^2\leq\left(\tfrac{\tau}{2}+\tfrac{1}{2}\tfrac{a_1\tau}{2\lambda_2}\right)\sum_{m=1}^{K}\mathrm{G}_m\|\delta_t e^{k-m+1}\|^2\leq\left(\tfrac{\tau}{2}+\tfrac{1}{2}\right)O(e^K). \tag{5.72}$$

Using equation (5.72), equation (5.70) reduces to

$$\left(\tfrac{1}{2}-\tfrac{\tau}{2}\right)O(e^K)\leq O(e^0)+\tfrac{\tau}{2}\sum_{m=1}^{K}\|\tfrac{d^2e^m}{dx^2}\|^2+\tfrac{\tau}{2\epsilon\lambda_2}\sum_{m=1}^{K}\|\mathrm{R}^m\|^2. \tag{5.73}$$

$$\text{Now,}\;\; \|e^n\|^2\leq 2\|e^0\|^2+2T\tau\sum_{m=1}^{n}\|\delta_t e^k\|^2, \tag{5.74}$$

$$\begin{aligned}
\|e^n\|^2 &\leq 2\|e^0\|^2 + T\tfrac{\lambda_2}{\epsilon}O(e^n) \\
&\leq 2\|e^0\|^2 + \tfrac{C}{1-\tau}\left(2O(e^0) + \tau\sum_{m=1}^{n}\|\tfrac{d^2e^m}{dx^2}\|^2 + \tfrac{\tau}{\lambda_2\epsilon}\sum_{m=1}^{n}\|\mathrm{R}^m\|^2\right) \\
&\leq 2\|e^0\|^2 + \tfrac{C}{1-\tau}\left(2O(e^0) + \tau n\max_{1\leq m\leq n}\|\frac{d^2e^m}{dx^2}\|^2 + \tfrac{\tau n}{\lambda_2\epsilon}\max_{1\leq m\leq n}\|\mathrm{R}^m\|^2\right) \\
&\leq 2\|e^0\|^2 + \tfrac{C}{1-\tau}\left(2O(e^0) + T\max_{1\leq m\leq n}\|\frac{d^2e^m}{dx^2}\|^2 + \tfrac{T}{\lambda_2\epsilon}\max_{1\leq m\leq n}\|\mathrm{R}^m\|^2\right),
\end{aligned} \tag{5.75}$$

where $C = \frac{T\lambda_2}{\epsilon}$.

Now, from equation (5.1), we have

$$\tfrac{d^2u^k}{dx^2} = \ {}_0D_t^{\alpha}u(t_k,x) - \lambda_1\ {}_0D_t^{\beta}u(t_k,x) - f(t_k,x). \tag{5.76}$$

Hence, using equations (5.6) and (5.7), we get the following relation

$$\|\tfrac{d^2e^k}{dx^2}\| \leq \mathcal{O}\left(\tau^{3-\alpha} + \tau^{3-\beta}\right), \quad k = 1, 2, ...; \tag{5.77}$$

Also, $e^0 = 0$ and $O(e^0) = 0$, and using $\frac{1}{1-\tau} = 1 + \tau + \tau^2 + ...$

From equation (5.77) and equation (5.75), we conclude that

$$\|e^n\| \leq \mathcal{C}\mathcal{O}\left(\tau^{min\{3-\alpha,3-\beta\}}\right) = \mathcal{C}\mathcal{O}(\tau^{3-\alpha}).$$

5.8 Numerical experiments

Three examples having exact solutions, are considered to demonstrate the efficiency and accuracy of the proposed method. The following error norms are calculated for the test examples.

$$\|u - \bar{u}\|_2 = \left(\sum_{i=1}^{M} h|u(T,x_i) - \bar{u}(T,x_i)|^2\right)^{\frac{1}{2}}.$$

Also, the point wise absolute error is calculated using the following formula

$$\text{Absolute error} = |u(t_k,x) - \bar{u}^k(x)|,$$

where $\bar{u}^k(x)$ is the numerical solution at k^{th} level of t and $u(t_k,x)$ is the value of exact solution at point t_k.

All the simulations given in Tables 5.1-5.8 are done with $N = 6, h = 1/60$ and $T = 1$

for different values of α and β. In Tables 5.1 and 5.8, the L_2 norms are calculated for example 5.8.1 and example 5.8.2, respectively. The pointwise errors for the first example at time $T = 0.5$ & $T = 1$ are provided in Tables 5.2 and 5.8. Similarly, for second example, the pointwise errors at $T = 0.5$ and $T = 1$ are given in Tables 5.8 and 5.8. Moreover, all the graphs are plotted with $N = 8$ and $M = 60$. The 2-D Figures 5.1 and 5.3 depict the numerical (right part) and exact solutions (left part) for examples 5.8.1 and 5.8.2, respectively. The 2-D Figures 5.2 and 5.4 (left part in Figure 5.2 and 5.44) show the absolute errors for $\alpha = 1.5$ and $\beta = 1.1$. Moreover, the Figures 5.2 and 5.4 (right part in Figure 5.2 and 5.4) show the combined errors for different values of α and β. L_2-error with convergence order is calculated in Table 5.7 for example 5.8.3. Results obtained from example 5.8.3 are compared with the results obtained in [168] (see Table 5.8). Also, all the simulation results in the Table 5.7 and 5.8 have been done for $N = 8$ and $T = 1$.

Example 5.8.1. Consider the following example

$$ {}_0D_t^{\alpha}u(t,x) +_0 D_t^{\beta}u(t,x) - \nabla^2 u(t,x) = f(t,x), \quad 1 < \beta < \alpha < 2, $$

with initial and boundary conditions as $u(0,x) = \left.\dfrac{\partial u(t,x)}{\partial t}\right|_{t=0} = u(t,0) = u(t,1) = 0$ and having exact solution $u(t,x) = t^3 x^{1+\alpha+\beta}(1-x)$.

Table 5.1: L_2-errors for different values of α and β, $T = 1,\ h = 1/60,\ N = 6$ for example 5.8.1.

$\boldsymbol{\tau}$	$\boldsymbol{\beta = 1.1,\ \alpha = 1.5}$ $\boldsymbol{\lVert u - \bar{u} \rVert_2}$	$\boldsymbol{\beta = 1.5,\ \alpha = 1.7}$ $\boldsymbol{\lVert u - \bar{u} \rVert_2}$	$\boldsymbol{\beta = 1.3,\ \alpha = 1.9}$ $\boldsymbol{\lVert u - \bar{u} \rVert_2}$
1/5	2.1207e-03	3.5284e-03	4.3242e-03
1/10	7.4896e-04	1.5083e-03	2.1053e-03
1/20	2.6056e-04	6.1595e-04	9.9411e-04
1/40	9.0647e-05	2.4875e-04	4.6395e-04
1/80	3.1686e-05	1.0634e-04	2.18226e-04

Table 5.2: Point wise errors for different values α and β, $T = 0.5,\ h = 1/60,\ N = 6$ for example 5.8.1.

$\boldsymbol{x}$	$\boldsymbol{\beta = 1.1,\ \alpha = 1.5}$	$\boldsymbol{\beta = 1.5,\ \alpha = 1.7}$	$\boldsymbol{\beta = 1.3,\ \alpha = 1.9}$
0.0	1.9395e-05	1.6941e-05	1.5444e-05
0.1	1.3101e-05	3.0244e-06	9.7490e-05
0.2	9.7717e-06	2.4265e-05	3.9946e-05
0.3	1.7114e-05	4.0473e-05	7.2547e-05
0.4	3.0580e-05	5.3239e-05	1.0893e-04
0.5	3.8147e-05	7.0291e-05	1.5179e-04
0.6	3.6081e-05	8.9887e-05	1.9168e-04
0.7	3.0244e-05	9.9083e-05	2.0739e-04
0.8	2.8124e-05	8.2749e-05	1.7708e-04
0.9	2.4261e-05	4.1461e-05	9.8272e-05
1.0	1.9339e-05	1.6941e-05	1.5444e-05

Table 5.3: Point wise errors for different values α and β, $T = 1,\ h = 1/60,\ N = 6$ for example 5.8.1.

$\boldsymbol{x}$	$\boldsymbol{\beta = 1.1,\ \alpha = 1.5}$	$\boldsymbol{\beta = 1.5,\ \alpha = 1.7}$	$\boldsymbol{\beta = 1.3,\ \alpha = 1.9}$
0.0	1.4318e-04	1.4488e-04	1.5273e-05
0.1	2.7351e-05	1.9429e-05	5.9256e-05
0.2	1.8064e-04	9.3332e-05	1.2003e-05
0.3	2.4767e-04	1.2725e-04	1.1358e-06
0.4	2.4181e-04	9.6608e-05	5.7931e-05
0.5	2.4972e-04	8.9973e-05	7.7229e-05
0.6	2.9624e-04	1.3467e-04	3.7941e-05
0.7	3.2027e-04	1.7215e-04	2.0247e-06
0.8	2.3667e-04	1.1951e-04	2.5315e-05
0.9	6.9064e-04	1.7861e-06	7.8543e-05
1.0	1.4318e-04	1.4488e-04	1.5273e-04

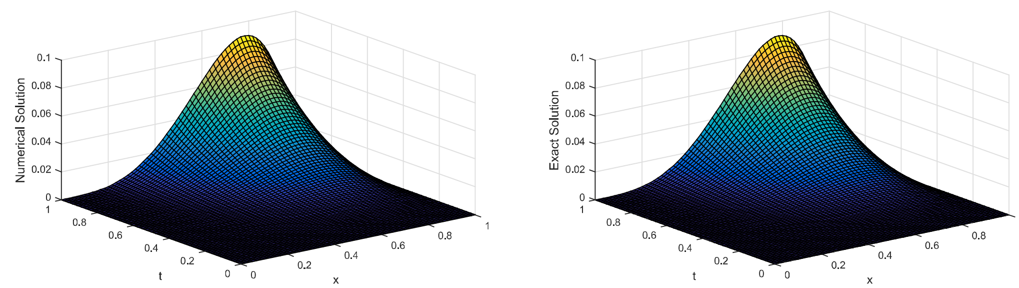

Figure 5.1: Numerical (left figure) and exact solutions (right figure) of example 5.8.1 for $N = 8, M = 60$, for $\alpha = 1.5$ and $\beta = 1.1$.

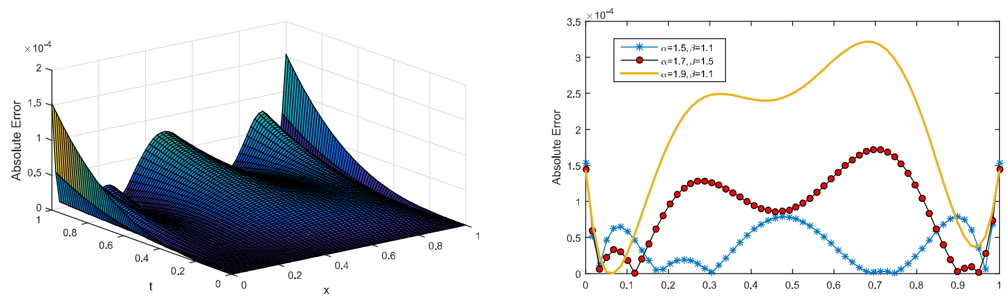

Figure 5.2: Absolute errors for $\alpha = 1.5$, $\beta = 1.1$ (left figure) and absolute errors at $T = 1$ for different value of α and β (right figure) for example 5.8.1 with $N = 8, M = 60$.

Example 5.8.2. Let us consider the following equation

$$ {}_0D_t^{\alpha}u(t,x) +_0 D_t^{\beta}u(t,x) - \nabla^2 u(t,x) = f(t,x), \quad 1 < \beta < \alpha < 2, $$

with initial and boundary conditions $u(0,x) = \left.\dfrac{\partial u(t,x)}{\partial t}\right|_{t=0} = u(t,0) = u(t,1) = 0$, whose exact solution is $u(t,x) = t^3 sin(\pi x)$.

Table 5.4: L_2-errors for different values of α and β, $T = 1,\ h = 1/60,\ N = 6$ for example 5.8.2.

$\boldsymbol{\tau}$	$\boldsymbol{\beta = 1.1,\ \alpha = 1.5}$ $\boldsymbol{\|u - \bar{u}\|_2}$	$\boldsymbol{\beta = 1.5,\ \alpha = 1.7}$ $\boldsymbol{\|u - \bar{u}\|_2}$	$\boldsymbol{\beta = 1.3,\ \alpha = 1.9}$ $\boldsymbol{\|u - \bar{u}\|_2}$
1/5	4.9258e-03	1.0675e-02	1.3099e-02
1/10	1.7420e-03	4.5929e-03	6.4236e-03
1/20	6.0051e-04	1.8787e-03	3.0445e-03
1/40	2.1213e-04	7.5135e-04	1.4198e-03
1/80	7.4564e-05	3.0108e-04	6.5962e-04

Table 5.5: Point wise errors for different values α and β, $T = 0.5,\ h = 1/60,\ N = 6$ for example 5.8.2.

$\boldsymbol{x}$	$\boldsymbol{\beta = 1.1,\ \alpha = 1.5}$	$\boldsymbol{\beta = 1.5,\ \alpha = 1.7}$	$\boldsymbol{\beta = 1.3,\ \alpha = 1.9}$
0.0	1.6226e-04	1.6397e-03	1.6669e-04
0.1	1.3110e-04	5.2243e-04	1.1578e-03
0.2	4.0225e-04	1.1148e-03	2.3593e-03
0.3	5.6310e-04	1.5900e-03	3.2572e-03
0.4	5.8951e-04	1.7961e-03	3.7548e-03
0.5	5.8076e-04	1.8491e-03	3.9079e-03
0.6	5.8951e-04	1.7961e-03	3.7548e-03
0.7	5.6310e-04	1.5900e-03	3.2572e-03
0.8	4.0225e-04	1.1482e-03	2.3593e-03
0.9	1.3110e-04	5.2243e-04	1.1578e-03
1.0	1.6226e-04	1.6397e-04	1.6664e-04

Table 5.6: Point wise errors for different values α and β, $T = 1,\ h = 1/60,\ N = 6$ for example 5.8.2.

$\boldsymbol{x}$	$\boldsymbol{\beta = 1.1,\ \alpha = 1.5}$	$\boldsymbol{\beta = 1.5,\ \alpha = 1.7}$	$\boldsymbol{\beta = 1.3,\ \alpha = 1.9}$
0.0	1.2919e-03	1.2952e-03	1.3039e-03
0.1	2.7665e-04	4.8520e-04	1.7565e-03
0.2	6.9069e-04	2.1429e-03	4.4692e-03
0.3	1.0249e-03	3.0242e-03	6.2226e-03
0.4	6.2700e-04	2.9759e-03	6.7379e-03
0.5	3.4769e-04	2.8168e-03	6.7712e-03
0.6	6.2700e-04	2.9759e-03	6.7379e-03
0.7	1.0249e-03	3.0242e-03	6.2263e-03
0.8	6.9069e-04	2.1429e-03	4.4692e-03
0.9	2.7665e-04	4.8520e-04	1.7056e-03
1.0	1.2919e-03	1.2915e-03	1.3003e-03

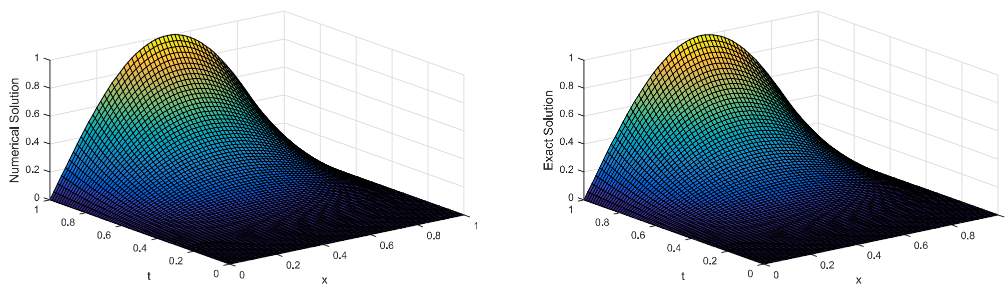

Figure 5.3: Numerical (left figure) and exact solutions (right) of example 5.8.2 for $N = 8, M = 60$, for $\alpha = 1.5$ and $\beta = 1.1$.

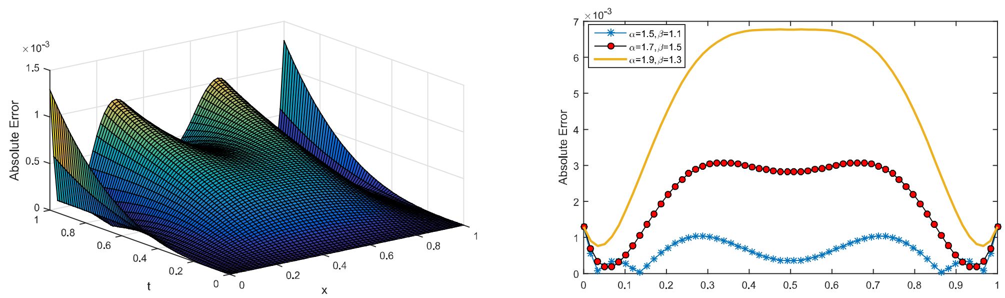

Figure 5.4: Absolute error for $\alpha = 1.5$, $\beta = 1.1$ (left figure) and absolute errors at $T = 1$ for different values of α and β (right figure) for example 5.8.2 with $N = 8, M = 60$.

Example 5.8.3. In this example, we consider the following problem from [168]. This example is the particular case of proposed problem for $\lambda_1 = 0, \lambda_2 = 1$.

$$ {}_0D_t^{\alpha}u(t,x) - \nabla^2 u(t,x) = f(t,x), \quad 1 < \alpha < 2, $$

with initial and boundary conditions as $u(0,x) = \left.\dfrac{\partial u(t,x)}{\partial t}\right|_{t=0} = u(t,0) = u(t,1) = 0$ and having the exact solution as $u(t,x) = t^3 x^{1+\alpha}(1-x)$.

Table 5.7: L_2-errors and the convergence rate for example 5.8.3 at time T = 1 and different values of α and $N = 8$.

$\boldsymbol{\tau}$	$\boldsymbol{\alpha = 1.1}$		$\boldsymbol{\alpha = 1.5}$		$\boldsymbol{\alpha = 1.9}$	
	$\|\|u - \bar{u}\|\|_2$	**C-Rate**	$\|\|u - \bar{u}\|\|_2$	**C-Rate**	$\|\|u - \bar{u}\|\|_2$	**C-Rate**
1/5	1.0383e-03	-	3.0299e-03	-	7.2954e-03	-
1/10	2.7873e-04	1.8971	1.0941e-03	1.4695	3.8455e-03	0.9238
1/20	7.6835e-05	1.8590	3.8289e-04	1.5147	1.9347e-03	0.9911
1/40	2.0810e-05	1.8845	1.3483e-04	1.5058	9.4090e-04	1.0400
1/80	5.5505e-06	1.9066	4.9630e-05	1.4419	4.4827e-04	1.0697
1/160	1.4555e-06	1.9311	1.7500e-05	1.5039	2.1126e-04	1.0853

Table 5.8: Comparison of L_2-errors with the method established in [168].

$\boldsymbol{\tau}$	$\boldsymbol{\alpha = 1.1}$		$\boldsymbol{\alpha = 1.5}$		$\boldsymbol{\alpha = 1.9}$	
	$\|\|u - \bar{u}\|\|_2$	**Error in [168]**	$\|\|u - \bar{u}\|\|_2$	**Error in [168]**	$\|\|u - \bar{u}\|\|_2$	**Error in [168]**
1/5	1.0383e-03	1.0381e-03	3.0299e-03	3.0298e-03	7.2954e-03	7.2953e-03
1/10	2.7873e-04	2.7833e-04	1.0941e-03	1.0939e-04	3.8455e-03	3.8455e-03
1/20	7.6835e-05	7.5432e-05	3.8289e-04	3.8251e-04	1.9347e-03	1.9347e-03
1/40	2.0810e-05	2.0454e-05	1.3483e-04	1.3386e-04	9.4090e-04	9.4082e-04
1/80	5.5505e-06	5.5238e-06	4.9630e-05	4.7079e-05	4.4827e-04	4.4818e-04
1/160	1.4555e-06	1.4716e-06	1.7500e-05	1.6577e-05	2.1126e-04	2.1117e-04

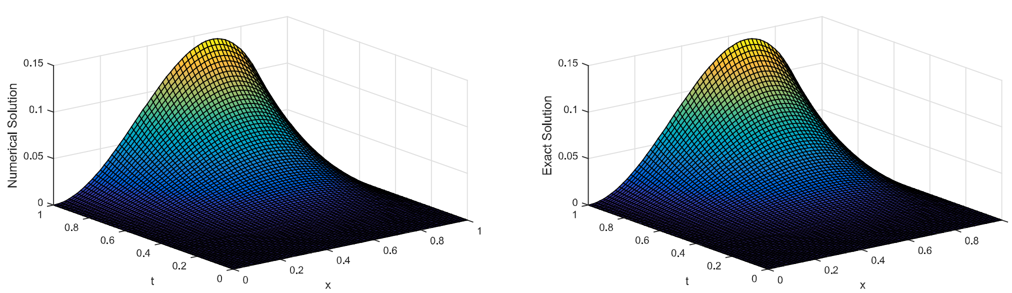

Figure 5.5: Numerical (left figure) and exact solutions (right figure) of example 5.8.3 for $N = 8, M = 60$, for $\alpha = 1.1$.

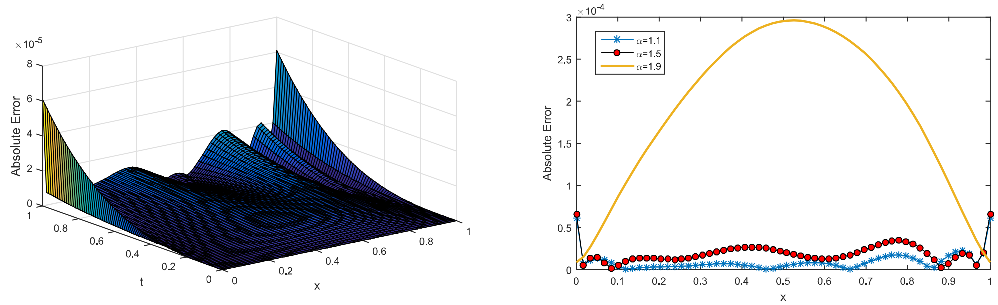

Figure 5.6: Absolute error for $\alpha = 1.1$, (left figure) and absolute errors at $T = 1$ for different values of α (right figure) for example 5.8.3 with $N = 8, M = 80$.

Further, some additional points about Example 5.8.3 are given below:

1. To investigate the order of convergence in the time direction, we fix $N = 8$ and varying the time step $\tau = \frac{1}{5}, \frac{1}{10}, \frac{1}{20}, \frac{1}{40}, \frac{1}{80}, \frac{1}{160}$. The logarithmic scale has been used in Figure 5.7 (left part) for both L_2-error and time step τ.

2. As the theoretical result about order of convergence is established in t direction, from the Figure 5.7 (left part) and Table 5.7, one can see that the order of convergence in t direction is of $\tau^{3-\alpha}$.

3. To analyze the behavior of error in the spatial direction, we use semi-log plot (logarithmic scale used only for error) between L_2-error and N by fixing the step size $\tau = 5 \times 10^{-3}$ with different values of α.

4. One can see from Figure 5.7 (right part), the error decay with the rate $\mathcal{O}\left(\frac{1}{2^{N+1}(N+1)!}\right)$ because the semi-log plot shows that the error decays is almost linear with respect to N.

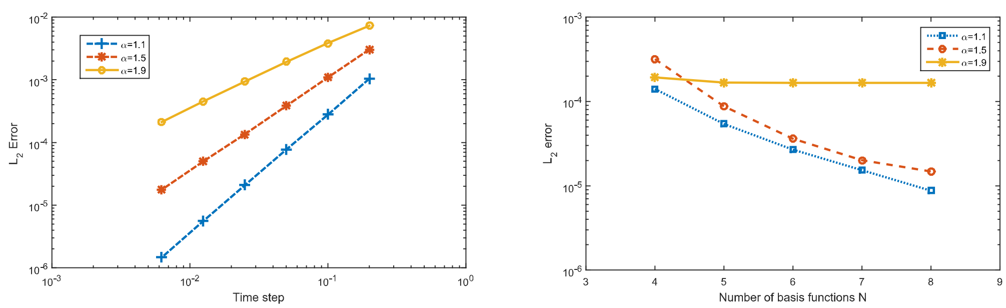

Figure 5.7: L_2-error versus time step τ for different values of α (Left figure) and L_2-error versus number of basis function N for different α, $\tau = 5 \times 10^{-3}$(Right figure).

5.9 Conclusion

In this chapter, a semi-discretization technique combined with operational matrix approach is proposed for solving the fractional wave equation which arises in the dielectric medium. The advantage of operational matrix approach over the ODE solver is that it converts the system of ODEs into the algebraic equations which are easy to solve. Three numerical examples having exact solutions are taken to demonstrate the accuracy of the numerical scheme. Also, one can see from Tables 5.1-5.8 that matrix based scheme gives high order accuracy for small number of basis functions (even for a small number of discretization point i.e., $h = 1/60$), which is very beneficial for the researchers from the computational point of view. Moreover, the comparison of the proposed scheme with the fully discrete scheme [168] is given in Tables 5.7-5.8 for example 5.8.3. The desired accuracy and fast convergence for small number of basis functions are obtained as compared to [168]. The log-log and semi-log graphs (Figure 5.7) are plotted to demonstrate the order of convergence in the time and spatial directions, respectively. Hence, this scheme has many attractive features, like simplicity, fast convergence, high order accuracy in less computational effort, and easy for programming. Also, one can apply this scheme for solving any class of the fractional order differential equations. Moreover, this method can be extended for solving higher dimensional PDEs, which is one of the goals and can be the topic for future study.

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