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1. Chakraborty, M. and Prasad, SD. (2019). Determination of bearing capacity factor Nc for circular piles in clays by using lower bound finite element analysis. *International Journal of Geotechnical Engineering*, 15 (10), 1325-1331.
2. Prasad, S.D. and Chakraborty, M. (2021). Bearing capacity of ring footing resting on two-layered soil. *Computers and Geotechnics*, 134 (104088).
3. Prasad, S.D. and Chakraborty, M. (2021). Behaviour of ring footing on two-layered soil due to various loading positions. *International Journal of Geomechanics*. 22(5), 04022049.
4. Prasad, S.D. and Chakraborty M., 2023. Upper Bound Collapse Load of Strip Footing that Rests on Unsaturated Sands. *International Journal of Geomechanics* 23 (4), 04023016.

UNDER REVIEW

1. Prasad, S.D. and Chakraborty, M. Effect of load eccentricity and load inclination on the ultimate bearing capacity of strip footing rested on unsaturated sands. *International Journal of Geomechanics* .

2. Prasad, S.D. and Chakraborty, M. Bearing Capacity of ring footing resting on rock mass under different loading positions. *Sadhana*

Appendix A

ALGORITHM FOR SOLVING NONLINEAR OPTIMIZATION PROBLEM

Following flowchart shows the nonlinear optimization problem algorithm for solving Equation (3.22) to obtain LB collapse load:

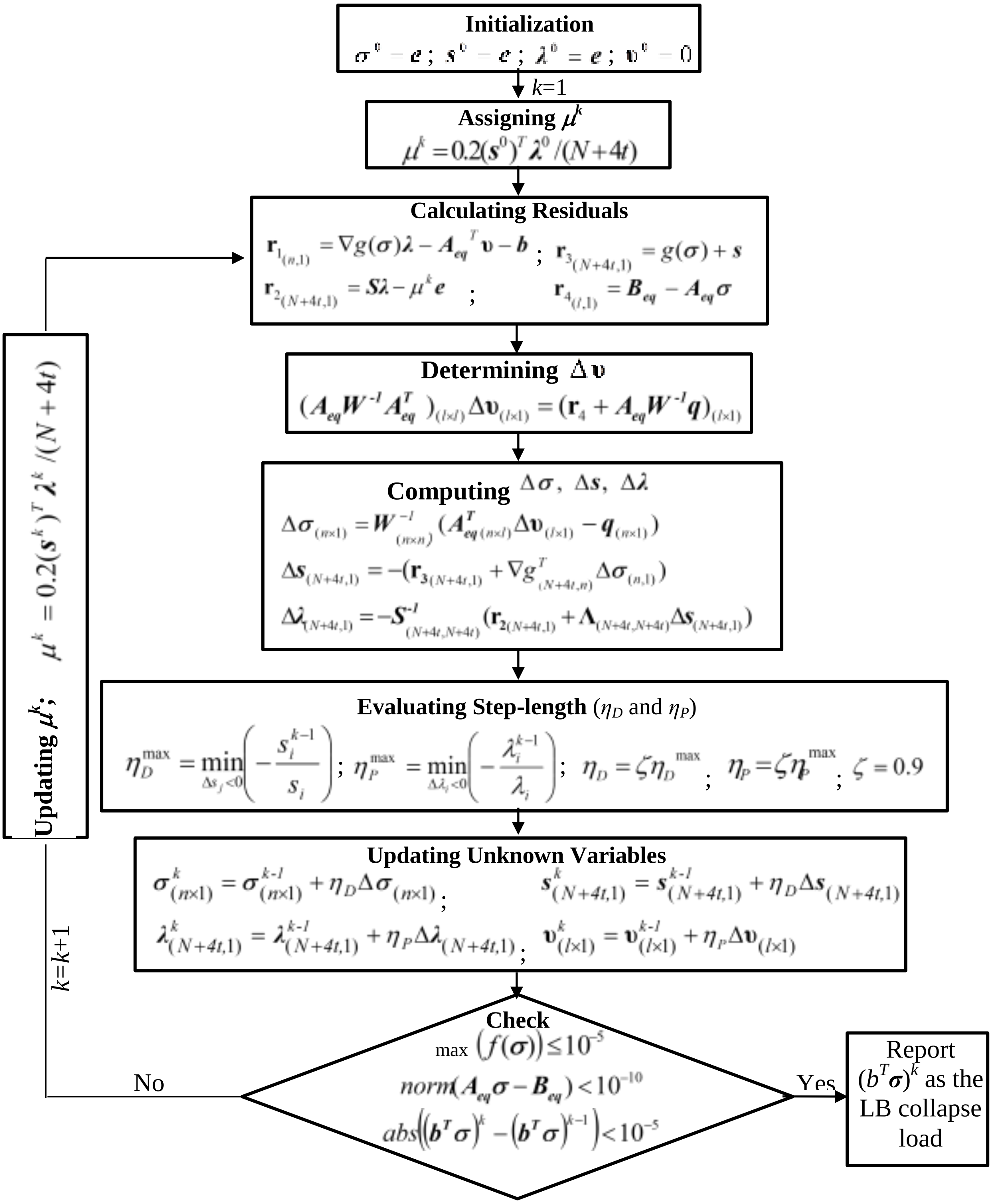

Figure A.1 Flowchart of the IPM based optimization strategy.

Appendix B

GRADIENT AND HESSIAN OF SMOOTHENED MOHR-COULOMB YIELD FUNCTION

B1. YIELD FUNCTION

$$f = \sigma_m \sin\phi + \sqrt{\bar{\sigma}^2 k(\theta)^2 + \varepsilon^2 \sin^2\phi} - c\cos\phi$$

$$k(\theta) = \left(\cos\theta - \frac{1}{\sqrt{3}}\sin\phi\sin\theta\right) \qquad \text{for} \quad |\theta| \le \theta_T$$

$$= \left(A + B\sin 3\theta + C\sin^2 3\theta\right) \qquad \text{for} \quad |\theta| > \theta_T$$

The definition of *A*, *B* and *C* are mentioned in Section 3.29.2.

B2. GRADIENT CALCULATION

$$\nabla f = \frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{\partial f}{\partial \sigma_m}\frac{\partial \sigma_m}{\partial \boldsymbol{\sigma}} + \frac{\partial f}{\partial \bar{\sigma}}\frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial \boldsymbol{\sigma}} \tag{B.1}$$

On simplification, $\nabla f = \frac{\partial f}{\partial \boldsymbol{\sigma}} = C_1\frac{\partial \sigma_m}{\partial \boldsymbol{\sigma}} + C_2\frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}} + C_3\frac{\partial J_3}{\partial \boldsymbol{\sigma}}$ (B.2)

where, $C_1 = \frac{\partial f}{\partial \sigma_m} = \sin\phi$ (B.3)

$$C_2 = \frac{\partial f}{\partial \bar{\sigma}} - \frac{\tan 3\theta}{\bar{\sigma}}\frac{\partial f}{\partial \theta} = \alpha_{11}\left(K - \tan 3\theta\frac{dK}{d\theta}\right) = \alpha_{11}P \tag{B.4}$$

$$\alpha_{11} = \frac{\bar{\sigma}K}{\sqrt{\bar{\sigma}^2K^2 + \varepsilon^2\sin^2\phi}} \quad ; \quad K = k(\theta) \quad \text{and}$$

$$P = K - \tan 3\theta\frac{dK}{d\theta} \qquad \text{for } |\theta| \le \theta_T$$

$$= A - 2B\sin 3\theta - 5C\sin^2 3\theta \qquad \text{for } |\theta| > \theta_T$$

$$C_3 = -\frac{\sqrt{3}}{2}\frac{1}{\cos 3\theta}\frac{1}{\bar{\sigma}^3}\frac{\partial f}{\partial \theta} = \alpha_{11}R \tag{B.5}$$

$$R = \left(-\frac{\sqrt{3}}{2\bar{\sigma}^2 \cos 3\theta} \right) \frac{dK}{d\theta} \qquad \text{for } |\theta| \le \theta_T$$
$$= \left(\frac{3\sqrt{3}}{2\bar{\sigma}^2} \right) (B + 2C \sin 3\theta) \qquad \text{for } |\theta| > \theta_T$$

$$\frac{\partial \sigma_m}{\partial \boldsymbol{\sigma}} = \frac{1}{3} \begin{Bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{Bmatrix} ; \quad \frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}} = \frac{1}{2\bar{\sigma}} \begin{Bmatrix} s_r \\ s_z \\ 2\tau_{rz} \\ s_\theta \end{Bmatrix} ; \quad \frac{\partial J_3}{\partial \boldsymbol{\sigma}} = \begin{Bmatrix} s_z s_\theta + \dfrac{\bar{\sigma}^2}{3} \\ s_r s_\theta + \dfrac{\bar{\sigma}^2}{3} \\ -2 s_\theta \tau_{rz} \\ s_z s_r - \tau_{rz}^2 + \dfrac{\bar{\sigma}^2}{3} \end{Bmatrix} \tag{B.6}$$

If $\bar{\sigma}$ tends to zero, the value of α_{11} will also approach to zero and the direction of the gradient will point along the positive hydrostatic axis. However, the third term in Equation (B.2) involves division by $\bar{\sigma}^2$. Hence, for smaller value of $\bar{\sigma}$, Equation (B.2) becomes undefined. In order to get rid of this indefiniteness, division by $\bar{\sigma}$ should be eliminated for all the equations. Considering this fact, the third term in Equation (B.2) should be computed as:

$$C_3 \frac{\partial J_3}{\partial \boldsymbol{\sigma}} = \overline{C_3}\, \overline{\frac{\partial J_3}{\partial \boldsymbol{\sigma}}} \tag{B.7}$$

where $\overline{C_3} = C_3 \bar{\sigma}^2$ and $\overline{\dfrac{\partial J_3}{\partial \boldsymbol{\sigma}}} = \dfrac{1}{\bar{\sigma}^2} \dfrac{\partial J_3}{\partial \boldsymbol{\sigma}}$

Equation (B.2) becomes $\nabla f = \dfrac{\partial f}{\partial \boldsymbol{\sigma}} = C_1 \dfrac{\partial \sigma_m}{\partial \boldsymbol{\sigma}} + C_2 \dfrac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}} + \overline{C_3}\, \overline{\dfrac{\partial J_3}{\partial \boldsymbol{\sigma}}}$

(B.8)

B.2. HESSIAN CALCULATION

$$\nabla^2 f = \frac{\partial C_2}{\partial \boldsymbol{\sigma}} \frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}} + C_2 \frac{\partial^2 \bar{\sigma}}{\partial \boldsymbol{\sigma}^2} + \frac{\partial C_3}{\partial \boldsymbol{\sigma}} \frac{\partial J_3}{\partial \boldsymbol{\sigma}} + C_3 \frac{\partial^2 J_3}{\partial \boldsymbol{\sigma}^2} \tag{B.9}$$

$$= A_1 \frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}} \otimes \frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}} + A_2 \left(\frac{\partial J_3}{\partial \boldsymbol{\sigma}} \otimes \frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}} + \frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}} \otimes \frac{\partial J_3}{\partial \boldsymbol{\sigma}} \right) + A_3 \frac{\partial^2 \bar{\sigma}}{\partial \boldsymbol{\sigma}^2} + A_4 \frac{\partial J_3}{\partial \boldsymbol{\sigma}} \otimes \frac{\partial J_3}{\partial \boldsymbol{\sigma}} + A_5 \frac{\partial^2 J_3}{\partial \boldsymbol{\sigma}^2}$$

Here $A_1 = (k_2 k_5 + k_3)$; $A_2 = k_2 k_4$; $A_3 = C_2$; $A_4 = (k_9 k_5 + k_8)$; $A_5 = C_3$

$$\begin{aligned} k_1 &= \frac{dK}{d\theta} - \frac{d^2K}{d\theta^2}\tan 3\theta - 3\frac{dK}{d\theta}\sec^2 3\theta && \text{for } |\theta| \le \theta_T \\ &= -6B\cos 3\theta - 15C\sin 6\theta && \text{for } |\theta| > \theta_T \end{aligned}$$

$$k_2 = \left(\alpha_{11} k_1 + \frac{(1-\alpha_{11}^2)P\bar{\sigma}}{Q}\frac{dK}{d\theta} \right) \quad ; \quad k_3 = \frac{(1-\alpha_{11}^2)}{Q} PK \ ;$$

$$k_4 = -\frac{\sqrt{3}}{2\bar{\sigma}^3 \cos 3\theta} \quad ; \qquad k_5 = \frac{3\sqrt{3}J_3}{2\bar{\sigma}^4 \cos 3\theta} \ ;$$

$$k_6 = \frac{\sqrt{3}}{\cos 3\theta \bar{\sigma}^3}\frac{dK}{d\theta} \quad ; \qquad k_7 = -\frac{\sqrt{3}}{2\cos 3\theta \bar{\sigma}^2}\left(\frac{d^2K}{d\theta^2} + 3\frac{dK}{d\theta}\tan 3\theta \right)$$

$$\begin{aligned} k_8 &= \alpha_{11} k_6 + \frac{(1-\alpha_{11}^2)}{Q} KR && \text{for } |\theta| \le \theta_T \\ &= \frac{3\sqrt{3}}{\bar{\sigma}^2}\left(B + 2C\sin 3\theta \right)\left(\frac{\alpha_{11}}{\bar{\sigma}} + \frac{1}{2}\frac{(1-\alpha_{11}^2)}{Q} K \right) && \text{for } |\theta| > \theta_T \end{aligned}$$

$$\begin{aligned} k_9 &= \frac{(1-\alpha_{11}^2)\bar{\sigma}R}{Q}\frac{dK}{d\theta} - \alpha_{11} k_7 && \text{for } |\theta| \le \theta_T \\ &= \frac{9\sqrt{3}\cos 3\theta}{\bar{\sigma}}\left(\frac{(1-\alpha_{11}^2)}{2Q}(B + 2C\sin 3\theta)^2 + \frac{C\alpha_{11}}{\bar{\sigma}} \right) && \text{for } |\theta| > \theta_T \end{aligned}$$

$$\frac{\partial^2 J_3}{\partial \boldsymbol{\sigma}^2} = \frac{1}{3}\begin{bmatrix} s_r - s_z - s_\theta & 2s_\theta & 2\tau_{rz} & 2s_z \\ 2s_\theta & s_z - s_r - s_\theta & 2\tau_{rz} & 2s_r \\ 2\tau_{rz} & 2\tau_{rz} & -6\tau_{rz} & -4\tau_{rz} \\ 2s_z & 2s_r & -4\tau_{rz} & s_\theta - s_z - s_r \end{bmatrix} \tag{B.10}$$

$$\frac{\partial^2 \bar{\sigma}}{\partial \boldsymbol{\sigma}^2} = \frac{1}{\bar{\sigma}}\left[\frac{1}{2}\frac{\partial \mathbf{s}}{\partial \boldsymbol{\sigma}} - \frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}} \otimes \frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}} \right] \tag{B.11}$$

$$\frac{\partial \boldsymbol{s}}{\partial \boldsymbol{\sigma}} = \begin{bmatrix} 2/3 & -1/3 & 0 & -1/3 \\ -1/3 & 2/3 & 0 & -1/3 \\ 0 & 0 & 2 & 0 \\ -1/3 & -1/3 & 0 & 2/3 \end{bmatrix}; \ \frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}} \otimes \frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}} = \frac{1}{4\bar{\sigma}^2} \begin{bmatrix} s_r^2 & s_r s_z & 2\tau_{rz} s_r & s_r s_\theta \\ s_r s_z & s_z^2 & 2\tau_{rz} s_z & s_z s_\theta \\ 2\tau_{rz} s_r & 2\tau_{rz} s_z & 4\tau_{rz}^{\ 2} & 2\tau_{rz} s_\theta \\ s_r s_\theta & s_z s_\theta & 2\tau_{rz} s_\theta & s_\theta^2 \end{bmatrix} \tag{B.12}$$

To avoid division by $\bar{\sigma}$, following equations were been used:

$$\frac{\overline{\partial^2 \bar{\sigma}}}{\partial \boldsymbol{\sigma}^2} = \bar{\sigma} \frac{\partial^2 \bar{\sigma}}{\partial \boldsymbol{\sigma}^2} \quad \text{and} \quad \frac{\overline{\partial^2 J_3}}{\partial \boldsymbol{\sigma}^2} = \frac{1}{\bar{\sigma}} \frac{\partial^2 J_3}{\partial \boldsymbol{\sigma}^2} \tag{B.13}$$

$$\therefore \nabla^2 f = A_1 \frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}} \otimes \frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}} + A_2 \bar{\sigma}^{2} \left(\frac{\overline{\partial J_3}}{\partial \boldsymbol{\sigma}} \otimes \frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}} + \frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}} \otimes \frac{\overline{\partial J_3}}{\partial \boldsymbol{\sigma}} \right) + \frac{A_3}{\bar{\sigma}} \frac{\overline{\partial^2 \bar{\sigma}}}{\partial \boldsymbol{\sigma}^2} + A_4 \bar{\sigma}^{4} \frac{\overline{\partial J_3}}{\partial \boldsymbol{\sigma}} \otimes \frac{\overline{\partial J_3}}{\partial \boldsymbol{\sigma}} + A_5 \bar{\sigma} \frac{\overline{\partial^2 J_3}}{\partial \boldsymbol{\sigma}^2}$$

Appendix C

GRADIENT AND HESSIAN OF SMOOTHENED TRESCA YIELD FUNCTION

C1. YIELD FUNCTION

$$f = \bar{\sigma} k(\theta) - c$$

$$k(\theta) = \cos\theta \qquad \text{for} \quad |\theta| \le \theta_T$$

$$= \left(A + B\sin 3\theta + C\sin^2 3\theta\right) \qquad \text{for} \quad |\theta| > \theta_T$$

The definition of *A*, *B* and *C* are mentioned in Section 3.2.9.2.

C2. GRADIENT CALCULATION

$$\nabla f = \frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{\partial f}{\partial \sigma_m}\frac{\partial \sigma_m}{\partial \boldsymbol{\sigma}} + \frac{\partial f}{\partial \bar{\sigma}}\frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial \boldsymbol{\sigma}} \tag{C.1}$$

On simplification, $$\nabla f = \frac{\partial f}{\partial \boldsymbol{\sigma}} = C_1\frac{\partial \sigma_m}{\partial \boldsymbol{\sigma}} + C_2\frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}} + C_3\frac{\partial J_3}{\partial \boldsymbol{\sigma}} \tag{C.2}$$

where, $$C_1 = \frac{\partial f}{\partial \sigma_m} = 0 \tag{C.3}$$

$$C_2 = \frac{\partial f}{\partial \bar{\sigma}} - \frac{\tan 3\theta}{\bar{\sigma}}\frac{\partial f}{\partial \theta} = P \tag{C.4}$$

$$P = K - \tan 3\theta \frac{dK}{d\theta} \qquad \text{for} \; |\theta| \le \theta_T$$

$$= A - 2B\sin 3\theta - 5C\sin^2 3\theta \qquad \text{for} \; |\theta| > \theta_T$$

$$C_3 = -\frac{\sqrt{3}}{2}\frac{1}{\cos 3\theta}\frac{1}{\bar{\sigma}^3}\frac{\partial f}{\partial \theta} = R \tag{C.5}$$

$$R = \left(-\frac{\sqrt{3}}{2\bar{\sigma}^2\cos 3\theta}\right)\frac{dK}{d\theta} \qquad \text{for} \; |\theta| \le \theta_T$$

$$= \left(\frac{3\sqrt{3}}{2\bar{\sigma}^2}\right)\left(B + 2C\sin 3\theta\right) \qquad \text{for} \; |\theta| > \theta_T$$

C.2. HESSIAN CALCULATION

$$\nabla^2 f = \frac{\partial C_2}{\partial \boldsymbol{\sigma}}\frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}} + C_2\frac{\partial^2 \bar{\sigma}}{\partial \boldsymbol{\sigma}^2} + \frac{\partial C_3}{\partial \boldsymbol{\sigma}}\frac{\partial J_3}{\partial \boldsymbol{\sigma}} + C_3\frac{\partial^2 J_3}{\partial \boldsymbol{\sigma}^2} \tag{C.6}$$

$$= A_6\frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}}\otimes\frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}} + A_7\left(\frac{\partial J_3}{\partial \boldsymbol{\sigma}}\otimes\frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}} + \frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}}\otimes\frac{\partial J_3}{\partial \boldsymbol{\sigma}}\right) + A_8\frac{\partial^2 \bar{\sigma}}{\partial \boldsymbol{\sigma}^2} + A_9\frac{\partial J_3}{\partial \boldsymbol{\sigma}}\otimes\frac{\partial J_3}{\partial \boldsymbol{\sigma}} + A_{10}\frac{\partial^2 J_3}{\partial \boldsymbol{\sigma}^2}$$

Here, $A_6 = k_{10}k_{12}$; $A_7 = k_{10}k_{11}$; $A_8 = C_2$; $A_9 = \left(k_{15}k_{12} + k_{13}\right)$; $A_{10} = C_3$

$$k_{10} = \frac{dK}{d\theta} - \frac{d^2K}{d\theta^2}\tan 3\theta - 3\frac{dK}{d\theta}\sec^2 3\theta \qquad \text{for } |\theta| \le \theta_T$$

$$= -6B\cos 3\theta - 15C\sin 6\theta \qquad \text{for } |\theta| > \theta_T$$

$$k_{11} = -\frac{\sqrt{3}}{2\bar{\sigma}^3\cos 3\theta}; \qquad k_{12} = \frac{3\sqrt{3}J_3}{2\bar{\sigma}^4\cos 3\theta};$$

$$k_{13} = \frac{\sqrt{3}}{\cos 3\theta\,\bar{\sigma}^3}\frac{dK}{d\theta} \qquad \text{for } |\theta| \le \theta_T$$

$$= \frac{3\sqrt{3}}{\bar{\sigma}^3}\left(B + 2C\sin 3\theta\right) \qquad \text{for } |\theta| > \theta_T$$

$$k_{14} = -\frac{\sqrt{3}}{2\cos 3\theta\,\bar{\sigma}^2}\left(\frac{d^2K}{d\theta^2} + 3\frac{dK}{d\theta}\tan 3\theta\right)$$

$$k_{15} = -k_{14} \qquad \text{for } |\theta| \le \theta_T$$

$$= \frac{9\sqrt{3}C\cos 3\theta}{\bar{\sigma}^2} \qquad \text{for } |\theta| > \theta_T$$

Appendix D

GRADIENT AND HESSIAN OF SMOOTHENED HOEK-BROWN YIELD FUNCTION

D.1. YIELD FUNCTION

$$\begin{aligned} f(\boldsymbol{\sigma}) &= (2\ddot{\sigma}\cos\theta)^{1/\alpha} - m_b\sigma_{ci}^{(1-\alpha)/\alpha}\ddot{\sigma}\left(\frac{\sin\theta}{\sqrt{3}} - \cos\theta\right) + m_b\sigma_{ci}^{(1-\alpha)/\alpha}\sigma_m - s\sigma_{ci}^{1/\alpha} \qquad \text{for}\,|\theta| < \theta_T \\ &= m_b\sigma_{ci}^{(1\text{-}\alpha)/\alpha}\sigma_m - m_b\sigma_{ci}^{(1\text{-}\alpha)/\alpha}\ddot{\sigma}\left(A + B\sin 3\theta + C\sin^2 3\theta\right) - s\sigma_{ci}^{1/\alpha} \qquad \text{for}\,|\theta| \geq \theta_T \end{aligned}$$

The expressions of *A*, *B* and *C* are mentioned in Equations (6.8), (6.9), and (6.10).

D.2. GRADIENT CALCULATION

$$\nabla f = \frac{\partial f}{\partial \sigma} = C_1\frac{\partial \sigma_m}{\partial \sigma} + C_2\frac{\partial \bar{\sigma}}{\partial \sigma} + C_3\frac{\partial J_m}{\partial \sigma} \tag{D.1}$$

(i) For $|\theta| < \theta_T$,

$$C_1 = \frac{\partial f}{\partial \sigma_m} = m_b\sigma_{ci}^{(1-\alpha)/\alpha} \tag{D.2}$$

$$C_2 = \frac{\partial f}{\partial \bar{\sigma}} = \frac{\tan 3\theta}{\bar{\sigma}}\frac{\partial f}{\partial \theta} = \frac{\bar{\sigma}}{\ddot{\sigma}}k_1 + \frac{\tan 3\theta}{\bar{\sigma}}k_2 \tag{D.3}$$

$$C_3 = -\frac{\sqrt{3}}{2}\frac{1}{\cos 3\theta}\frac{1}{\bar{\sigma}^3}\frac{\partial f}{\partial \theta} = \frac{\sqrt{3}}{2\cos 3\theta\bar{\sigma}^3}k_2 \tag{D.4}$$

where

$$k_1 = -m_b\sigma_{ci}^{(1-\alpha)/\alpha}\left(\frac{\sin\theta}{\sqrt{3}} - \cos\theta\right) + (8\ddot{\sigma}\cos^2\theta)$$

$$k_2 = m_b\sigma_{ci}^{(1-\alpha)/\alpha}\ddot{\sigma}\left(\frac{\cos\theta}{\sqrt{3}} + \sin\theta\right) + (4\ddot{\sigma}^2\sin 2\theta)$$

(ii) For $|\theta| \geq \theta_T$,

$$C_1 = m_b\sigma_{ci}^{(1-\alpha)/\alpha} - m_b\sigma_{ci}^{(1-\alpha)/\alpha}\ddot{\sigma}(k_5 + k_6\sin 3\theta + k_7\sin^2 3\theta) \tag{D.5}$$

$$\begin{aligned} C_2 &= -m_b\sigma_{ci}^{(1-\alpha)/\alpha}(A + B\sin 3\theta + C\sin^2 3\theta)\frac{\bar{\sigma}}{\ddot{\sigma}} \\ &\quad + \frac{3\tan 3\theta}{\bar{\sigma}}m_b\sigma_{ci}^{(1-\alpha)/\alpha}\ddot{\sigma}(B\cos 3\theta + C\sin 6\theta) \end{aligned} \tag{D.6}$$

$$C_3 = \frac{3\sqrt{3}m_b\sigma_{ci}^{(1-\alpha)/\alpha}\ddot{\sigma}}{2\bar{\sigma}^3}(B + 2C\sin 3\theta) \tag{D.7}$$

where;

$$k_3 = \frac{R\sigma_m - s\sigma_{ci}^{(1-\alpha)/\alpha}}{162R\ddot{\sigma}^2\cos^2 3\theta} \qquad k_4 = \frac{\sin 3\theta}{162R\ddot{\sigma}^2\cos^2 3\theta}$$

$$k_5 = \frac{\partial A}{\partial \sigma_m} = 2T\sin^2 3\theta\left[\frac{R}{162R\ddot{\sigma}^2\cos^3 3\theta} + k_3 \times \frac{324}{\ddot{\sigma}^3}\cos^3 3\theta\frac{\partial\ddot{\sigma}}{\partial\sigma_m}\right] + k_3 \times 2\sin^2 3\theta \times \frac{\partial T}{\partial \sigma_m} +$$

$$N_1 T\sin 3\theta \times \left[0.5k_6 - (R\sigma_m - s\sigma_{ci}^{(1-\alpha)/\alpha} \times 1944\frac{\cos^3 3\theta}{\ddot{\sigma}^5} \times \frac{\partial\ddot{\sigma}}{\partial\sigma_m}\right] + \frac{k_3}{2} \times \sin 3\theta\left[N_1\frac{\partial T}{\sigma_m} + T\frac{\partial N_1}{\partial\sigma_m}\right] -$$

$$\left[\frac{R\sigma_m - s\sigma_{ci}^{(1-\alpha)/\alpha}}{R\ddot{\sigma}^2}\right]$$

$$k_6 = \frac{\partial B}{\partial \sigma_m} = -2k_4 \times T + 2\left(\frac{R\sigma_m - s\sigma_{ci}^{(1-\alpha)/\alpha}}{T}\right)T \times \frac{2\sin 3\theta}{162\ddot{\sigma}^3\cos^2 3\theta} \times \frac{\partial\ddot{\sigma}}{\partial\sigma_m} -$$

$$2\left(\frac{R\sigma_m - s\sigma_{ci}^{(1-\alpha)/\alpha}}{R}\right)k_4\frac{\partial T}{\partial\sigma_m} - \frac{R\sigma_m - s\sigma_{ci}^{(1-\alpha)/\alpha}}{3R\ddot{\sigma}^2\cos 3\theta} \times \frac{\partial N_1}{\partial\sigma_m} -$$

$$N_1\left[\frac{1}{3\ddot{\sigma}^2\cos 3\theta} - \frac{R\sigma_m - s\sigma_{ci}^{(1-\alpha)/\alpha}}{3R\ddot{\sigma}^3\cos 3\theta} \times \frac{\partial\ddot{\sigma}}{\partial\sigma_m}\right]$$

$$k_7 = \frac{\partial C}{\partial \sigma_m} = \frac{1}{162R\ddot{\sigma}^2\cos^2 3\theta} - \frac{2(R\sigma_m - s\sigma_{ci}^{(1-\alpha)/\alpha})}{162R\ddot{\sigma}^3 3\theta} \times \frac{\partial\ddot{\sigma}}{\partial\sigma_m}$$

D.3. HESSIAN CALCULATION

$$\nabla^2 f = \frac{\partial C_2}{\partial\sigma}\frac{\partial\bar{\sigma}}{\partial\sigma} + C_2\frac{\partial^2\bar{\sigma}}{\partial\sigma^2} + C_2\frac{\partial C_3}{\partial\sigma}\frac{\partial J_3}{\partial\sigma} + C_3\frac{\partial^2 J_3}{\partial\sigma^2}$$

(D.8)

$$\frac{\partial C_2}{\partial\sigma} = \frac{\partial C_2}{\partial\bar{\sigma}}\frac{\partial\bar{\sigma}}{\partial\sigma} + \frac{\partial C_2}{\partial\theta}\frac{\partial\theta}{\partial\sigma} \qquad \text{(D.9)}$$

$$\frac{\partial C_3}{\partial\sigma} = \frac{\partial C_3}{\partial\bar{\sigma}}\frac{\partial\bar{\sigma}}{\partial\sigma} + \frac{\partial C_3}{\partial\theta}\frac{\partial\theta}{\partial\sigma} \qquad \text{(D.10)}$$

(i) For $|\theta|<\theta_{T}$, $\frac{\partial C_2}{\partial\sigma} = \left(\frac{\varepsilon^2}{\bar{\sigma}^3}\right)k_1 - \frac{\tan 3\theta}{\bar{\sigma}^2}k_2 + \left(\frac{\bar{\sigma}}{\ddot{\sigma}}\right)^2 k_8 + \frac{\tan 3\theta}{\overleftrightarrow{\sigma}}k_9 = k_{11}$ (D.11)

$$\frac{\partial C_3}{\partial\theta} = -\left(\frac{\bar{\sigma}}{\overleftrightarrow{\sigma}}\right)k_9 + \frac{\tan 3\theta}{\bar{\sigma}}k_{10} + \frac{3\sec^2 3\theta}{\bar{\sigma}}k_2 = k_{12} \qquad \text{(D.12)}$$

$$\frac{\partial C_3}{\partial\bar{\sigma}} = \frac{\sqrt{3}}{2\cos 3\theta\bar{\sigma}^3}\left(\frac{-3k_2}{\bar{\sigma}} - \frac{\bar{\sigma}}{\ddot{\sigma}}k_9\right) = k_{13} \qquad \text{(D.13)}$$

$$\frac{\partial C_3}{\partial\bar{\sigma}} = \frac{\sqrt{3}}{2\cos 3\theta\bar{\sigma}^3}(3k_2\tan 3\theta + k_{10}) = k_{14} \qquad \text{(D.14)}$$

where $k_8 = 8\cos_2\theta$; $k_9 = 4\ddot{\sigma}\sin 2\theta + \dfrac{k_2}{\ddot{\sigma}}$

$$k_{10} = m_b\sigma_{ci}^{(1-\alpha)/\alpha}\ddot{\sigma}\left(\frac{\sin\theta}{\sqrt{3}} - \cos\theta\right) + 8\ddot{\sigma}^{1/\alpha}\cos 2\theta$$

(ii) For $|\theta| \geq \theta_{T,}$

$$\frac{\partial C_1}{\partial\sigma_m} = -m_b\sigma_{ci}^{(1-\alpha)/\alpha}\ddot{\sigma}(k_{19} + k_{20}\sin 3\theta + k_{21}\sin^2 3\theta) = k_{22} \tag{D.15}$$

$$\frac{\partial C_1}{\partial\bar{\sigma}} = -m_b\sigma_{ci}^{(1-\alpha)/\alpha}\frac{\bar{\sigma}}{\ddot{\sigma}}(k_5 + k_6\sin 3\theta + k_7\sin^2 3\theta) = k_{23} \tag{D.16}$$

$$\frac{\partial C_1}{\partial\theta} = -3m_b\sigma_{ci}^{(1-\alpha)/\alpha}\ddot{\sigma}(k_6\cos 3\theta + k_7\sin 6\theta) = k_{24} \tag{D.17}$$

$$\begin{aligned}\frac{\partial C_2}{\partial\sigma_m} &= -m_b\sigma_{ci}^{(1-\alpha)/\alpha}\frac{\bar{\sigma}}{\ddot{\sigma}}(k_5 + k_6\sin 3\theta + k_7\sin^2 3\theta) + \\ &\frac{3\tan 3\theta}{\bar{\sigma}}m_b\sigma_{ci}^{(1-\alpha)/\alpha}\ddot{\sigma}(k_6\cos 3\theta + k_7\sin 6\theta = k_{25}\end{aligned} \tag{D.18}$$

$$\begin{aligned}\frac{\partial C_2}{\partial\bar{\sigma}} &= -m_b\sigma_{ci}^{(1-\alpha)/\alpha}(A + B\sin 3\theta + C\sin^2 3\theta)\frac{\varepsilon^2}{\bar{\sigma}^3} - \\ &\frac{\varepsilon^2}{\bar{\sigma}^2\ddot{\sigma}}3\tan 3\theta m_b\sigma_{ci}^{(1-\alpha)/\alpha}(B\cos 3\theta + C\sin 6\theta) = k_{26}\end{aligned} \tag{D.19}$$

$$\begin{aligned}\frac{\partial C_2}{\partial\theta} &= 3m_b\sigma_{ci}^{(1-\alpha)/\alpha}(B\cos 3\theta + C\sin 6\theta)\left(-\frac{\bar{\sigma}}{\ddot{\sigma}} + 3\frac{\ddot{\sigma}}{\bar{\sigma}}\sec^2\theta\right) \\ &-\frac{9\tan 3\theta}{\bar{\sigma}}m_b\sigma_{ci}^{(1-\alpha)/\alpha}\ddot{\sigma}(B\cos 3\theta - 2C\cos 6\theta) = k_{27}\end{aligned} \tag{D.20}$$

$$\frac{\partial C_3}{\partial\sigma_m} = \frac{3\sqrt{3}m_b\sigma_{ci}^{(1-\alpha)/\alpha}\ddot{\sigma}(k_6 + 2k_7\sin 3\theta)}{2\bar{\sigma}^3} = k_{28} \tag{D.21}$$

$$\frac{\partial C_3}{\partial\sigma_m} = \frac{3\sqrt{3}m_b\sigma_{ci}^{(1-\alpha)/\alpha}(B + 2C\sin 3\theta)}{2}\left(\frac{\bar{\sigma}}{\ddot{\sigma}} - \frac{3}{\bar{\sigma}^4}\right) = k_{29} \tag{D.22}$$

$$\frac{\partial C_3}{\partial\theta} = \frac{9\sqrt{3}m_b\sigma_{ci}^{(1-\alpha)/\alpha}\ddot{\sigma}C\cos 3\theta}{\bar{\sigma}^3} = k_{30} \tag{D.23}$$

where,

$k_{15} = \dfrac{R}{(162R\ddot{\sigma}^2\cos^2 3\theta)^2}$; $k_{16} = \dfrac{R\sigma_m - s\sigma_{ci}^{(1-\alpha)/\alpha}}{293R\ddot{\sigma}^4\cos^2 3\theta}$; $k_{17} = \dfrac{R\sigma_m - s\sigma_{ci}^{(1-\alpha)/\alpha}}{R}$;

$$k_{18} = \frac{2\sin 3\theta}{162\ddot{\sigma}^3\cos^2 3\theta}$$

$$k_{19}=\frac{\partial^2 A}{\partial\sigma_m^2}=\left[k_{15}+k_3\times\frac{324}{\ddot{\sigma}^3}\cos^2 3\theta\frac{\partial\ddot{\sigma}}{\partial\sigma_m}\right]\times 2\sin^2 3\theta\times\frac{\partial T}{\partial\sigma_m}+2T\sin^2 3\theta\left[-2k_{15}+k_3\right]\times$$

$$\left[\frac{324}{\ddot{\sigma}^3}\cos^2 3\theta\frac{\partial^2\ddot{\sigma}}{\partial\sigma_m^2}-\frac{972}{\ddot{\sigma}^4}\cos^2 3\theta\frac{\partial\ddot{\sigma}}{\partial\sigma_m}\right]+\frac{324}{\ddot{\sigma}^3}\cos^2 3\theta\frac{\partial\ddot{\sigma}}{\partial\sigma_m}\times\left[k_{15}+4k_3\times k_{15}\right]+$$

$$k_3\times 2\sin^2 3\theta\times\frac{\partial^2 T}{\partial\sigma_m^2}+2\sin^2 3\theta\times\frac{\partial T}{\partial\sigma_m}\times\left[k_{15}-2k_3\right]+\left[k_{16}-(R\sigma_m-s\sigma_{ci}^{(1-\alpha)/\alpha})^2\times 1944\frac{\cos^3 3\theta}{\ddot{\sigma}^5}\times\frac{\partial\ddot{\sigma}}{\partial\sigma_m}\right]\times$$

$$\sin 3\theta\left[N_1\frac{\partial T}{\partial\sigma_m}+\frac{\partial N_1}{\partial\sigma_m}\right]+N_1T\sin 3\theta\times\left[0.5k_{26}-k_3\right]+(R\sigma_m-s\sigma_{ci}^{(1-\alpha)/\alpha})^2\times$$

$$1944\left[\cos^3 3\theta\times\frac{5\partial\ddot{\sigma}}{\ddot{\sigma}^6\partial\sigma_m}-\frac{\cos^3 3\theta}{\ddot{\sigma}^5}\frac{\partial^2\ddot{\sigma}}{\partial\sigma_m^2}-\frac{\cos^3 3\theta}{\ddot{\sigma}^5}\frac{\partial\ddot{\sigma}}{\partial\sigma_m}\times 2R(R\sigma_m-s\sigma_{ci}^{(1-\alpha)/\alpha})\right]$$

$$+(0.25k_{16})\sin 3\theta\left[N_1\frac{\partial^2 T}{\partial\sigma_m^2}+\frac{\partial T}{\partial\sigma_m}\frac{\partial N_1}{\partial\sigma_m}+T\frac{\partial^2 N_1}{\partial\sigma_m}+\frac{\partial T}{\partial\sigma_m}\frac{\partial N_1}{\partial\sigma_m}\right]$$

$$+\left[0.25k_{16}\times 2R-k_3\right]\times\sin 3\theta\left[N_1\frac{\partial T}{\partial\sigma_m}+T\frac{\partial N_1}{\partial\sigma_m}\right]-\left[-\frac{1}{\ddot{\sigma}^2}\frac{\partial\ddot{\sigma}}{\partial\sigma_m}+\frac{(R\sigma_m-s\sigma_{ci}^{(1-\alpha)/\alpha}}{R\ddot{\sigma}^3}-\frac{1}{\ddot{\sigma}^2}\right]$$

$$k_{20}=\frac{\partial^2 B}{\partial\sigma_m^2}=2T\times k_{18}\frac{\partial\ddot{\sigma}}{\partial\sigma_m}-2\left(\frac{\sin 3\theta}{162\ddot{\sigma}^2\cos^2 3\theta}\right)\times\frac{\partial T}{\partial\sigma_m}+2k_{17}Tk_{18}\frac{\partial^2\ddot{\sigma}}{\partial\sigma_m}+$$

$$2Tk_{18}\frac{\partial\ddot{\sigma}}{\partial\sigma_m}+2\times k_{17}\times T\times\left\{k_{18}\times\frac{\partial^2\ddot{\sigma}}{\partial\sigma_m^2}-\frac{\partial\ddot{\sigma}}{\partial\sigma_m}\times 3k_{18}\frac{\partial\ddot{\sigma}}{\partial\sigma_m}\right\}-2k_{17}k_{18}\times\frac{\partial^2 T}{\partial\sigma_m^2}-$$

$$k_{18}\frac{\partial T}{\partial\sigma_m}+4\times k_{18}\times k_{17}\times\frac{\partial T}{\partial\sigma_m}\times\frac{\partial\ddot{\sigma}}{\partial\sigma_m}-k_3\times\frac{\partial^2 N_1}{\partial\sigma_m^2}-\frac{\partial N_1}{\partial\sigma_m}\left[\frac{1}{3\ddot{\sigma}^2\cos 3\theta}-2k_3\times\frac{\partial\ddot{\sigma}}{\partial\sigma_m}\right]-$$

$$\left[\frac{1}{3\ddot{\sigma}^2\cos 3\theta}-2k_3\times\frac{\partial\ddot{\sigma}}{\partial\sigma_m}\right]\times\frac{\partial N_1}{\partial\sigma_m}-N_1\left[\begin{array}{l}-\dfrac{2}{3\ddot{\sigma}^3\cos 3\theta}\dfrac{\partial\ddot{\sigma}}{\partial\sigma_m}-\left\{\dfrac{2}{3\ddot{\sigma}^3\cos 3\theta}+6k_3\times\dfrac{\partial\ddot{\sigma}}{\partial\sigma_m}\right\}\times\\ \dfrac{\partial\ddot{\sigma}}{\partial\sigma_m}-2\times k_3\times\dfrac{\partial^2\ddot{\sigma}}{\partial\sigma_m^2}\end{array}\right]$$

$$k_{21}=\frac{\partial^2 C}{\partial\sigma_m^2}=\frac{2}{162\ddot{\sigma}^3\cos^2 3\theta}\times\frac{\partial\ddot{\sigma}}{\partial\sigma_m}-\left[2k_3\times\frac{\partial^2\ddot{\sigma}}{\partial\sigma_m^2}+\frac{\partial\ddot{\sigma}}{\partial\sigma_m}\left\{\frac{2}{162\ddot{\sigma}^3\cos^2 3\theta}-6k_3\frac{\partial\ddot{\sigma}}{\partial\sigma_m}\right\}\right]$$

$$k_{31}=\frac{-\sqrt{3}}{2\cos 3\theta\bar{\sigma}^3};\quad k_{32}=\frac{3\sqrt{3}J_3}{2\cos 3\theta\bar{\sigma}^4}$$

The expressions of *N1, N2, R, T, M, P1, P2, P3, P4, P5, P6* and *C* are mentioned in Chapter

6

(i) For $|\theta|<\theta_T$

$$\begin{aligned}\nabla^2 f &= \frac{\partial C_2}{\partial \boldsymbol{\sigma}}\frac{\partial \overline{\sigma}}{\partial \boldsymbol{\sigma}} + C_2\frac{\partial^2 \overline{\sigma}}{\partial \boldsymbol{\sigma}^2} + \frac{\partial C_3}{\partial \boldsymbol{\sigma}}\frac{\partial J_3}{\partial \boldsymbol{\sigma}} + C_3\frac{\partial^2 J_3}{\partial \boldsymbol{\sigma}^2} \\ &= (k_{11}+k_{12}k_{32})\frac{\partial \overline{\sigma}}{\partial \boldsymbol{\sigma}}\otimes\frac{\partial \overline{\sigma}}{\partial \boldsymbol{\sigma}} + k_{12}k_{31}\frac{\partial J_3}{\partial \boldsymbol{\sigma}}\otimes\frac{\partial \overline{\sigma}}{\partial \boldsymbol{\sigma}} + (k_{13}+k_{14}k_{32})\frac{\partial \overline{\sigma}}{\partial \boldsymbol{\sigma}}\otimes\frac{\partial J_3}{\partial \boldsymbol{\sigma}} + \\ &\quad k_{14}k_{31}\frac{\partial J_3}{\partial \boldsymbol{\sigma}}\otimes\frac{\partial J_3}{\partial \boldsymbol{\sigma}} \end{aligned} \tag{D.24}$$

(ii) For $|\theta|\geq\theta_T$

$$\begin{aligned}\nabla^2 f &= \frac{\partial C_1}{\partial \boldsymbol{\sigma}}\frac{\partial \sigma_m}{\partial \boldsymbol{\sigma}} + \frac{\partial C_2}{\partial \boldsymbol{\sigma}}\frac{\partial \overline{\sigma}}{\partial \boldsymbol{\sigma}} + C_2\frac{\partial^2 \overline{\sigma}}{\partial \boldsymbol{\sigma}^2} + \frac{\partial C_3}{\partial \boldsymbol{\sigma}}\frac{\partial J_3}{\partial \boldsymbol{\sigma}} + C_3\frac{\partial^2 J_3}{\partial \boldsymbol{\sigma}^2} \\ &= k_{22}\frac{\partial \sigma_m}{\partial \boldsymbol{\sigma}}\otimes\frac{\partial \sigma_m}{\partial \boldsymbol{\sigma}} + (k_{23}+k_{24}k_{33})\frac{\partial \overline{\sigma}}{\partial \boldsymbol{\sigma}}\otimes\frac{\partial \sigma_m}{\partial \boldsymbol{\sigma}} + k_{24}k_{31}\frac{\partial J_3}{\partial \boldsymbol{\sigma}}\otimes\frac{\partial \sigma_m}{\partial \boldsymbol{\sigma}} + C_2\frac{\partial^2 \overline{\sigma}}{\partial \boldsymbol{\sigma}^2} + C_3\frac{\partial^2 J_3}{\partial \boldsymbol{\sigma}^2} + \\ &\quad k_{25}\frac{\partial \sigma_m}{\partial \boldsymbol{\sigma}}\otimes\frac{\partial \overline{\sigma}}{\partial \boldsymbol{\sigma}} + (k_{26}+k_{27}k_{32})\frac{\partial \overline{\sigma}}{\partial \boldsymbol{\sigma}}\otimes\frac{\partial \overline{\sigma}}{\partial \boldsymbol{\sigma}} + k_{27}k_{31}\frac{\partial J_3}{\partial \boldsymbol{\sigma}}\otimes\frac{\partial \overline{\sigma}}{\partial \boldsymbol{\sigma}} + k_{28}\frac{\partial \sigma_m}{\partial \boldsymbol{\sigma}}\otimes\frac{\partial J_3}{\partial \boldsymbol{\sigma}} + \\ &\quad (k_{29}+k_{30}k_{32})\frac{\partial \overline{\sigma}}{\partial \boldsymbol{\sigma}}\otimes\frac{\partial J_3}{\partial \boldsymbol{\sigma}} + k_{30}k_{31}\frac{\partial J_3}{\partial \boldsymbol{\sigma}}\otimes\frac{\partial J_3}{\partial \boldsymbol{\sigma}} \end{aligned} \tag{D.25}$$

To avoid division by $\overline{\sigma}$, following equations were been used:

$$\overline{C}_3 = C_3\overline{\sigma}^2 \quad \text{and} \quad \overline{\frac{\partial J_3}{\partial \boldsymbol{\sigma}}} = \frac{\bar{1}}{\overline{\boldsymbol{\sigma}}^{2}}\frac{\partial J_3}{\partial \boldsymbol{\sigma}}$$

$$\overline{\frac{\partial^2 \overline{\sigma}}{\Im \sigma^2}} = \overline{\sigma}\frac{\partial^2 \overline{\sigma}}{\partial \sigma^2} \qquad \text{and} \qquad \overline{\frac{\partial^2 J_3}{\partial \boldsymbol{\sigma}^{2}}} = \frac{\bar{1}}{\overline{\boldsymbol{\sigma}}}\frac{\partial^2 J_3}{\partial \boldsymbol{\sigma}^{2}}$$

Appendix E

CRANK-NICOLSON FINITE DIFFERENCE IMPLICIT TECHNIQUE

E.1. Deriving the governing differential equation

Richards' equation: $\frac{\partial}{\partial y}\left[k(h_m)\left(\frac{\partial h_m}{\partial y}+1\right)\right]=\frac{\partial \theta}{\partial t}$ (E.1)

Gardener's one parameter HCF model: $k=k_s\exp(\alpha h_m)$ (E.2)

Gardener's SWRC model: $\Theta_n=S_e=\frac{\theta-\theta_r}{\theta_s-\theta_r}=\exp(\alpha h_m)$ (E.3)

Equation (E.3) can be further simplified as follows:

$$\theta=\theta_r+(\theta_s-\theta_r)\exp(\alpha h_m) \tag{E.4}$$

Performing partial differentiation of Equation (E.4) with respect to time (*t*), following expression can be obtained:

$$\frac{\partial \theta}{\partial t}=\alpha(\theta_s-\theta_r)\exp(\alpha h_m)\frac{\partial h_m}{\partial t} \tag{E.5}$$

Substituting Equation (E.2) and Equation (E.5) into Equation (E.1), following differential equation can be obtained:

$$k_s\left[\exp(\alpha h_m)\frac{\partial^2 h_m}{\partial y^2}+\alpha\exp(\alpha h_m)\left(\frac{\partial h_m}{\partial y}\right)^2\right]+\alpha k_s\exp(\alpha h_m)\frac{\partial h_m}{\partial y}=\alpha(\theta_s-\theta_r)\exp(\alpha h_m)\frac{\partial h_m}{\partial t} \tag{E.6}$$

$$\Rightarrow k_s\exp(\alpha h_m)\left[\left\{\frac{\partial^2 h_m}{\partial y^2}+\alpha\left(\frac{\partial h_m}{\partial y}\right)^2\right\}+\alpha\frac{\partial h_m}{\partial y}\right]=\alpha(\theta_s-\theta_r)\exp(\alpha h_m)\frac{\partial h_m}{\partial t} \tag{E.7}$$

$$\Rightarrow\left[\frac{\partial^2 h_m}{\partial y^2}+\alpha\left(\frac{\partial h_m}{\partial y}\right)^2\right]+\alpha\frac{\partial h_m}{\partial y}=\frac{\alpha(\theta_s-\theta_r)}{k_s}\frac{\partial h_m}{\partial t} \tag{E.8}$$

E.2. Numerical Scheme for solving the IVP

GD SWRC: $\left[\dfrac{\partial^2 h_m}{\partial y^2}+\alpha\left(\dfrac{\partial h_m}{\partial y}\right)\right]+\partial\dfrac{\partial h_m}{\partial y}=\lambda\dfrac{\partial h_m}{\partial t}$; $\lambda=\dfrac{\alpha(\theta_s-\theta_r)}{k_s}$

$$\left[(1-\theta)\frac{\partial^2 h_m}{\partial y^2}\bigg|_{t+\Delta t}+\theta\frac{\partial h_m}{\partial y^2}\bigg|_{t+\Delta t}+\alpha\left(\frac{\partial h_m}{\partial y}\bigg|_t\right)^2\right]+\alpha\frac{\partial h_m}{\partial y}\bigg|_t=\lambda\frac{\partial h_m}{\partial t} \tag{E.9}$$

In terms of finite difference terms, the generalized equations for any arbitrary i^{th} node can be written as:

$$\left[\frac{h_{m,i-1}^{t+\Delta t}-2h_{m,i}^{t+\Delta t}+h_{m,i+1}^{t+\Delta t}}{2\Delta y^2}+\frac{h_{m,i-1}^{t}-2h_{m,i}^{t}+h_{m,i+1}^{t}}{2\Delta y^2}+\alpha\left(\frac{h_{m,i+1}^{t}-h_{m,i+1}^{t}}{\Delta y}\right)^2\right]+\alpha\left(\frac{h_{m,i+1}^{t+\Delta t}-h_{m,i+1}^{t+\Delta t}}{\Delta y}\right)=\lambda\frac{h_{m,i}^{t+\Delta t}-h_{m,i}^{t}}{\Delta t} \tag{E.10}$$

$$\Rightarrow -m_1h_{m,i-1}^{t+\Delta t}+(1+2m_1+m_2)h_{m,i}^{t+\Delta t}-(m_1+m_2)h_{m,i+1}^{t+\Delta t}=h_{m,i}^{t}+m_1\left(h_{m,i-1}^{t}-2h_{m,i}^{t}+h_{m,i+1}^{t}\right)+\alpha m_1\left(h_{m,i+1}^{t}-h_{m,i}^{t}\right)^2$$

Here, $h_{m,i+1}^{t}, h_{m,i}^{t}$, and $h_{m,i-1}^{t}$ are the matric suction head at time t corresponding to i+1, i, and i-1 grid-points.

$$m_1=\frac{\Delta t}{2\lambda\Delta y^2}, m_2=\frac{\alpha\Delta t}{\lambda\Delta y};\ L_1=\frac{k_s}{\alpha(\theta_s-\theta_r)}$$

Top boundary constraint yields the following algebraic relation:

$$k_s\exp(\alpha h_m)\left(\frac{\partial h_m}{\partial y}+1\right)\bigg|_{@(n+1)^{th}\,node}=q$$

$$\Rightarrow\left(\frac{\partial h_m}{\partial y}\bigg|_{n+1}+1\right)=\frac{q}{k_s}\exp(-\alpha h_{m,n+1})$$

$$\Rightarrow\frac{\partial h_m}{\partial y}\bigg|_{n+1}=Q\exp(-\alpha h_{m,n+1})-1$$

$$\Rightarrow\frac{h_{m,n+2}^{t+\Delta t}-h_{m,n}^{t+\Delta t}}{2\Delta y}=Q\exp\left(-\alpha h_{m,n+1}^{t}\right)-1;\ Q\text{ (flow ratio)}=q/k_s\ ;$$

$$\Rightarrow h_{m,n+2}^{t+\Delta t} = h_{m,n}^{t+\Delta t} + 2\Delta y Q \exp(-\alpha h_{m,n+1}^{t}) - 2\Delta y \qquad \text{(E.11)}$$

The simultaneous set of linear equation can be rewritten as: $\boldsymbol{A}\boldsymbol{H}_m^{t+\Delta t} = \boldsymbol{B}^t$ (E.12)

$$\boldsymbol{H}_m^{t+\Delta t} = \left[h_{m,1}^{t+\Delta t} \quad h_{m,2}^{t+\Delta t} \quad \ldots \quad \ldots \quad h_{m,i-1}^{t+\Delta t} \quad h_{m,i}^{t+\Delta t} \quad h_{m,i+1}^{t+\Delta t} \quad \ldots \quad \ldots \quad h_{m,n}^{t+\Delta t} \quad h_{m,n+1}^{t+\Delta t} \right]_{1\times(n+1)}^{T}$$

$$\boldsymbol{B}^{\boldsymbol{t}} = \left[B_1 \quad B_2 \quad \ldots \quad \ldots \quad B_{i-1} \quad B_i \quad B_{i+1} \quad \ldots \quad \ldots \quad B_n \quad B_{n+1} \right]_{1\times(n+1)}^{T}$$

$$B_1 = 0; \quad B_i = h_{m,i}^{t} + m_1\left(h_{m,i-1}^{t} - 2h_{m,i}^{t} + h_{m,i+1}^{t}\right) + \alpha m_1\left(h_{m,i+1}^{t} - h_{m,i}^{t}\right)^2;$$

$$B_n = h_{m,n}^{t} + m_1\left(h_{m,n-1}^{t} - 2h_{m,n}^{t} + h_{m,n+1}^{t}\right) + \alpha m_1\left(h_{m,n+1}^{t} - h_{m,n}^{t}\right)^2$$

$$B_{n+1} = h_{m,n+1}^{t} + p\left(m_1 + m_2\right) + m_1\left(2h_{m,n}^{t} - 2h_{m,n+1}^{t} + p\right) + \alpha m_1\left[h_{m,n}^{t} + p - h_{m,n+1}^{t}\right]^2$$

$$p = 2\Delta y(Q\exp(-\alpha h_{m,n+1}^{t}) - 1)$$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -m_1 & \left(1+2m_1+m_2\right) & -\left(m_1+m_2\right) & 0 & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & -m_1 & \left(1+2m_1+m_2\right) & -\left(m_1+m_2\right) & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & -m_1 & \left(1+2m_1+m_2\right) & -\left(m_1+m_2\right) \\ 0 & 0 & 0 & 0 & 0 & -\left(2m_1+m_2\right) & \left(1+2m_1+m_2\right) \end{bmatrix}_{\left(\substack{n+1 \\ \times n+1}\right)}$$

Eq. (E.12) is further solved by employing Gauss Elimination scheme.

$$\begin{array}{ccc} \boldsymbol{A}\boldsymbol{H}_m^{t+\Delta t} & = & \boldsymbol{B}^t \\ \boldsymbol{A}\boldsymbol{H}_m^{t+2\Delta t} & = & \boldsymbol{B}^{t+\Delta t} \\ \ldots & \Bigg\downarrow & \ldots \\ \ldots & & \ldots \\ \ldots & & \ldots \\ \boldsymbol{A}\boldsymbol{H}_m^{t+n\Delta t} & = & \boldsymbol{B}^{t+(n-1)\Delta t} \end{array}$$