Chapter 5

Fixed time synchronization of octonion valued neural networks with time varying delays

5.1 Introduction

In this chapter, some more dynamics like fixed time synchronization (FTS) of OVNNs have been studied. Synchronization of the chaotic systems has became a fascinating topic since the pioneering work of Pecora and Carroll [54] in 1990. It has many applications in image encryption [34], secure communication [55], cryptography [56] etc. There are different kinds of synchronization with different controllers discussed in [57, 58, 59, 60, 61, 62]. In FTS, the synchronization problem of the non-linear systems can be converted into the stability problem of error system, and by using some fixed time stability results one can investigate the FTS [63, 64, 65, 66]. So far most of the researchers have studied the FTS in high dimensional models up to QVNNs, the FTS problems are not yet been studied on OVNNs and therefore, it is imperative to explore the FTS problem for OVNNs.

5.2 Problem Formulation and Preliminaries

In this section the following OVNNs model with time-varying delay is introduced as

$$\dot{p}_i(t) = -d_i p_i(t) + \sum_{j=1}^{n} a_{ij} g_j(p_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(p_j(t-\tau_0(t))) + k_i, \quad i = 1, 2, ..., n, \tag{5.1}$$

where n is the number of neurons in the network; $p_i(t) \in \mathbb{O}$ represents the state of i-th neuron at time t; $d_i \in \mathbb{R}$ with $d_i > 0$ is state decaying constant; $a_{ij}, b_{ij} \in \mathbb{O}$ represent instantaneous and delayed synaptic connection strengths from j-th to i-th neuron of the network; $\tau_0(t) \geq 0$ be time-varying delay; $k_i \in \mathbb{O}$ is an external input to each neuron of the network; the mapping $g_j(\cdot) : \mathbb{O} \to \mathbb{O}$ denotes the octonion-valued activation functions for each $j = 1, 2, ..., n$. Let us define the activation functions as

$$g_j(p_j(t)) = \sum_{b=0}^{7} g_j^{(b)}(p_j^{(b)}(t))\omega_b, \ \Big(g_j(p_j(t)) - g_j(q_j(t))\Big)^{(b)} = \Big(g_j^{(b)}(p_j^{(b)}(t)) - g_j^{(b)}(q_j^{(b)}(t))\Big), \tag{5.2}$$

where, w_b are octonion units and $g_j^{(b)}(\cdot)$ are real valued functions, $b = 0, 1, 2, ..., 7$.

Assumption 5.1. For all $j = 1, 2, ..., n$, the activation functions $g_j^{(b)}(\cdot)$ satisfy the following inequalites:

$$|g_j^{(b)}(x) - g_j^{(b)}(y)| \leq M_j |x - y|, \ \forall \ x, y \in \mathbf{R},$$

where $b = 0, 1, ..., 7$, and M_j's are constants.

Remark 5.1. The activation functions are important factors those affect the dynamical behaviour of the designed NNs. From Assumption (5.1), since the activation functions are continuous and satisfy Lipschitz condition then there exists a unique solution of the system (5.1) (see [67]). Here one can use sigmoid function, piece-wise linear function and hyperbolic tangent function as the activation functions.

Now, by using multiplication between octonions, the system (5.1) can be separated into eight real-valued system of equations as

$$\begin{aligned}
\dot{p_i}^{(0)}(t) = &- d_i p_i^{(0)}(t) + \sum_{j=1}^{n}\Big[a_{ij}^{(0)} g_j^{(0)}(p_j^{(0)}(t)) - a_{ij}^{(1)} g_j^{(1)}(p_j^{(1)}(t)) - a_{ij}^{(2)} g_j^{(2)}(p_j^{(2)}(t)) - a_{ij}^{(3)} g_j^{(3)}(p_j^{(3)}(t)) \\
&\qquad - a_{ij}^{(4)} g_j^{(4)}(p_j^{(4)}(t)) - a_{ij}^{(5)} g_j^{(5)}(p_j^{(5)}(t)) - a_{ij}^{(6)} g_j^{(6)}(p_j^{(6)}(t)) - a_{ij}^{(7)} g_j^{(7)}(p_j^{(7)}(t))\Big] \\
&+ \sum_{j=1}^{n}\Big[b_{ij}^{(0)} g_j^{(0)}(p_j^{(0)}(t-\tau_0(t))) - b_{ij}^{(1)} g_j^{(1)}(p_j^{(1)}(t-\tau_0(t))) - b_{ij}^{(2)} g_j^{(2)}(p_j^{(2)}(t-\tau_0(t))) - \\
&\quad b_{ij}^{(3)} g_j^{(3)}(p_j^{(3)}(t-\tau_0(t))) - b_{ij}^{(4)} g_j^{(4)}(p_j^{(4)}(t-\tau_0(t))) - b_{ij}^{(5)} g_j^{(5)}(p_j^{(5)}(t-\tau_0(t))) - \\
&\quad b_{ij}^{(6)} g_j^{(6)}(p_j^{(6)}(t-\tau_0(t))) - b_{ij}^{(7)} g_j^{(7)}(p_j^{(7)}(t-\tau_0(t)))\Big] + k_i^{(0)}.
\end{aligned}$$

With the aid of equation (1.17), we can express $\dot{p_i}^{(0)}(t)$ as

$$\dot{p_i}^{(0)}(t) = -d_i p_i^{(0)}(t) + \sum_{j=1}^{n}(a_{ij} g_j(p_j(t)))^{(0)} + \sum_{j=1}^{n}(b_{ij} g_j(p_j(t-\tau_0)))^{(0)} + k_i^{(0)}.$$

Therefore, the eight real valued systems of equations can be written as

$$\dot{p_i}^{(b)}(t) = -d_i p_i^{(b)}(t) + \sum_{j=1}^{n}(a_{ij} g_j(p_j(t)))^{(b)} + \sum_{j=1}^{n}(b_{ij} g_j(p_j(t-\tau_0)))^{(b)} + k_i^{(b)},$$

$$b = 0, 1, 2, ..., 7. \qquad (5.3)$$

The response system corresponding to drive system (5.3) is given by

$$\dot{s_i}^{(b)}(t) = -d_i s_i^{(b)}(t) + \sum_{j=1}^{n} (a_{ij} g_j(s_j(t)))^{(b)} + \sum_{j=1}^{n} (b_{ij} g_j(s_j(t-\tau_0)))^{(b)} + k_i^{(b)} + u_i^{(b)}(t), \qquad b = 0, 1, 2, ..., 7, \tag{5.4}$$

where $u_i^{(b)}(t)$ are the controllers that will be defined later on. Let $e(t) = s(t) - p(t)$ is the error function, where $e(t) = (e_1(t), e_2(t), ..., e_n(t)) \in \mathbb{O}^n$. By subtracting equation (5.3) from equation (5.4), we get the following system of equations as

$$\begin{aligned}
\dot{e_i}^{(0)}(t) =& -d_i e_i^{(0)}(t) + \sum_{j=1}^{n} \Big[a_{ij}^{(0)}(g_j^{(0)}(s_j^{(0)}(t)) - g_j^{(0)}(p_j^{(0)}(t))) - a_{ij}^{(1)}(g_j^{(1)}(s_j^{(1)}(t)) - g_j^{(1)}(p_j^{(1)}(t))) - \\
& a_{ij}^{(2)}(g_j^{(2)}(s_j^{(2)}(t)) - g_j^{(2)}(p_j^{(2)}(t))) - a_{ij}^{(3)}(g_j^{(3)}(s_j^{(3)}(t)) - (g_j^{(3)}(p_j^{(3)}(t))) - a_{ij}^{(4)}(g_j^{(4)}(s_j^{(4)}(t)) - \\
& (g_j^{(4)}(p_j^{(4)}(t))) - a_{ij}^{(5)}(g_j^{(5)}(s_j^{(5)}(t)) - g_j^{(5)}(p_j^{(5)}(t))) - a_{ij}^{(6)}(g_j^{(6)}(s_j^{(6)}(t)) - g_j^{(6)}(p_j^{(6)}(t))) - \\
& a_{ij}^{(7)}(g_j^{(7)}(s_j^{(7)}(t)) - g_j^{(7)}(p_j^{(7)}(t))) \Big] + \sum_{j=1}^{n} \Big[b_{ij}^{(0)}(g_j^{(0)}(s_j^{(0)}(t-\tau_0(t))) - g_j^{(0)}(p_j^{(0)}(t-\tau_0(t)))) - \\
& b_{ij}^{(1)}(g_j^{(1)}(s_j^{(1)}(t-\tau_0(t))) - g_j^{(1)}(p_j^{(1)}(t-\tau_0(t)))) - b_{ij}^{(2)}(g_j^{(2)}(s_j^{(2)}(t-\tau_0(t))) - \\
& g_j^{(2)}(p_j^{(2)}(t-\tau_0(t)))) - b_{ij}^{(3)}(g_j^{(3)}(s_j^{(3)}(t-\tau_0(t))) - g_j^{(3)}(p_j^{(3)}(t-\tau_0(t)))) - \\
& b_{ij}^{(4)}(g_j^{(4)}(s_j^{(4)}(t-\tau_0(t))) - (g_j^{(4)}(p_j^{(4)}(t-\tau_0(t)))) - b_{ij}^{(5)}(g_j^{(5)}(s_j^{(5)}(t-\tau_0(t))) - \\
& g_j^{(5)}(p_j^{(5)}(t-\tau_0(t)))) - b_{ij}^{(6)}(g_j^{(6)}(s_j^{(6)}(t-\tau_0(t))) - g_j^{(6)}(s_j^{(6)}(t-\tau_0(t)))) - \\
& b_{ij}^{(7)}(g_j^{(7)}(s_j^{(7)}(t-\tau_0(t))) - g_j^{(7)}(p_j^{(7)}(t-\tau_0(t)))) \Big] + u_i^{(0)}(t).
\end{aligned}$$

By using equations (1.17) and (5.2), the above $\dot{e_i}^{(0)}(t)$ can be written as

$$\begin{aligned}
\dot{e_i}^{(0)}(t) =& -d_i e_i^{(0)}(t) + \sum_{j=1}^{n} \Big[a_{ij}(g_j(s(t)) - g_j(p(t))) \Big]^{(0)} + \\
& \sum_{j=1}^{n} \Big[b_{ij}(g_j(s(t-\tau_0(t))) - g_j(p(t-\tau_0(t)))) \Big]^{(0)} + u_i^{(0)}(t).
\end{aligned}$$

In the similar way, all the eight equations of the error system can be expressed as

$$\begin{aligned} \dot{e}_i{}^{(b)}(t) = &- d_i e_i^{(b)}(t) + \sum_{j=1}^{n} \Big[a_{ij}(g_j(s(t)) - g_j(p(t))) \Big]^{(b)} + \\ &\sum_{j=1}^{n} \Big[b_{ij}(g_j(s(t-\tau_0(t))) - g_j(p(t-\tau_0(t)))) \Big]^{(b)} + u_i^{(b)}(t), \end{aligned} \tag{5.5}$$

where $b = 0, 1, 2, ..., 7$. The target here is to synchronize the drive-response systems (5.3) and (5.4) in fixed time. This is same as stabilizing the corresponding error system (5.5) in fixed time.

Remark 5.2. Since there are sufficient criteria for the fixed time stability of the system, by using those criteria on error system, one can investigate the synchronization of drive-response systems at a fixed time.

Lemma 5.1. [28, 68] If there is a radially unbounded and continuous function $V(\cdot) : \mathbf{R}^n \to [0, \infty)$ and $e(t)$ is any solution of system (5.5), such that

1. $V(e(t)) = 0$ iff $e(t) = 0$;
2. for some $\lambda, \mu > 0$, $\alpha \in (0, 1)$ and $\beta \in (1, +\infty)$, $e(t)$ satisfies

   $\dot{V}(t) \leq -\lambda V^{\alpha}(e(t)) - \mu V^{\beta}(e(t))$;

then the origin of the system (5.5) is fixed time stable. Moreover, the following estimate admits $V(t) = 0, t \geq T(e_0)$, with the settling time bound by $T(e_0) \leq \frac{1}{\lambda(1-\alpha)} + \frac{1}{\mu(\beta-1)}, \forall\ e_0 \in \mathbf{R}^n$.

Lemma 5.2. [69] For $i = 1, 2, .., n$, let w_i be any non-negative real number, $\mathrm{p} \in (0, 1]$ and $\mathrm{q} \in (1, +\infty)$, then

$\sum_{i=1}^{n} \mathrm{w}_i^{\mathrm{p}} \geq (\sum_{i=1}^{n} \mathrm{w}_i)^{\mathrm{p}}, \quad \sum_{i=1}^{n} \mathrm{w}_i^{\mathrm{q}} \geq n^{1-\mathrm{q}}(\sum_{i=1}^{n} \mathrm{w}_i)^{\mathrm{q}}.$

Remark 5.3. The main distinction between finite-time stability and fixed-time stability, as defined in subsections 1.5.7 and 1.5.8, is whether the settling time is independent of the initial value. Definitely, in the fixed time stability the settling time is independent of the initial value.

5.3 Main results

In this section, some sufficient criteria are derived by designing the suitable non-linear controllers to reach the FTS of the drive-response systems (5.3) and (5.4). Now for $i = 1, 2, ..., n$, the controllers may be designed as

$$u_i^{(b)}(t) = -\lambda_{1i} e_i^{(b)}(t) - sign(e_i^{(b)}(t))\big(\lambda_{2i}|e_i^{(b)}(t-\tau_0(t))| + \lambda_{3i}|e_i^{(b)}(t)|^{\alpha} + \lambda_{4i}|e_i^{(b)}(t)|^{\beta}\big), \quad b = 0, 1, ..., 7, \tag{5.6}$$

where, $0 < \alpha < 1, \beta > 1$, $\lambda_{1i} > 0$, λ_{2i}, λ_{3i}, λ_{4i} are the parameters to be designed later.

Remark 5.4. It is obvious that the controllers of this chapter are different from those have been designed in [68, 70]. The controllers (5.6) have two parts, the first part is linear and the other is non-linear. Since the non-linear terms in the non-linear part play such a large influence in synchronization speed, the appropriate coefficients can be properly set based on the actual demand. Moreover, delay terms are also included in the non-linear part of the controllers (5.6), which indicate the delay in synchronization. In [68], the authors have investigated the FTS of QVNNs in which controllers are independent of time-delay terms, therefore the designed controllers (5.6) are more general than those in [68]. In [70], the preassigned-time bipartite synchronization of complex networks with quantized coupling and stochastic perturbations have been investigated without the use of linear term in the controllers.

Theorem 5.1. If the Assumption 5.1 holds, the drive-response systems (5.3) and (5.4) achieve FTS with controllers defined in (5.6), if the following inequalities hold.

$$\begin{aligned} & d_i + \lambda_{1i} - \sum_{j=1}^{n}(|a_{ji}^{(0)}| + |a_{ji}^{(1)}| + ...+|a_{ji}^{(7)}|)M_i > 0, \\ & \lambda_{2i} - \sum_{j=1}^{n}(|b_{ji}^{(0)}| + |b_{ji}^{(1)}| + ... + |b_{ji}^{(7)}|)M_i > 0, \\ & \min_i \lambda_{3i} > 0 \text{ and } \min_i(\lambda_{4i})(8n)^{1-\beta} > 0. \end{aligned} \tag{5.7}$$

Moreover, the estimated settling time T_{set} is given by

$$T_{set} \leq T(e_0) \leq \frac{1}{\min_i(\lambda_{3i})(1-\alpha)} + \frac{1}{\min_i(\lambda_{4i})(8n)^{1-\beta}(\beta-1)}, \quad i = 1, 2, ..., n. \tag{5.8}$$

Proof. Let us consider the Lyapunov function as

$$V(t) = \sum_{b=0}^{7} V^{(b)}(t), \quad where \quad V^{(b)}(t) = \sum_{i=1}^{n}|e_i^{(b)}(t)|, \ b = 0, 1, ..., 7.$$

Let us calculate the derivative of $V^{(0)}(t)$ as

$$
\begin{aligned}
&\dot{V}^{(0)}(t)=\sum_{i=1}^{n} sign(e_i^{(0)}(t))\dot{e}_i^{(0)}(t)\\
&=\sum_{i=1}^{n} sign(e_i^{(0)}(t))\bigg(-d_ie_i^{(0)}(t)+\sum_{j=1}^{n}\bigg[a_{ij}^{(0)}(g_j^{(0)}(s_j^{(0)}(t))-g_j^{(0)}(p_j^{(0)}(t)))-a_{ij}^{(1)}(g_j^{(1)}(s_j^{(1)}(t))-\\
&\quad g_j^{(1)}(p_j^{(1)}(t)))-a_{ij}^{(2)}(g_j^{(2)}(s_j^{(2)}(t))-g_j^{(2)}(p_j^{(2)}(t)))-a_{ij}^{(3)}(g_j^{(3)}(s_j^{(3)}(t))-(g_j^{(3)}(p_j^{(3)}(t)))-\\
&\quad a_{ij}^{(4)}(g_j^{(4)}(s_j^{(4)}(t))-(g_j^{(4)}(p_j^{(4)}(t)))-a_{ij}^{(5)}(g_j^{(5)}(s_j^{(5)}(t))-g_j^{(5)}(p_j^{(5)}(t)))-a_{ij}^{(6)}(g_j^{(6)}(s_j^{(6)}(t))-\\
&\quad g_j^{(6)}(p_j^{(6)}(t)))-a_{ij}^{(7)}(g_j^{(7)}(s_j^{(7)}(t))-g_j^{(7)}(p_j^{(7)}(t)))\bigg]+\sum_{j=1}^{n}\bigg[b_{ij}^{(0)}(g_j^{(0)}(s_j^{(0)}(t-\tau_0(t)))-\\
&\quad g_j^{(0)}(p_j^{(0)}(t-\tau_0(t))))-b_{ij}^{(1)}(g_j^{(1)}(s_j^{(1)}(t-\tau_0(t)))-g_j^{(1)}(p_j^{(1)}(t-\tau_0(t))))-b_{ij}^{(2)}(g_j^{(2)}(s_j^{(2)}(t-\tau_0(t)))\\
&\quad -g_j^{(2)}(p_j^{(2)}(t-\tau_0(t))))-b_{ij}^{(3)}(g_j^{(3)}(s_j^{(3)}(t-\tau_0(t)))-g_j^{(3)}(p_j^{(3)}(t-\tau_0(t))))-b_{ij}^{(4)}\\
&\quad \times(g_j^{(4)}(s_j^{(4)}(t-\tau_0(t)))-(g_j^{(4)}(p_j^{(4)}(t-\tau_0(t))))-b_{ij}^{(5)}(g_j^{(5)}(s_j^{(5)}(t-\tau_0(t)))-g_j^{(5)}(p_j^{(5)}(t-\tau_0(t))))\\
&\quad -b_{ij}^{(6)}(g_j^{(6)}(s_j^{(6)}(t-\tau_0(t)))-g_j^{(6)}(s_j^{(6)}(t-\tau_0(t))))-b_{ij}^{(7)}(g_j^{(7)}(s_j^{(7)}(t-\tau_0(t)))-\\
&\quad g_j^{(7)}(p_j^{(7)}(t-\tau_0(t))))\bigg]+u_i^{(0)}(t)\bigg)\\
&\leq-\sum_{i=1}^{n}d_i|e_i^{(0)}(t)|+\sum_{i=1}^{n}\sum_{j=1}^{n}\bigg(|a_{ij}^{(0)}|M_j|e_j^{(0)}(t)|+|a_{ij}^{(1)}|M_j|e_j^{(1)}(t)|+|a_{ij}^{(2)}|M_j|e_j^{(2)}(t)|+|a_{ij}^{(3)}|M_j\\
&\quad \times|e_j^{(3)}(t)|+|a_{ij}^{(4)}|M_j|e_j^{(4)}(t)|+|a_{ij}^{(5)}|M_j|e_j^{(5)}(t)|+|a_{ij}^{(6)}|M_j|e_j^{(6)}(t)|+|a_{ij}^{(7)}|M_j|e_j^{(7)}(t)|+|b_{ij}^{(0)}|M_j\\
&\quad \times|e_j^{(0)}(t-\tau_0(t))|+|b_{ij}^{(1)}|M_j|e_j^{(1)}(t-\tau_0(t))|+|b_{ij}^{(2)}|M_j|e_j^{(2)}(t-\tau_0(t))|+|b_{ij}^{(3)}|M_j|e_j^{(3)}(t-\tau_0(t))|+\\
&\quad |b_{ij}^{(4)}|M_j|e_j^{(4)}(t-\tau_0(t))|+|b_{ij}^{(5)}|M_j|e_j^{(5)}(t-\tau_0(t))|+|b_{ij}^{(6)}|M_j|e_j^{(6)}(t-\tau_0(t))|+|b_{ij}^{(7)}|M_j\\
&\quad \times|e_j^{(7)}(t-\tau_0(t))|\bigg)+\sum_{i=1}^{n}sign(e_i^{(0)}(t))\bigg(-\lambda_{1i}e_i^{(0)}(t)-sign(e_i^{(0)}(t))\big(\lambda_{2i}|e_i^{(0)}(t-\tau_0(t))|+\\
&\quad \lambda_{3i}|e_i^{(0)}(t)|^{\alpha}+\lambda_{4i}|e_i^{(0)}(t)|^{\beta}\big)\bigg)\\
&\leq-\sum_{i=1}^{n}\bigg(d_i-\sum_{j=1}^{n}|a_{ji}^{(0)}|M_i+\lambda_{1i}\bigg)|e_i^{(0)}(t)|+\sum_{i=1}^{n}\sum_{j=1}^{n}\bigg(|a_{ji}^{(1)}|M_i|e_i^{(1)}(t)|+|a_{ji}^{(2)}|M_i|e_i^{(2)}(t)|+|a_{ji}^{(3)}|M_i\\
&\quad \times|e_i^{(3)}(t)|+|a_{ji}^{(4)}|M_i|e_i^{(4)}(t)|+|a_{ji}^{(5)}|M_i|e_i^{(5)}(t)|+|a_{ji}^{(6)}|M_i|e_i^{(6)}(t)|+|a_{ji}^{(7)}|M_i|e_i^{(7)}(t)|\\
&\quad +|b_{ji}^{(0)}|M_i|e_i^{(0)}(t-\tau_0(t))|+|b_{ji}^{(1)}|M_i|e_i^{(1)}(t-\tau_0(t))|+|b_{ji}^{(2)}|M_i|e_i^{(2)}(t-\tau_0(t))|+|b_{ji}^{(3)}|M_i\\
&\quad \times|e_i^{(3)}(t-\tau_0(t))|+|b_{ji}^{(4)}|M_i|e_i^{(4)}(t-\tau_0(t))|+|b_{ji}^{(5)}|M_i|e_i^{(5)}(t-\tau_0(t))|+|b_{ji}^{(6)}|M_i|e_i^{(6)}(t-\tau_0(t))|\\
&\quad +|b_{ji}^{(7)}|M_i|e_i^{(7)}(t-\tau_0(t))|-\sum_{i=1}^{n}\lambda_{2i}|e_i^{(0)}(t-\tau_0(t))|\bigg)-\sum_{i=1}^{n}\big(\lambda_{3i}|e_i^{(0)}(t)|^{\alpha}+\lambda_{4i}|e_i^{(0)}(t)|^{\beta}\big).
\end{aligned}
$$

Similarly for $b = 1, 2, ..., 7$, the derivatives of $V^{(b)}$ can be defined as

$$\begin{aligned}\dot{V}^{(1)}(t) \leq &-\sum_{i=1}^{n}\Big(d_i - \sum_{j=1}^{n}|a_{ji}^{(0)}|M_i + \lambda_{1i}\Big)|e_i^{(1)}(t)| + \sum_{i=1}^{n}\sum_{j=1}^{n}\Big(|a_{ji}^{(1)}|M_i|e_i^{(0)}(t)| + |a_{ji}^{(2)}|M_i|e_i^{(3)}(t)|+\\ &|a_{ji}^{(3)}|M_i|e_i^{(2)}(t)| + |a_{ji}^{(4)}|M_i|e_i^{(5)}(t)| + |a_{ji}^{(5)}|M_i|e_i^{(4)}(t)| + |a_{ji}^{(6)}|M_i|e_i^{(7)}(t)| + |a_{ji}^{(7)}|M_i|e_i^{(6)}(t)|\\ &+ |b_{ji}^{(0)}|M_i|e_i^{(1)}(t-\tau_0(t))| + |b_{ji}^{(1)}|M_i|e_i^{(0)}(t-\tau_0(t))| + |b_{ji}^{(2)}|M_i|e_i^{(3)}(t-\tau_0(t))| + |b_{ji}^{(3)}|M_i|\\ &\times e_i^{(2)}(t-\tau_0(t))| + |b_{ji}^{(4)}|M_i|e_i^{(5)}(t-\tau_0(t))| + |b_{ji}^{(5)}|M_i|e_i^{(4)}(t-\tau_0(t))| + |b_{ji}^{(6)}|M_i|\\ &\times e_i^{(7)}(t-\tau_0(t))| + |b_{ji}^{(7)}|M_i|e_i^{(6)}(t-\tau_0(t))|\Big) - \sum_{i=1}^{n}\lambda_{2i}|e_i^{(1)}(t-\tau_0(t))| -\\ &\sum_{i=1}^{n}\big(\lambda_{3i}|e_i^{(1)}(t)|^{\alpha} + \lambda_{4i}|e_i^{(1)}(t)|^{\beta}\big),\end{aligned}$$

$$\begin{aligned}\dot{V}^{(2)}(t) \leq &-\sum_{i=1}^{n}\Big(d_i - \sum_{j=1}^{n}|a_{ji}^{(0)}|M_i + \lambda_{1i}\Big)|e_i^{(2)}(t)| + \sum_{i=1}^{n}\sum_{j=1}^{n}\Big(|a_{ji}^{(2)}|M_i|e_i^{(0)}(t)| + |a_{ji}^{(1)}|M_i|e_i^{(3)}(t)|+\\ &|a_{ji}^{(3)}|M_i|e_i^{(1)}(t)| + |a_{ji}^{(4)}|M_i|e_i^{(6)}(t)| + |a_{ji}^{(6)}|M_i|e_i^{(4)}(t)| + |a_{ji}^{(5)}|M_i|e_i^{(7)}(t)| + |a_{ji}^{(7)}|M_i|e_i^{(5)}(t)|\\ &+ |b_{ji}^{(0)}|M_i|e_i^{(2)}(t-\tau_0(t))| + |b_{ji}^{(2)}|M_i|e_i^{(0)}(t-\tau_0(t))| + |b_{ji}^{(1)}|M_i|e_i^{(3)}(t-\tau_0(t))| + |b_{ji}^{(3)}|M_i\\ &\times |e_i^{(1)}(t-\tau_0(t))| + |b_{ji}^{(4)}|M_i|e_i^{(6)}(t-\tau_0(t))| + |b_{ji}^{(6)}|M_i|e_i^{(4)}(t-\tau_0(t))| + |b_{ji}^{(5)}|M_i\\ &\times |e_i^{(7)}(t-\tau_0(t))| + |b_{ji}^{(7)}|M_i^{(5)}|e_i^{(5)}(t-\tau_0(t))|\Big) - \sum_{i=1}^{n}\lambda_{2i}|e_i^{(2)}(t-\tau_0(t))| -\\ &\sum_{i=1}^{n}\big(\lambda_{3i}|e_i^{(2)}(t)|^{\alpha} + \lambda_{4i}|e_i^{(2)}(t)|^{\beta}\big),\end{aligned}$$

$$\begin{aligned}\dot{V}^{(3)}(t) \leq &-\sum_{i=1}^{n}\Big(d_i - \sum_{j=1}^{n}|a_{ji}^{(0)}|M_i + \lambda_{1i}\Big)|e_i^{(3)}(t)| + \sum_{i=1}^{n}\sum_{j=1}^{n}\Big(|a_{ji}^{(3)}|M_i|e_i^{(0)}(t)| + |a_{ji}^{(1)}|M_i|e_i^{(2)}(t)|+\\ &|a_{ji}^{(2)}|M_i|e_i^{(1)}(t)| + |a_{ji}^{(4)}|M_i|e_i^{(7)}(t)| + |a_{ji}^{(7)}|M_i|e_i^{(4)}(t)| + |a_{ji}^{(5)}|M_i|e_i^{(6)}(t)| + |a_{ji}^{(6)}|M_i|e_i^{(5)}(t)|\\ &+ |b_{ji}^{(0)}|M_i|e_i^{(3)}(t-\tau_0(t))| + |b_{ji}^{(3)}|M_i|e_i^{(0)}(t-\tau_0(t))| + |b_{ji}^{(1)}|M_i|e_i^{(2)}(t-\tau_0(t))| + |b_{ji}^{(2)}|M_i|\\ &\times e_i^{(1)}(t-\tau_0(t))| + |b_{ji}^{(4)}|M_i|e_i^{(7)}(t-\tau_0(t))| + |b_{ji}^{(7)}|M_i|e_i^{(4)}(t-\tau_0(t))| + |b_{ji}^{(5)}|M_i|\\ &\times e_i^{(6)}(t-\tau_0(t))| + |b_{ji}^{(6)}|M_i|e_i^{(5)}(t-\tau_0(t))|\Big) - \sum_{i=1}^{n}\lambda_{2i}|e_i^{(2)}(t-\tau_0(t))| -\\ &\sum_{i=1}^{n}\big(\lambda_{3i}|e_i^{(3)}(t)|^{\alpha} + \lambda_{4i}|e_i^{(3)}(t)|^{\beta}\big),\end{aligned}$$

$$
\begin{aligned}
\dot{V}^{(4)}(t) \leq & -\sum_{i=1}^{n}\Bigg(d_i-\sum_{j=1}^{n}|a_{ji}^{(0)}|M_i+\lambda_{1i}\Bigg)|e_i^{(4)}(t)|+\sum_{i=1}^{n}\sum_{j=1}^{n}\Bigg(|a_{ji}^{(4)}|M_i|e_i^{(0)}(t)|+|a_{ji}^{(1)}|M_i|e_i^{(5)}(t)|+\\
& |a_{ji}^{(5)}|M_i|e_i^{(1)}(t)|+|a_{ji}^{(2)}|M_i|e_i^{(6)}(t)|+|a_{ji}^{(6)}|M_i|e_i^{(2)}(t)|+|a_{ji}^{(3)}|M_i|e_i^{(7)}(t)|+|a_{ji}^{(7)}|M_i|e_i^{(3)}(t)|\\
& +|b_{ji}^{(0)}|M_i|e_i^{(4)}(t-\tau_0(t))|+|b_{ji}^{(4)}|M_i|e_i^{(0)}(t-\tau_0(t))|+|b_{ji}^{(1)}|M_i|e_i^{(5)}(t-\tau_0(t))|+|b_{ji}^{(5)}|M_i\\
& \times|e_i^{(1)}(t-\tau_0(t))|+|b_{ji}^{(2)}|M_i|e_i^{(6)}(t-\tau_0(t))|+|b_{ji}^{(6)}|M_i|e_i^{(2)}(t-\tau_0(t))|+|b_{ji}^{(3)}|M_i|\\
& \times e_i^{(7)}(t-\tau_0(t))|+|b_{ji}^{(7)}|M_i|e_i^{(3)}(t-\tau_0(t))|\Bigg)-\sum_{i=1}^{n}\lambda_{2i}|e_i^{(4)}(t-\tau_0(t))|-\\
& \sum_{i=1}^{n}\big(\lambda_{3i}|e_i^{(4)}(t)|^{\alpha}+\lambda_{4i}|e_i^{(4)}(t)|^{\beta}\big),
\end{aligned}
$$

$$
\begin{aligned}
\dot{V}^{(5)}(t) \leq & -\sum_{i=1}^{n}\Bigg(d_i-\sum_{j=1}^{n}|a_{ji}^{(0)}|M_i+\lambda_{1i}\Bigg)|e_i^{(5)}(t)|+\sum_{i=1}^{n}\sum_{j=1}^{n}\Bigg(|a_{ji}^{(5)}|M_i|e_i^{(0)}|+|a_{ji}^{(1)}|M_i|e_i^{(4)}(t)|+\\
& |a_{ji}^{(4)}|M_i|e_i^{(1)}(t)|+|a_{ji}^{(2)}|M_i|e_i^{(7)}(t)|+|a_{ji}^{(7)}|M_i|e_i^{(2)}(t)|+|a_{ji}^{(3)}|M_i|e_i^{(6)}(t)|+|a_{ji}^{(6)}|M_i|e_i^{(3)}(t)|\\
& +|b_{ji}^{(0)}|M_i|e_i^{(5)}(t-\tau_0(t))|+|b_{ji}^{(5)}|M_i|e_i^{(0)}(t-\tau_0(t))|+|b_{ji}^{(1)}|M_i|e_i^{(4)}(t-\tau_0(t))|+|b_{ji}^{(4)}|M_i|\\
& \times e_i^{(1)}(t-\tau_0(t))|+|b_{ji}^{(2)}|M_i|e_i^{(7)}(t-\tau_0(t))|+|b_{ji}^{(7)}|M_i|e_i^{(2)}(t-\tau_0(t))|+|b_{ji}^{(3)}|M_i\\
& \times|e_i^{(6)}(t-\tau_0(t))|+|b_{ji}^{(6)}|M_i|e_i^{(3)}(t-\tau_0(t))|\Bigg)-\sum_{i=1}^{n}\lambda_{2i}|e_i^{(5)}(t-\tau_0(t))|-\\
& \sum_{i=1}^{n}\big(\lambda_{3i}|e_i^{(5)}(t)|^{\alpha}+\lambda_{4i}|e_i^{(5)}(t)|^{\beta}\big),
\end{aligned}
$$

$$
\begin{aligned}
\dot{V}^{(6)}(t) \leq & -\sum_{i=1}^{n}\Bigg(d_i-\sum_{j=1}^{n}|a_{ji}^{(0)}|M_i+\lambda_{1i}\Bigg)|e_i^{(6)}(t)|+\sum_{i=1}^{n}\sum_{j=1}^{n}\Bigg(|a_{ji}^{(6)}|M_i|e_i^{(0)}(t)|+|a_{ji}^{(2)}|M_i|e_i^{(4)}(t)|+\\
& |a_{ji}^{(4)}|M_i|e_i^{(2)}(t)|+|a_{ji}^{(1)}|M_i|e_i^{(7)}(t)|+|a_{ji}^{(7)}|M_i|e_i^{(1)}(t)|+|a_{ji}^{(3)}|M_i|e_i^{(5)}(t)|+|a_{ji}^{(5)}|M_i|e_i^{(3)}(t)|\\
& +|b_{ji}^{(0)}|M_i|e_i^{(6)}(t-\tau_0(t))|+|b_{ji}^{(6)}|M_i|e_i^{(0)}(t-\tau_0(t))|+|b_{ji}^{(2)}|M_i|e_i^{(4)}(t-\tau_0(t))|+|b_{ji}^{(4)}|M_i\\
& \times|e_i^{(2)}(t-\tau_0(t))|+|b_{ji}^{(1)}|M_i|e_i^{(7)}(t-\tau_0(t))|+|b_{ji}^{(7)}|M_i|e_i^{(1)}(t-\tau_0(t))|+|b_{ji}^{(3)}|\\
& \times M_i|e_i^{(5)}(t-\tau_0(t))|+|b_{ji}^{(5)}|M_i|e_i^{(3)}(t-\tau_0(t))|\Bigg)-\sum_{i=1}^{n}\lambda_{2i}|e_i^{(6)}(t-\tau_0(t))|-\\
& \sum_{i=1}^{n}\big(\lambda_{3i}|e_i^{(6)}(t)|^{\alpha}+\lambda_{4i}|e_i^{(6)}(t)|^{\beta}\big),
\end{aligned}
$$

$$
\begin{aligned}
\dot{V}^{(7)}(t) \leq & -\sum_{i=1}^{n}\bigg(d_i-\sum_{j=1}^{n}|a_{ji}^{(0)}|M_i+\lambda_{1i}\bigg)|e_i^{(7)}(t)|+\sum_{i=1}^{n}\sum_{j=1}^{n}\bigg(|a_{ji}^{(7)}|M_i|e_i^{(0)}(t)|+|a_{ji}^{(1)}|M_i|e_i^{(6)}(t)|+ \\
& |a_{ji}^{(6)}|M_i|e_i^{(1)}(t)|+|a_{ji}^{(2)}|M_i|e_i^{(5)}(t)|+|a_{ji}^{(5)}|M_i|e_i^{(2)}(t)|+|a_{ji}^{(3)}|M_i|e_i^{(4)}(t)|+|a_{ji}^{(4)}|M_i|e_i^{(3)}(t)| \\
& +|b_{ji}^{(0)}|M_i|e_i^{(7)}(t-\tau_0(t))|+|b_{ji}^{(7)}|M_i|e_i^{(0)}(t-\tau_0(t))|+|b_{ji}^{(1)}|M_i|e_i^{(6)}(t-\tau_0(t))|+|b_{ji}^{(6)}| \\
& \times M_i|e_i^{(1)}(t-\tau_0(t))|+|b_{ji}^{(2)}|M_i|e_i^{(5)}(t-\tau_0(t))|+|b_{ji}^{(5)}|M_i|e_i^{(2)}(t-\tau_0(t))|+|b_{ji}^{(3)}|M_i \\
& \times|e_i^{(4)}(t-\tau_0(t))|+|b_{ji}^{(4)}|M_i|e_i^{(3)}(t-\tau_0(t))|\bigg)-\sum_{i=1}^{n}\lambda_{2i}|e_i^{(7)}(t-\tau_0(t))|- \\
& \sum_{i=1}^{n}\big(\lambda_{3i}|e_i^{(7)}(t)|^{\alpha}+\lambda_{4i}|e_i^{(7)}(t)|^{\beta}\big).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\dot{V} &= \sum_{b=0}^{7}\dot{V}^{(b)}(t) \\
&\leq -\sum_{i=1}^{n}\bigg(d_i+\lambda_{1i}-\sum_{j=1}^{n}\big(|a_{ji}^{(0)}|+|a_{ji}^{(1)}|+|a_{ji}^{(2)}|+...+|a_{ji}^{(7)}|\big)M_i\bigg)|e_i^{(0)}(t)| \\
&\quad -\sum_{i=1}^{n}\bigg(\lambda_{2i}-\sum_{j=1}^{n}\big(|b_{ji}^{(0)}|+|b_{ji}^{(1)}|+|b_{ji}^{(2)}|+...+|b_{ji}^{(7)}|\big)M_i\bigg)|e_i^{(0)}(t-\tau_0(t)| \\
&\quad -\sum_{i=1}^{n}\bigg(d_i+\lambda_{1i}-\sum_{j=1}^{n}\big(|a_{ji}^{(0)}|+|a_{ji}^{(1)}|+|a_{ji}^{(2)}|+...+|a_{ji}^{(7)}|\big)M_i\bigg)|e_i^{(1)}(t)| \\
&\quad -\sum_{i=1}^{n}\bigg(\lambda_{2i}-\sum_{j=1}^{n}\big(|b_{ji}^{(0)}|+|b_{ji}^{(1)}|+|b_{ji}^{(2)}|+...+|b_{ji}^{(7)}|\big)M_i\bigg)|e_i^{(1)}(t-\tau_0(t)| \\
&\quad -\sum_{i=1}^{n}\bigg(d_i+\lambda_{1i}-\sum_{j=1}^{n}\big(|a_{ji}^{(0)}|+|a_{ji}^{(1)}|+|a_{ji}^{(2)}|+...+|a_{ji}^{(7)}|\big)M_i\bigg)|e_i^{(2)}(t)| \\
&\quad -\sum_{i=1}^{n}\bigg(\lambda_{2i}-\sum_{j=1}^{n}\big(|b_{ji}^{(0)}|+|b_{ji}^{(1)}|+|b_{ji}^{(2)}|+...+|b_{ji}^{(7)}|\big)M_i\bigg)|e_i^{(2)}(t-\tau_0(t)| \\
&\quad -\sum_{i=1}^{n}\bigg(d_i+\lambda_{1i}-\sum_{j=1}^{n}\big(|a_{ji}^{(0)}|+|a_{ji}^{(1)}|+|a_{ji}^{(2)}|+...+|a_{ji}^{(7)}|\big)M_i\bigg)|e_i^{(3)}(t)| \\
&\quad -\sum_{i=1}^{n}\bigg(\lambda_{2i}-\sum_{j=1}^{n}\big(|b_{ji}^{(0)}|+|b_{ji}^{(1)}|+|b_{ji}^{(2)}|+...+|b_{ji}^{(7)}|\big)M_i\bigg)|e_i^{(3)}(t-\tau_0(t)| \\
&\quad -\sum_{i=1}^{n}\bigg(d_i+\lambda_{1i}-\sum_{j=1}^{n}\big(|a_{ji}^{(0)}|+|a_{ji}^{(1)}|+|a_{ji}^{(2)}|+...+|a_{ji}^{(7)}|\big)M_i\bigg)|e_i^{(4)}(t)|
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{i=1}^{n}\left(\lambda_{2i} - \sum_{j=1}^{n}\big(|b_{ji}^{(0)}| + |b_{ji}^{(1)}| + |b_{ji}^{(2)}| + ... + |b_{ji}^{(7)}|\big)M_i\right)|e_i^{(4)}(t-\tau_0(t)| \\
&- \sum_{i=1}^{n}\left(d_i + \lambda_{1i} - \sum_{j=1}^{n}\big(|a_{ji}^{(0)}| + |a_{ji}^{(1)}| + |a_{ji}^{(2)}| + ... + |a_{ji}^{(7)}|\big)M_i\right)|e_i^{(5)}(t)| \\
&- \sum_{i=1}^{n}\left(\lambda_{2i} - \sum_{j=1}^{n}\big(|b_{ji}^{(0)}| + |b_{ji}^{(1)}| + |b_{ji}^{(2)}| + ... + |b_{ji}^{(7)}|\big)M_i\right)|e_i^{(5)}(t-\tau_0(t)| \\
&- \sum_{i=1}^{n}\left(d_i + \lambda_{1i} - \sum_{j=1}^{n}\big(|a_{ji}^{(0)}| + |a_{ji}^{(1)}| + |a_{ji}^{(2)}| + ... + |a_{ji}^{(7)}|\big)M_i\right)|e_i^{(6)}(t)| \\
&- \sum_{i=1}^{n}\left(\lambda_{2i} - \sum_{j=1}^{n}\big(|b_{ji}^{(0)}| + |b_{ji}^{(1)}| + |b_{ji}^{(2)}| + ... + |b_{ji}^{(7)}|\big)M_i\right)|e_i^{(6)}(t-\tau_0(t)| \\
&- \sum_{i=1}^{n}\left(d_i + \lambda_{1i} - \sum_{j=1}^{n}\big(|a_{ji}^{(0)}| + |a_{ji}^{(1)}| + |a_{ji}^{(2)}| + ... + |a_{ji}^{(7)}|\big)M_i\right)|e_i^{(7)}(t)| \\
&- \sum_{i=1}^{n}\left(\lambda_{2i} - \sum_{j=1}^{n}\big(|b_{ji}^{(0)}| + |b_{ji}^{(1)}| + |b_{ji}^{(2)}| + ... + |b_{ji}^{(7)}|\big)M_i\right)|e_i^{(7)}(t-\tau_0(t)| \\
&- \sum_{i=1}^{n}\lambda_{3i}\big(|e_i^{(0)}(t)|^{\alpha} + |e_i^{(1)}(t)|^{\alpha} + ... + |e_i^{(7)}(t)|^{\alpha}\big) - \sum_{i=1}^{n}\lambda_{4i}\big(|e_i^{(0)}(t)|^{\beta} + |e_i^{(1)}(t)|^{\beta} + ... \\
&+ |e_i^{(7)}(t)|^{\beta}\big).
\end{aligned}
$$

By using Lemma 5.2, we obtain

$$
\begin{aligned}
\dot{V}(t) &\leq - \sum_{i=1}^{n}\lambda_{3i}\big(|e_i^{(0)}(t)|^{\alpha} + |e_i^{(1)}(t)|^{\alpha} + ... + |e_i^{(7)}(t)|^{\alpha}\big) - \sum_{i=1}^{n}\lambda_{4i}\big(|e_i^{(0)}|^{\beta} + |e_i^{(1)}(t)|^{\beta} + ...+ \\
&\quad |e_i^{(7)}(t)|^{\beta}\big) \\
&\leq - \sum_{i=1}^{n}\lambda_{3i}\big(|e_i^{(0)}(t)| + |e_i^{(1)}(t)| + ... + |e_i^{(7)}(t)|\big)^{\alpha} - \sum_{i=1}^{n}\lambda_{4i}8^{1-\beta}\big(|e_i^{(0)}(t)| + |e_i^{(1)}(t)| + ...+ \\
&\quad |e_i^{(7)}(t)|\big)^{\beta} \\
&\leq - \min_i(\lambda_{3i})\left(\sum_{i=1}^{n}\big(|e_i^{(0)}(t)| + |e_i^{(1)}(t)| + ... + |e_i^{(7)}(t)|\big)\right)^{\alpha} - \min_i(\lambda_{4i})(8n)^{1-\beta} \\
&\quad \times \left(\sum_{i=1}^{n}\big(|e_i^{(0)}(t)| + |e_i^{(1)}(t)| + ... + |e_i^{(7)}(t)|\big)\right)^{\beta} \\
&\leq - \min_i(\lambda_{3i})V^{\alpha}(e(t)) - \min_i(\lambda_{4i})(8n)^{1-\beta}V^{\beta}(e(t)). \qquad (5.9)
\end{aligned}
$$

Therefore, by using the Lemma 5.1, it is seen that the origin of the error system (5.5) is fixed time stable and the settling time estimation is given by

$$T_{set} \leq T(e_0) \leq \frac{1}{\min_i(\lambda_{3i})(1-\alpha)} + \frac{1}{\min_i(\lambda_{4i})(8n)^{1-\beta}(\beta-1)}, \quad i = 1, 2, ..., n.$$

□

5.4 Numerical Example

In this section, a numerical example has been demonstrated to show the effectiveness of the derived results.

Example 5.1. *Let us consider the following OVNNs* (5.1) *for* $n = 2$ *as*

$$\dot{p}_i(t) = -d_i p_i(t) + \sum_{j=1}^{2} a_{ij} g_j(p_j(t)) + \sum_{j=1}^{2} b_{ij} g_j(p_j(t - \tau_0(t))) + k_i \ ,$$

where, $i = 1, 2;\ d_1 = 2, d_2 = 1;$

$$\begin{aligned}
a_{11} =& \omega_0 + \omega_1 + 1.5\omega_2 + 1.3\omega_3 + 2\omega_4 + 1.2\omega_5 + 1.5\omega_6 + 1.3\omega_7,\\
a_{12} =& -\omega_0 - \omega_1 - 2.3\omega_2 - 3.2\omega_3 - \omega_4 - 4.5\omega_5 - 2.3\omega_6 - 3.2\omega_7,\\
a_{21} =& \omega_0 + 1.7\omega_1 + 1.4\omega_2 + 1.6\omega_3 + \omega_4 + 1.7\omega_5 + \omega_6 + 0.6\omega_7,\\
a_{22} =& 0.8\omega_0 + 0.9\omega_1 + 0.6\omega_2 + 0.5\omega_3 + 0.8\omega_4 + 0.9\omega_5 + 0.6\omega_6 + 0.5\omega_7,\\
b_{11} =& \omega_0 - 0.9\omega_1 + 2.1\omega_2 + \omega_3 + \omega_4 - 0.9\omega_5 + 2.1\omega_6 + \omega_7,\\
b_{12} =& 2\omega_0 + 2\omega_1 - \omega_2 + 1.1\omega_3 + 2\omega_4 + 2\omega_5 - \omega_6 + 1.1\omega_7,\\
b_{21} =& \omega_0 - \omega_1 + 0.4\omega_2 + 0.6\omega_3 + 3\omega_4 - \omega_5 + 0.4\omega_6 + 0.6\omega_7,\\
b_{22} =& 0.2\omega_0 - \omega_1 + 2\omega_2 + 1\omega_3 + 0.2\omega_4 - \omega_5 + 2\omega_6 + \omega_7,\\
k_1 =& 1.2\omega_0 - 1.2\omega_1 + 1.3\omega_2 + 1.4\omega_3 + 1.2\omega_4 - 1.2\omega_5 + 1.3\omega_6 + 1.4\omega_7,\\
k_2 =& 1.4\omega_0 + 1.3\omega_1 + 1.4\omega_2 - 1.3\omega_3 + 1.4\omega_4 + 1.3\omega_5 + 1.4\omega_6 - 1.3\omega_7,
\end{aligned}$$

Here $\tau_0(t) = t - cos^2(t)$. *and* $g_1^{(b)}(x) = g_2^{(b)}(x) = tanhx$ *are the the activation functions, where* $b = 0, 1, 2, ..., 7$. *Clearly Assumption 5.1 holds when* $M_j = 1$ *and* $j = 1, 2$. *Now we choose the initial conditions as*

$$p_1(0) = -1.6w_0 + 1.5w_1 + 1.0w_2 + 0.3w_3 - 2.0w_4 + 1.1w_5 + 0.1w_6 + 0.9w_7,$$

$$s_1(0) = 1.5w_0 + 5.1w_1 + 0.3w_2 + 0.5w_3 - 0.6w_4 - 0.9w_5 + 0.4w_6 + 0.5w_7,$$

$$p_2(0) = -0.5w_0 + 0.6w_1 - 0.5w_2 - 0.1w_3 + 0.3w_4 + 0.8w_5 + 0.5w_6 + 0.1w_7,$$

$$s_2(0) = 0.5w_0 + 0.5w_1 - 0.1w_2 + 0.2w_3 + 0.4w_4 + 0.6w_5 + 0.2w_6 - 0.5w_7.$$

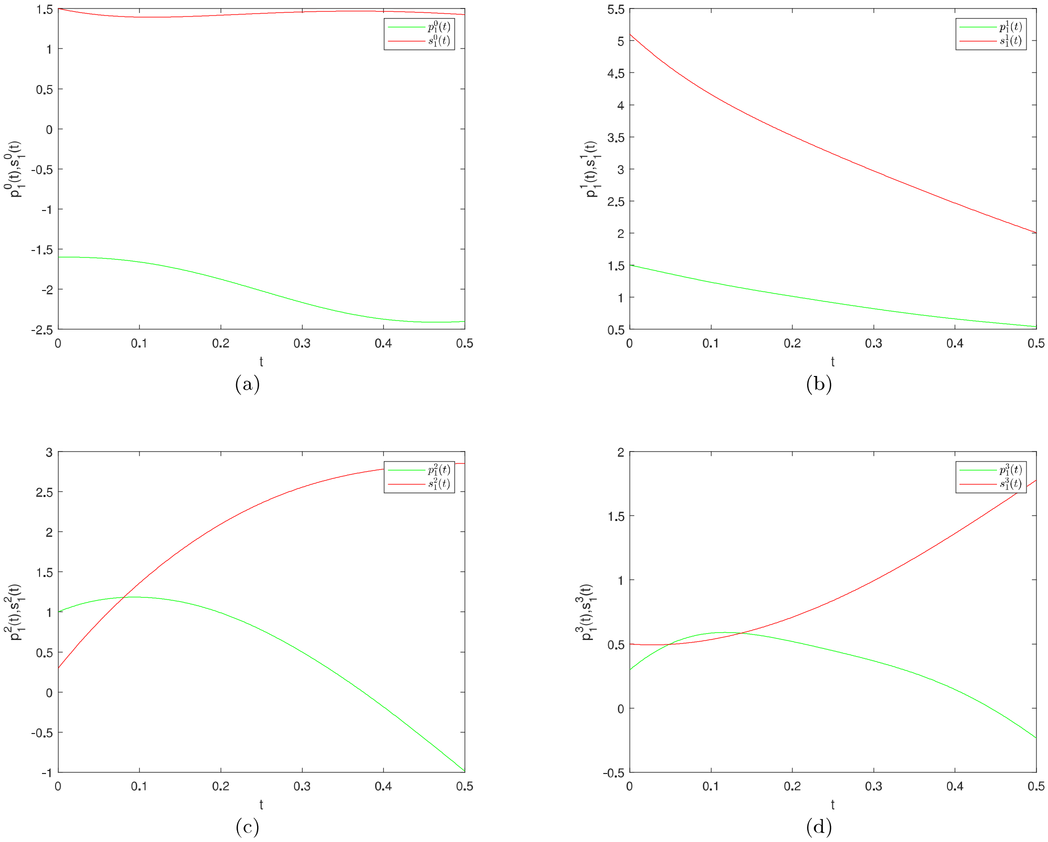
Clearly from the Figures 5.1(a)-5.1(d), we can say that without controllers the state trajectories can not be synchronized. Taking $\lambda_{11} = 20$, $\lambda_{12} = 24$, $\lambda_{21} = 21$, $\lambda_{22} = 21$, $\lambda_{31} = 1$, $\lambda_{32} = 1$, $\lambda_{41} = 1$ *and* $\lambda_{42} = 1$, $\alpha = 0.5$, $\beta = 1.5$, *according to equation*

(5.6), *the controllers can be designed as*

$$u_1^{(b)}(t) = -20e_1^{(b)}(t) - sign(e_1^{(b)}(t))\big(21|e_1^{(b)}(t-\tau_0(t))| + |e_1^{(b)}(t)|^{0.5} + |e_1^{(b)}(t)|^{1.5}\big),$$

$$u_2^{(b)}(t) = -24e_2^{(b)}(t) - sign(e_2^{(b)}(t))\big(21|e_2^{(b)}(t-\tau_0(t))| + |e_2^{(b)}(t)|^{0.5} + |e_2^{(b)}(t)|^{1.5}\big).$$

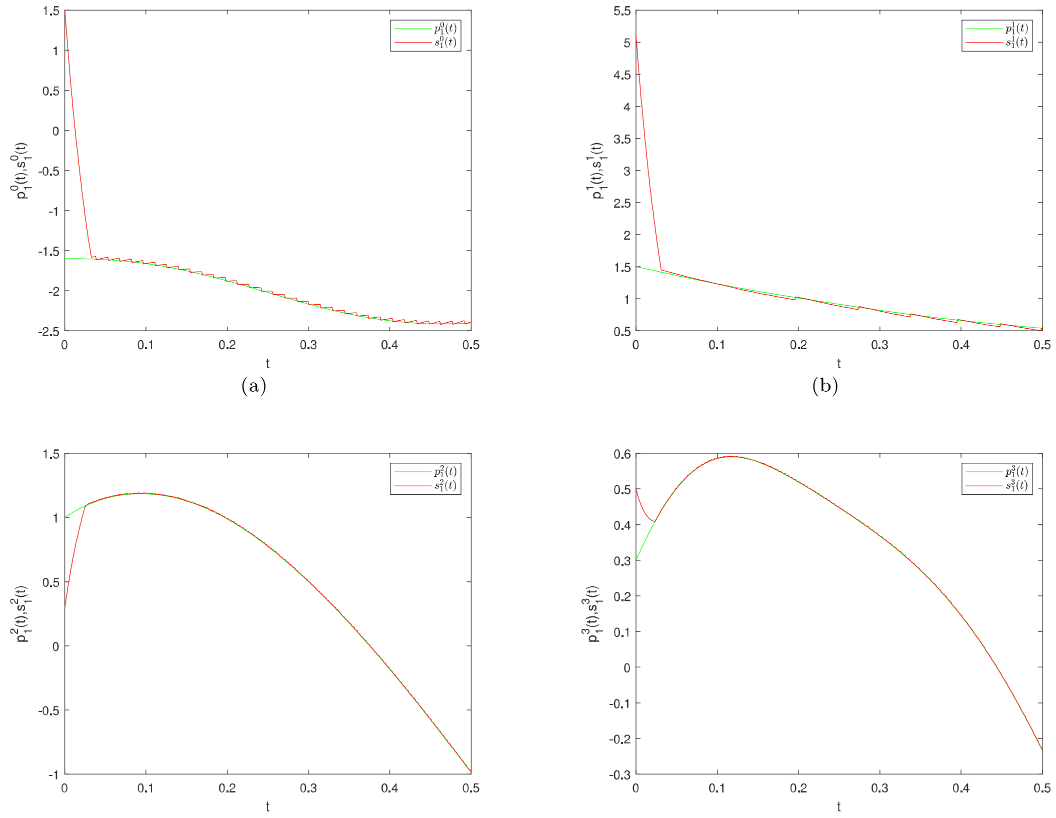
After simple calculation, it is observed that all the sufficient conditions of Theorem 5.1 are satisfied. Hence the drive system (5.3) *and response system* (5.4) *will be synchronized in fixed time as shown in Figures* (5.2(a))-(5.2(d)). *Here the estimated settling time is bounded by 2.5.*



(a) (b)

(c) (d)

FIGURE 5.1: Figures (a), (b), (c) and (d), respectively demonstrate the state trajectories of $p_1^{(0)}$ and $s_1^{(0)}$, $p_1^{(1)}$ and $s_1^{(1)}$, $p_1^{(2)}$ and $s_1^{(2)}$, $p_1^{(3)}$ and $s_1^{(3)}$ of the drive-response systems (5.3) and (5.4) without control.



(a) (b) (c) (d)

FIGURE 5.2: Figures (a), (b), (c) and (d), respectively demonstrate the state trajectories of $p_1^{(0)}$ and $s_1^{(0)}$, $p_1^{(1)}$ and $s_1^{(1)}$, $p_1^{(2)}$ and $s_1^{(2)}$, $p_1^{(3)}$ and $s_1^{(3)}$ of the drive-response systems (5.3) and (5.4) with control.

From the Figures 5.1(a)-5.1(d), it is easily observed that the drive-response systems (5.3) *and* (5.4) *do not synchronize when the controllers in* (5.6) *are not used. After using the controllers* (5.6)*, the drive-response systems achieve synchronization state as shown in Figures 5.2(a)-5.2(d) and hence the simulation results confirm the validity of the Theorem 5.1. Moreover, It should be noted that we can change the synchronization speed by taking the appropriate coefficients of the controllers given in* (5.6).

Remark 5.5. Song et *al.* [7] have discussed the advantage of higher dimensional NNs as compared to lower dimensional NNs. Here the authors have shown that to store 12 by 12-pixel color image, QVNNs need only 144 neurons whereas CVNNs need 432 neurons to store the same image. The above discussion shows that higher-dimensional NNs have more storage capacity as compared to lower-dimensional NNs. Since OVNNs have more dimensions as compared to CVNNs and QVNNs, then obviously it has larger storage capacity than CVNNs and QVNNs. In [5], the authors have discussed the superiority of OVNNs, where they have described that the OVNNs require only 72 neurons to store and retrieve the 12 by 12-pixel image, which clearly indicates the high storage capacity of OVNNs. The application give in [5], can be further extended to FTS which will make this result of FTS even more useful in the future. Using the said application in future, we may find the behavior of retrieving the 12 by 12-pixel images corresponding to drive and response systems, which will be the same within the fixed time, and this fixed time will be independent of the initial conditions.

5.5 Conclusion

In this chapter, FTS of OVNNs with time varying delays have been investigated. Firstly, OVNNs are decomposed into eight real-valued systems by using the multiplication rules of octonion units. Then by using some lemmas and by designing suitable novel controller, several sufficient criteria are proposed for FTS of OVNNs with time varying delays. Finally, a numerical example is presented to illustrate the effectiveness of the derived results. The present chapter can be extended further with mixed delay terms and also with discontinuous activation functions. The results obtained in this chapter also can be applied to octonion-valued memristor neural networks with time-varying delays

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