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Discrete-Time Implementation of Continuous Terminal Algorithm With Implicit-Euler Method

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ABSTRACT This paper proposes an alternative implementation for a continuous terminal algorithm (CTA) proposed by Torres-Gonzalez et al. The original CTA is a continuous version of the twisting algorithm (TA), which mitigates the chattering by integrating the signum functions with increased relative higher order. However, the discrete-time version of CTA resulting from conventional explicit discretization method still suffer from some magnitude of chattering. The chattering is obvious when the gains of CTA and the time-step sizes are set large. We propose an implicit Euler integration method, which totally suppresses the chattering and keeps the properties of the continuous version of CTA, such as finite time convergence and high accuracy. The efficiency of this discrete-time implementation is illustrated by comparing it to the conventional explicit method.

INDEX TERMS Implicit-Euler discretization, discrete-time sliding mode control, terminal sliding model control, chattering-free implementation, discrete-time homogenous systems.

I. INTRODUCTION

Terminal Sliding mode control (TSMC) has been recognized as one of potentially useful control schemes due to its finite-time convergence, tracking accuracy, robustness against parametric uncertainty and external disturbances and simple tuning of parameters [1]–[3]. In practice, the main drawback of TSMC is numerical chattering which could cause damages to the actuators of systems and deteriorate the control performance. Several solutions have been proposed to alleviate the numerical chattering, such as higher order sliding mode (HOSM) [3], [4], adaptive sliding mode designs [5]–[8], and implicit Euler methods [9], [10].

Recently, some kinds of continuous terminal algorithms (CTAs) have been proposed [11]–[13] by using the homogeneity properties. Their structure are very similar to the famous super-twisting algorithms (STA). However, the relative degrees of the CTAs is higher than STA. The control input becomes smoother because the discontinuous signum

functions are integrated, and this is the reason why they are called “continuous” terminal algorithm. However, the implementation of CTAs with traditional explicit Euler methods still suffer from numerical chattering, as shown the Example section of [11]. On the other hand, sliding-mode control systems have been associated with differential inclusions. This paper tries to propose discrete-time implementation algorithms of CTAs based on the differential inclusions.

The proposed algorithms are based on the implicit-Euler discretization. In contrast to explicit-Euler implementations, which are prone to chattering, the implicit-Euler implementations do not result in chattering [4], [8], [14]. Huber et al proposed implicit-Euler implementation of twisting control [14] based on ZOH (Zero-Order Hold) discretization and AVI (Affine Variational Inequality) solver. The twisting control is one of HOSM with simple structure. Xiong et al proposed a simpler calculation for the twisting control based on the implicit Euler discretization. However, as far as the authors are aware, there is few work that proposes implicit Euler discretization of the other HOSM algorithms such as the CTAs proposed in [11]–[13]. The difficulty lies in that the implicit

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Euler discretization results in a complicate nonlinear implicit functions and stability analysis. This paper tries to explore the implicit Euler discretizations of CTAs in a chattering-free manner while keeping its robustness.

This paper is organized as follows. Section II introduces some mathematical preliminaries that will be used in this paper. New discrete-time algorithms of the continuous terminal algorithm are proposed in Section III, respectively. The effectiveness of the proposed algorithms are illustrated in Section V through simulation results. Some concluding remarks are given in Section VI.

II. MATHEMATICAL PRELIMINARIES

Let $\mathcal{A}$ be a closed interval of real numbers. This paper uses the following function:

$$\text{proj}_{\mathcal{A}}(x) \triangleq \arg \min_{\xi \in \mathcal{A}} (\xi - x)^2. \quad (1)$$

This function $\text{proj}_{\mathcal{A}}$ can be referred to as the projection onto the set $\mathcal{A}$. If the set $\mathcal{A}$ is written as $\mathcal{A} = [A, B]$ where $A \leq B$, this function can be written as follows:

$$\text{proj}_{[A,B]}(x) = \begin{cases} B & \text{if } x > B \\ x & \text{if } x \in [A, B] \\ A & \text{if } x < A. \end{cases} \quad (2)$$

This paper also uses the following definition of signum function:

$$\text{sgn}(x) \triangleq \begin{cases} [-1, 1] & \text{if } x = 0 \\ x/|x| & \text{if } x \neq 0. \end{cases} \quad (3)$$

This kind of set-valued definition has been employed by many previous papers [9], [10], [15]. With a non-negative scalar $F \geq 0$, the following relation is satisfied:

$$x \in F\text{sgn}(y - x) \iff x = \text{proj}_{[-F, F]}(y), \quad (4)$$

of which proofs are found in previous papers [9], [16].

The remainder of this paper employs the following lemmas:

Lemma 1: With $A > B > 0$, let us define $\mathcal{C} \triangleq [A - B, A + B]$. Then, for all $x, y \in \mathbb{R}$, the following two statements are equivalent to each other:

$$y \in [-A, A] + B\text{sgn}(x - y) \quad (5)$$

$$y \in [\text{proj}(-\mathcal{C}, \S), \text{proj}(\mathcal{C}, \S)]. \quad (6)$$

Lemma 2: With $A > B > 0$, let us define $\mathcal{C} \triangleq [A - B, A + B]$. Then, for all $x, y, z \in \mathbb{R}$, the following two statements are equivalent to each other:

$$z \in A\text{sgn}(x - z) + B\text{sgn}(y - z) \quad (7)$$

$$z = \text{proj}([\text{proj}(-\mathcal{C}, \S), \text{proj}(\mathcal{C}, \S)]). \quad (8)$$

Remark 1: Lemma 2 has been presented by Jin *et al.* [17] but a strict proof had not been presented.

III. CONTINUOUS TWISTING ALGORITHMS AND ITS IMPLEMENTATION

To solve the double integrator with significant uncertainty is given as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &\in u + \delta(t) \end{aligned} \quad (9)$$

where $\delta(t) \in \mathbb{R}$ represents the matched exogenous/model uncertainty. A design of continuous twisting algorithm as proposed in [12] follows:

$$\begin{aligned} u &= -\mathcal{L}^{\frac{2}{3}}k_1[x_1]^{\frac{1}{3}} - \mathcal{L}^{\frac{1}{2}}k_1[x_2]^{\frac{1}{2}} + \eta \\ \dot{\eta} &\in -\mathcal{L}k_3[x_1]^0 - \mathcal{L}k_4[x_2]^0 \end{aligned} \quad (10)$$

where $k_i > 0$, $i = 1, 2, 3, 4$. This control is able to drive the system trajectories of (9) to the origin in finite-time and reject the bounded derivative of disturbance $\dot{\delta}(t) = \Delta(t)$ and $|\Delta(t)| \leq L$ with some unknown positive constant. By putting (10) in (9), we get

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\mathcal{L}^{\frac{2}{3}}k_1[x_1]^{\frac{1}{3}} - \mathcal{L}^{\frac{1}{2}}k_1[x_2]^{\frac{1}{2}} + \eta + \delta(t) \\ \dot{\eta} &\in -\mathcal{L}k_3[x_1]^0 - \mathcal{L}k_4[x_2]^0 \end{aligned} \quad (11)$$

Although, a fictions state is defined with the second equation of (11) as

$$x_3 = \eta + \delta(t) \quad (12)$$

Therefore, the 3-order dynamical system for closed-loop system is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\mathcal{L}^{\frac{2}{3}}k_1[x_1]^{\frac{1}{3}} - \mathcal{L}^{\frac{1}{2}}k_1[x_2]^{\frac{1}{2}} + x_3 \\ \dot{x}_3 &\in -\mathcal{L}k_3[x_1]^0 - \mathcal{L}k_4[x_2]^0 + \Delta(t) \end{aligned} \quad (13)$$

The third line of (13) may be associated with the differential inclusion due to sign function with disturbance term i.e., $\dot{x}_3 \in -\mathcal{L}k_3[x_1]^0 - \mathcal{L}k_4[x_2]^0 + [-L, L]$. The solution is homogeneous in nature of degree $k = -1$ with weights $r = [3, 2, 1]^T$. However in nominal case i.e., $\delta(t) = 0$, the system trajectory x_3 is similar to η for all $t \geq 0$.

Let us scale the state x :

$$z_1 = \mathcal{L}x_1, \quad z_2 = \mathcal{L}x_2, \quad z_3 = \mathcal{L}x_3 \quad (14)$$

with a new constant $0 < \mathcal{L} \in \mathbb{R}$. Then the system (13) changes into the following new system:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -k_{p1}[z_1]^{\frac{1}{3}} - k_{p2}[z_2]^{\frac{1}{2}} + z_3 \\ \dot{z}_3 &= -k_{p3}[z_1]^0 - k_{p4}[z_2]^0 + \Delta_p(t) \end{aligned} \quad (15)$$

where

$$\begin{aligned} k_{p1} &= \mathcal{L}^{\frac{2}{3}}k_1 & k_{p2} &= \mathcal{L}^{\frac{1}{2}}k_2 \\ k_{p3} &= \mathcal{L}k_3 & k_{p4} &= \mathcal{L}k_4. \end{aligned} \quad (16)$$

To implement the continuous twisting algorithm, some discretization methods can be applied to (15). Let us recall

some of them. Consider the discrete-time formulation of (9) in one of traditional Euler's approaches as follows:

$$\begin{aligned} z_{1,k+1} &= z_{1,k} + h z_{2,k} \\ z_{2,k+1} &= z_{2,k} - h k_{p1} [x_{1,k}]^{\frac{1}{3}} - h k_{p2} [x_{2,k}]^{\frac{1}{2}} + h z_{3,k} \\ z_{3,k+1} &= z_{3,k} - h k_{p3} [x_{1,k}]^0 - h k_{p4} [z_{2,k}]^0 + h \Delta_{p,k} \end{aligned} \quad (17)$$

where $x_{i,k} := x_i(t_k)$, $h = t_{k+1} - t_k > 0$ represents time step interval and the disturbance term $\Delta_{p,k}$ is expressed as

$$\delta_{k+1} = \delta_k + h \Delta_{p,k}. \quad (18)$$

The above discrete-time formulation (17) is an approximation of (9)(10), which can leads to different properties from the continuous one. Here, alternative ways for relatively smoother discrete-time results are conducted with the implementation of implicit-Euler approach.

IV. PROPOSED IMPLICIT EULER IMPLEMENTATION

Before introducing the proposed implicit Euler implementation of (17), let us rewrite it as:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= u_1 + \eta + \delta(t) \end{aligned} \quad (19)$$

where $u_1 = -k_{p1} [z_1]^{\frac{1}{3}} - k_{p2} [z_2]^{\frac{1}{2}}$ and η is an intermediate variable used to compensate the disturbance $\delta(t)$. It is assumed that both η and $\delta(t)$ are differentiable. Their derivatives are $\dot{\eta} = -k_{p3} [z_1]^0 - k_{p4} [z_2]^0$ and $\dot{\eta} = \Delta(t)$, respectively.

The implicit Euler discretization of (19) is as follows:

$$\begin{aligned} z_{1,k+1} &= z_{1,k} + h z_{2,k+1} \\ z_{2,k+1} &= z_{2,k} + h u_{1,k} + h \eta_{k+1} + h \delta_{k+1} \end{aligned} \quad (20)$$

where

$$u_{1,k} \in -k_{p1} [z_{1,k+1}]^{\frac{1}{3}} - k_{p2} [z_{2,k+1}]^{\frac{1}{2}}. \quad (21)$$

The proposed implicit Euler implementation is divided into two stages and both stages are based on the discrete system (20).

A. STAGE I

The first stage begins with the normal version of the discrete system (20):

$$\begin{aligned} \tilde{z}_{1,k+1} &= z_{1,k} + h \tilde{z}_{2,k+1} \\ \tilde{z}_{2,k+1} &= z_{2,k} + h u_{1,k} \end{aligned} \quad (22)$$

where $u_{1,k}$ is the normal version:

$$u_{1,k} \in -k_{p1} [\tilde{z}_{1,k+1}]^{\frac{1}{3}} \text{sgn}(\tilde{z}_{1,k+1}) - k_{p2} [\tilde{z}_{2,k+1}]^{\frac{1}{2}} \text{sgn}(\tilde{z}_{2,k+1}) \quad (23)$$

with intermediate variables $\tilde{z}_{i,k+1}$ and $\tilde{z}_{i,k}$, $i \in \{1, 2\}$. Submitting (23) into (22), one has the following expression:

$$\tilde{z}_{1,k+1} = z_{1,k} + h \tilde{z}_{2,k+1}$$

$$\begin{aligned} \tilde{z}_{2,k+1} &\in z_{2,k} - k_{p1} [\tilde{z}_{1,k+1}]^{\frac{1}{3}} \text{sgn}(\tilde{z}_{1,k+1}) \\ &\quad - k_{p2} [\tilde{z}_{2,k+1}]^{\frac{1}{2}} \text{sgn}(\tilde{z}_{2,k+1}), \end{aligned} \quad (24)$$

from which one has:

$$\begin{aligned} \tilde{z}_{2,k+1} &\in z_{2,k} - k_{p1} [\tilde{z}_{1,k+1}]^{\frac{1}{3}} \text{sgn} \left(\frac{z_{1,k}}{h} + \tilde{z}_{2,k+1} \right) \\ &\quad - k_{p2} [\tilde{z}_{2,k+1}]^{\frac{1}{2}} \text{sgn}(\tilde{z}_{2,k+1}). \end{aligned} \quad (25)$$

By applying Lemma 2, one has:

$$\begin{aligned} \tilde{z}_{2,k+1} &= \text{proj} \left([\text{proj}(-\mathcal{A}_k, -z_{2,k}), \right. \\ &\quad \left. \text{proj}(\mathcal{A}_k, -z_{2,k})], -z_{2,k} - \frac{z_{1,k}}{h} \right) \end{aligned} \quad (26)$$

where $\mathcal{A}_k \triangleq [(k_{p1} [\tilde{z}_{1,k+1}]^{\frac{1}{3}} - k_{p2} [\tilde{z}_{2,k+1}]^{\frac{1}{2}}), (k_{p1} [\tilde{z}_{1,k+1}]^{\frac{1}{3}} + k_{p2} [\tilde{z}_{2,k+1}]^{\frac{1}{2}})]$. In conclusion, one has:

$$\begin{aligned} u_{1,k} &= \frac{1}{h} \text{proj} \left([\text{proj}(-\mathcal{A}_k, -z_{2,k}), \right. \\ &\quad \left. \text{proj}(\mathcal{A}_k, -z_{2,k})], -z_{2,k} - \frac{z_{1,k}}{h} \right) \end{aligned} \quad (27a)$$

$$\tilde{z}_{2,k+1} = z_{2,k} + h u_{1,k} \quad (27b)$$

$$\tilde{z}_{1,k+1} = z_{1,k} + h \tilde{z}_{2,k+1}. \quad (27c)$$

B. STAGE II

Let us introduce the second stage of the proposed implementation. One can rewrite (20) as follows:

$$z_{1,k+1} = z_{1,k} + h z_{2,k+1} \quad (28a)$$

$$z_{2,k+1} = z_{2,k} + h u_{1,k} + h \eta_{k+1} + h \delta_{k+1}. \quad (28b)$$

where $u_{1,k}$ is obtained from (27). Another control input η_{k+1} is discretized by using the implicit Euler method as follows:

$$\eta_{k+1} \in \eta_k - h k_{p3} [\tilde{z}_{1,k+1}]^0 - h k_{p4} [\tilde{z}_{2,k+1}]^0 \quad (29)$$

where $\tilde{z}_{i,k+1}$, $i \in \{1, 2\}$ are the intermediate variables and its updating will be introduced later.

Let us introduce $z_{3,k+1} \triangleq \eta_{k+1} + \delta_{k+1}$. Then (28) can be rewrite as:

$$z_{1,k+1} = z_{1,k} + h z_{2,k+1} \quad (30a)$$

$$z_{2,k+1} = z_{2,k} + h u_{1,k} + h z_{3,k+1} \quad (30b)$$

$$z_{3,k+1} = z_{3,k} - h k_{p3} [\tilde{z}_{1,k+1}]^0 - h k_{p4} [\tilde{z}_{2,k+1}]^0 + h \Delta_k \quad (30c)$$

where $z_{3,k} \triangleq \eta_k + \delta_k$ and $\delta_{k+1} = \delta_k + h \Delta_k$.

Here, in a similar way, we first consider the corresponding normal version of (30) as follows:

$$\bar{z}_{1,k+1} = z_{1,k} + h \bar{z}_{2,k+1} \quad (31a)$$

$$\bar{z}_{2,k+1} = z_{2,k} + h u_{1,k} + h \bar{z}_{3,k+1} \quad (31b)$$

$$\bar{z}_{3,k+1} = z_{3,k} - h k_{p3} [\bar{z}_{1,k+1}]^0 - h k_{p4} [\bar{z}_{2,k+1}]^0 \quad (31c)$$

From (31)(a)-(b), one has:

$$\bar{z}_{2,k+1} = \left(\frac{z_{2,k} + h u_{1,k}}{h} + \bar{z}_{3,k+1} \right) h \quad (32a)$$

$$\bar{z}_{1,k+1} = \left(\frac{z_{1,k} + h z_{2,k} + h^2 u_{1,k}}{h^2} + \bar{z}_{3,k+1} \right) h^2. \quad (32b)$$

Substituting $\bar{x}_{2,k+1}$ and $\bar{x}_{1,k+1}$ from (32) into (31)(c), we can derive the reduced form of the following inclusion as:

$$\bar{z}_{3,k+1} \in z_{3,k} - h k_{p,3} \left[\frac{z_{1,k}}{h^2} + \frac{z_{2,k} + h u_{1,k}}{h} + \bar{z}_{3,k+1} \right]^0 \quad (33a)$$

$$- h k_{p,4} \left[\frac{z_{2,k} + h u_{1,k}}{h} + \bar{z}_{3,k+1} \right]^0. \quad (33b)$$

By following the Lemma 2, the solution of (33a) can be obtained with as follows:

$$\bar{z}_{3,k+1} = z_{3,k} + \text{proj} \left(\left[\text{proj}(-\mathcal{B}_k, -y_{1,k}), \text{proj}(\mathcal{B}_k, -y_{1,k}) \right], -y_{2,k} \right) \quad (34)$$

where $\mathcal{B}_k \triangleq [h(k_{p,3} - k_{p,4}), h(k_{p,3} + h_{p,4})]$ and

$$y_{1,k} \triangleq \frac{z_{2,k} + h u_{1,k}}{h} + z_{3,k} \quad (35a)$$

$$y_{2,k} \triangleq \frac{z_{1,k}}{h^2} + \frac{z_{2,k} + h u_{1,k}}{h} + z_{3,k}. \quad (35b)$$

Then, the intermediate variables $\bar{z}_{i,k+1}, i \in \{1, 2, 3\}$ are updated as follows:

$$y_{1,k} := \frac{z_{2,k} + h u_{1,k}}{h} + z_{3,k} \quad (36a)$$

$$y_{2,k} := \frac{z_{1,k}}{h^2} + \frac{z_{2,k} + h u_{1,k}}{h} + z_{3,k} \quad (36b)$$

$$\bar{z}_{3,k+1} = z_{3,k} + \text{proj} \left(\left[\text{proj}(-\mathcal{B}_k, -y_{1,k}), \text{proj}(\mathcal{B}_k, -y_{1,k}) \right], -y_{2,k} \right) \quad (36c)$$

$$\bar{z}_{2,k+1} = \left(\frac{z_{2,k} + h u_{1,k}}{h} + \bar{z}_{3,k+1} \right) h \quad (36d)$$

$$\bar{z}_{1,k+1} = \left(\frac{z_{1,k} + h^2 z_{2,k} + h u_{1,k}}{h^2} + \bar{z}_{3,k+1} \right) h^2. \quad (36e)$$

Finally, the implementation of the CTA controller $u_k := u_{1,k} + \eta_{k+1}$ is as follows:

$$u_{1,k} = \frac{1}{h} \text{proj} \left(\left[\text{proj}(-\mathcal{A}_k, -z_{2,k}), \text{proj}(\mathcal{A}_k, -z_{2,k}) \right], -z_{2,k} - \frac{z_{1,k}}{h} \right) \quad (37a)$$

$$\eta_{k+1} := \eta_k + \text{proj} \left(\left[\text{proj}(-\mathcal{B}_k, -y_{1,k}), \text{proj}(\mathcal{B}_k, -y_{1,k}) \right], -y_{2,k} \right) \quad (37b)$$

$$u_k = u_{1,k} + \eta_{k+1} \quad (37c)$$

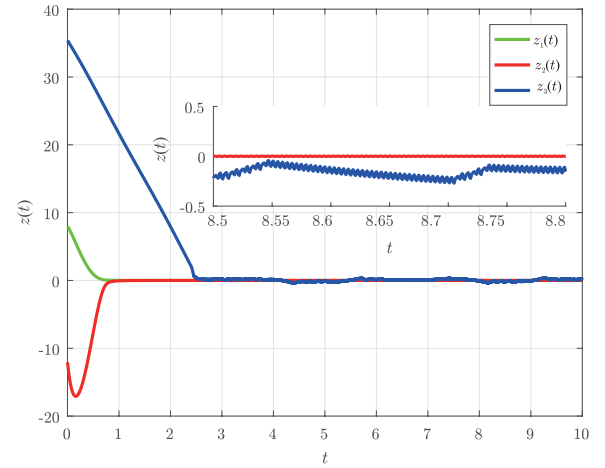


FIGURE 1. The convergence of states z_1, z_2 and z_3 with the conventional explicit Euler method (39).

V. SIMULATIONS

The proposed algorithms are compared with the conventional explicit Euler implementations through some simulations. The time evolution of the states of system (9) is illustrated in Fig.1. The disturbance is set as $\delta(t) = 35 + 0.6\cos(2t) + 0.4\sin(\sqrt{10}t)$ and its derivation is $\Delta(t) = 1.2\cos(2t) + 0.4\sqrt{10}\sin(\sqrt{10}t)$. The gains for the CTA that achieve finite-time convergence to zero of z_1 and z_2 are

$$k_{p,1} = 160.236, \quad k_{p,2} = 60.3738 \\ k_{p,3} = 28.5, \quad k_{p,4} = 15$$

and the scaling factor $L = 5$.

In simulations of the conventional implementation and the proposed implementation of the CTA, the following discrete system is employed:

$$\begin{aligned} z_{1,k+1} &= z_{1,k} + h z_{2,k} \\ z_{2,k+1} &= z_{2,k} + h u_k + h \delta_k \\ z_{3,k+1} &= \eta_k + \delta_k. \end{aligned} \quad (38)$$

where $\delta_k = 35 + 0.6\cos(2t(k)) + 0.4\sin(\sqrt{10}t(k))$. In the simulation of the conventional explicit Euler method, the following expression is used:

$$u_k = u_{1,k} + \eta_k \quad (39a)$$

$$u_{1,k} = -h k_{p1} [x_{1,k}]^{\frac{1}{3}} - h k_{p2} [x_{2,k}]^{\frac{1}{2}} + h \eta_k \quad (39b)$$

$$\eta_{k+1} = \eta_k - h k_{p3} [x_{1,k}]^0 - h k_{p4} [x_{2,k}]^0, \quad (39c)$$

while in the simulation of the proposed implicit Euler method, the expression (37) is employed. In both cases, the time step size is chosen as $h = 0.001$ s and the initial states are set as: $z_1(0) = 8, z_2(0) = -12$ and $z_3(0) = 35$.

The states z_1, z_2 , and z_3 with the conventional explicit Euler method are shown in Fig. 1. Fig. 1 shows that the states z_1, z_2 and z_3 converges to zeros in finite time. However, the inset plot in Fig. 1 shows that z_3 deviates from zero a little and the chattering appears with a small magnitude in z_3 . Similar phenomenons happen in the inset plot of Fig. 2. In Fig. 2, the variable η compensates the disturbance $\delta(t)$,

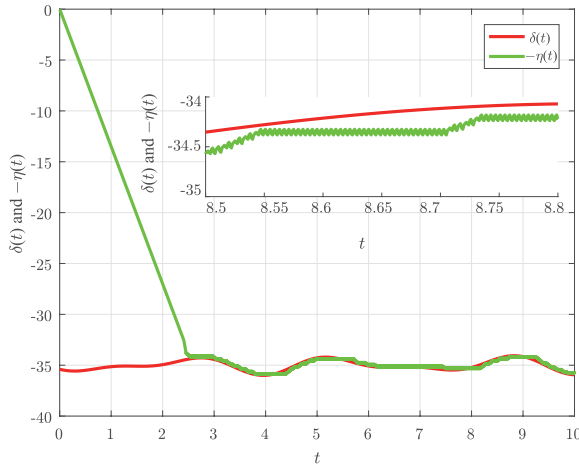


FIGURE 2. The convergence of $-\eta$ to the disturbance $\delta(t)$ with the conventional explicit Euler method (39).

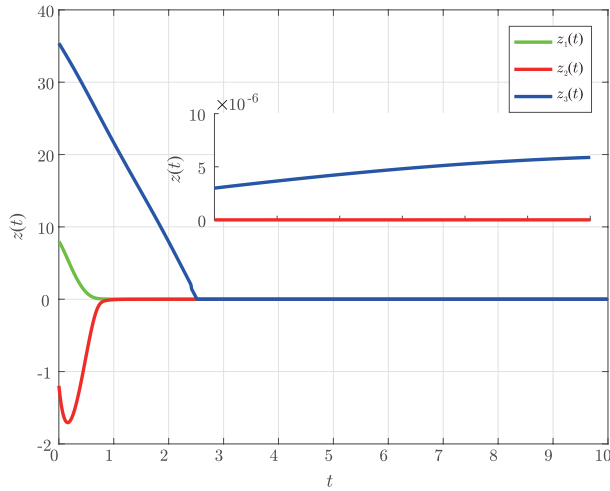


FIGURE 3. The convergence of states z_1 , z_2 and z_3 with the proposed implicit Euler method (37).

i.e., $-\eta = \delta(t)$. However, the inset plot in Fig. 2 shows that $-\eta$ deviates from $\delta(t)$ a little and a small of magnitude of chattering happens in η .

The states z_1 , z_2 , and z_3 with the proposed implicit Euler method are shown in Fig. 3. Fig. 3 shows that the states z_1 , z_2 and z_3 converges to zeros in finite time. The time of convergence is the same as that of the conventional explicit Euler method, about 2.5s. This means that the proposed implicit discretization does not change the convergence property of the CTA. Furthermore, the inset plot of Fig. 3 shows that z_3 converges to zero with a precision about 10^{-6} and there is no chattering. One can also observe that $\eta(t)$ compensates $\delta(t)$ with a high precision and no chattering happens.

The precision of the system states before scaling, i.e., $x_i = z_i/L$, $i \in \{1, 2, 3\}$, is shown Fig. 5. Within the period $8 \leq t \leq 10$ s, Fig. 5 shows that the precision of x_i is about

$$|x_1| \leq v_1 h^3, \quad |x_2| \leq v_2 h^2, \quad |x_3| \leq v_3 h \quad (40)$$

with constants

$$v_1 = 600, \quad v_2 = 610, \quad v_3 = 80. \quad (41)$$

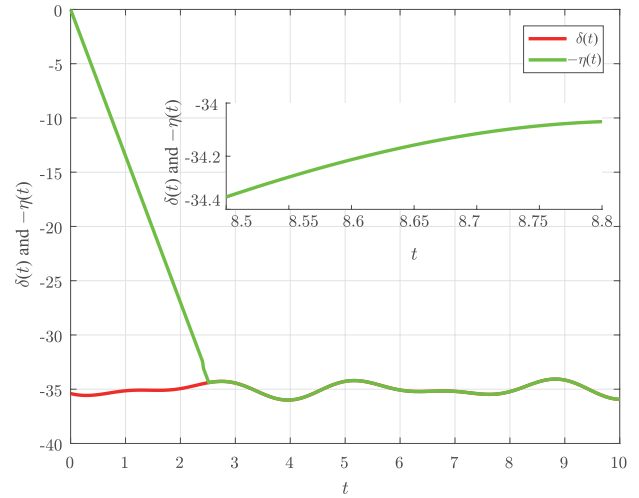


FIGURE 4. The convergence of $-\eta$ to the disturbance $\delta(t)$ with the proposed implicit Euler method (37).

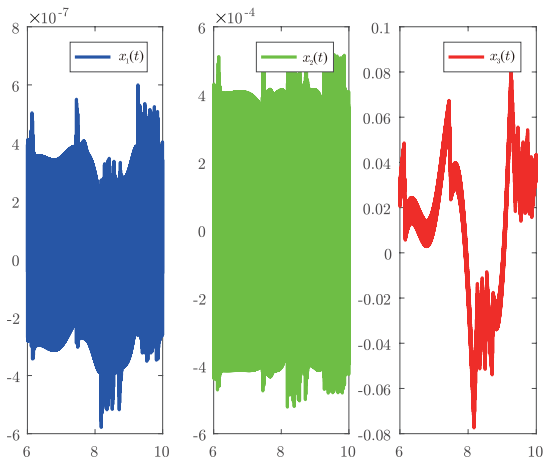


FIGURE 5. The precision of the state variables x_1 , x_2 and x_3 with the conventional explicit Euler method (39) and the time step size is $h = 0.001$ s.

One can also note that the proposed implicit Euler method achieve a much higher precision, as shown in Fig. 5. It satisfies the following inequalities:

$$|x_1| \leq v_1 h^4, \quad |x_2| \leq v_2 h^3, \quad |x_3| \leq v_3 h^2 \quad (42)$$

with constants

$$v_1 = 500, \quad v_2 = 1.5, \quad v_3 = 1.5. \quad (43)$$

Fig. 6 and Fig. 8 show the control input $u(t)$ with the conventional explicit Euler and proposed implicit Euler method, respectively. They illustrate that both the methods have almost the same magnitude of control input and this is very important for some actuators. Furthermore, the inset plot in Fig 6 clearly shows that there is chattering in the control input when the CTA implemented with the conventional explicit Euler method. In contrast, the control input signal calculated by using the proposed implicit Euler method is very smooth, as shown in the inset plot in Fig 6. Only one small chattering happens when z_3 goes to zero. The smooth input is desired for practical applications.

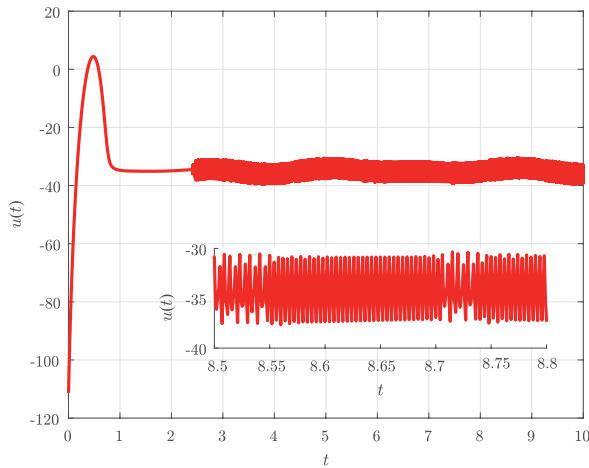


FIGURE 6. The control input signal $u(t)$ applied to (38) with the conventional explicit Euler method (39) and the disturbance is set as $\eta(t) = 35 + 0.6\sin(2t) + 0.4\sin(\sqrt{10}t)$.

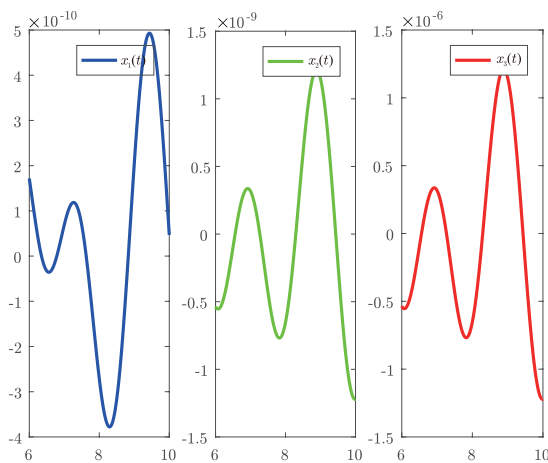


FIGURE 7. The precision of the state variables x_1 , x_2 and x_3 with the proposed implicit Euler method (37) and the time step size is $h = 0.001s$.

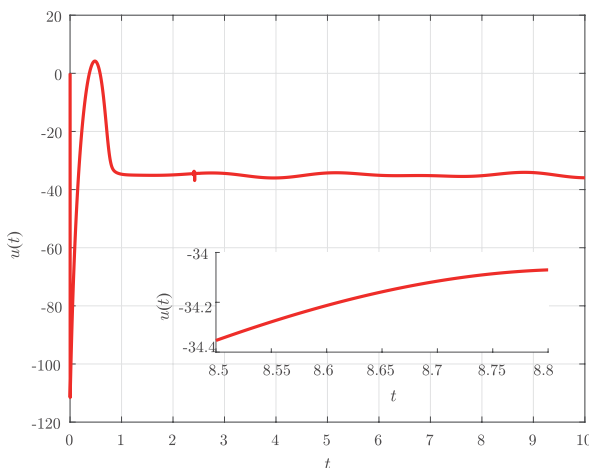


FIGURE 8. The control input signal $u(t)$ applied to (38) with the proposed implicit Euler method (37) and the disturbance is set as $\eta(t) = 35 + 0.6\sin(2t) + 0.4\sin(\sqrt{10}t)$.

To show the influence of the time-step size h on the conventional explicit Euler and the proposed implicit Euler method, in Fig. 9 and Fig. 10, the two methods are illustrated with

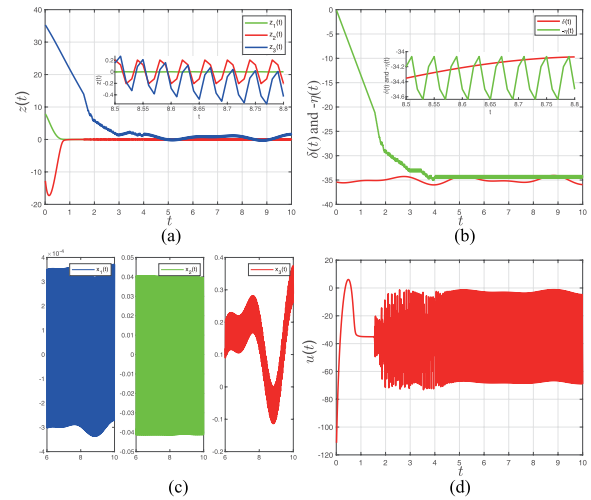


FIGURE 9. The performance of the conventional explicit Euler method (39) with the time step size $h = 0.01s$ and $\eta(t) = 35 + 0.6\sin(2t) + 0.4\sin(\sqrt{10}t)$.

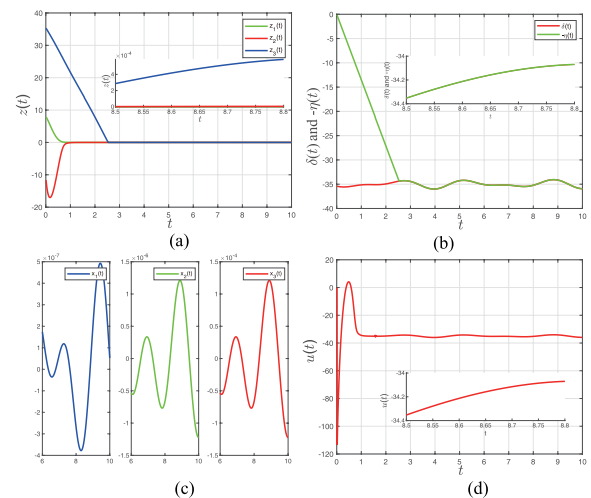


FIGURE 10. The performance of the proposed implicit Euler method (37) with the time step size $h = 0.01s$ and $\eta(t) = 35 + 0.6\sin(2t) + 0.4\sin(\sqrt{10}t)$.

$h = 0.01s$ and the same disturbance as in Fig. IV-B-8. One can see that, as the time-step size decrease, the chattering of the conventional explicit Euler become significant and the magnitude proportionally increases. On the contrary, the proposed implicit Euler method has little effects by the time-step size of h . Fig. 10(c) shows that the control precision satisfies (42) with constants

$$v_1 = 50, \quad v_2 = 1.5, \quad v_3 = 1.5. \quad (44)$$

Fig. 10(d) shows that the control input $u(t)$ produced by the proposed implicit Euler method is not sensitive to the time-step size.

VI. CONCLUSION

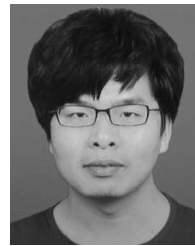
This paper proposed implicit-Euler implementations of continuous terminal algorithm (CTA) for second-order systems. Stability properties of the discretized CTA are provided.

Trajectory of a typical second-order system was taken as an illustrated example. Comparison shows that the proposed implicit-Euler implementation is superior than the conventional explicit-Euler implementation in term of the suppressions of chattering.

Proofs of finite-time convergence of the discretized by the proposed implicit Euler method may need to be sought in future study. It should also demonstrate the efficiency of the proposed implicit Euler method in real platforms with experiments.

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