Study of Reaction-Advection-Diffusion Equations arising in Porous Media

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By

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Chapter 6

Overall conclusions and future work

6.1 Overall conclusions

Several objectives have been accomplished in this thesis work. Firstly, operational matrices for partial derivative, integration and variable-order fractional derivative are used in the shifted Legendre collocation method to solve the non-linear space-time variable-order reaction-advection-diffusion problem. Through error analysis, the effectiveness of the suggested model is verified by contrasting the outcomes of the suggested approach with the current analytical results. A graphical representation of the effects of the reaction and advection terms on the solution profile with varying values of the variable order space and time derivatives has been provided. The main result found through the study is that the non-linearity of the variable order derivative is directly correlated with the diffusivity of solute concentration.

Secondly, the operational matrices for derivatives are used when combined with a numerical approach called the shifted Legendre collocation method to solve the nonlinear space-time fractional-order RADE problem. Through error analyses of the data produced between the suggested technique and the current analytical method, the efficiency and accuracy of the method are verified. Next, the FRADE-defined problem of pollutant transport in groundwater is resolved using the suggested methodology. The impact of the order of the spatial and time derivatives on the solution profile, as well as the effect of the advection term on the profile, are clearly displayed for variations in space and time. The explanation of the damping of the solution profile as the system moves from standard order to fractional order is another area of interest for the study.

In the thesis an effort has been made to investigate the nonlinear FRADE by applying NNM to approximate its solution under the given initial and boundary conditions. To find the weights of neurons, this method involves training with a finite amount of sample points. The solutions of PDEs under unknown conditions are subsequently obtained by using the trained neural network. The findings clarify why the concentration of the solution profile drops when the system's time derivative moves from fractional order towards the standard order. The most important finding is that, in the standard order situation ($\beta = 2$), the solute travels a shorter distance in the soil column as the spatial order derivative decreases to a fractional-order system. The graphical representations of the data clearly show that a greater soil column length will be covered by the solute concentration as a result.

Finally, the two-dimensional non-linear Riesz space-fractional order reaction-diffusion equation is solved numerically using the fast compact implicit integration factor (FcIIF) approach with non-uniform time meshes. It is found that for second order schemes, this technique provides excellent accuracy and stability. Graphs and tables demonstrate the efficacy of our suggested approach when applied to current problems with analytical answers. The accuracy of the solution is higher as compared to that of the methods using uniform meshes. The reason is that the numerical strategy works with diagonal matrices as well as non-uniform meshes lowers both the computing cost and the computational time. The said approach can also be used to solve other types of fractional order partial differential equations, which are significant in the real world of physics.

6.2 Future work

A precise analytical solution for the multidimensional fractional-order linear problem does not exist. It is particularly challenging to solve for nonlinear equations in systems of fractional orders. For the solution of these kinds of equations, the approximate analytical techniques and numerical techniques are very usefull. It is a very challenging job to achieve a computationally efficient solutions for those evolution equations for many specific cases. The mathematical software viz., MATHEMATICA, MATLAB, MAPLE are needed during numerical computations. There is still plenty of scope for researchs into nonlinear FRADE subject to various initial and boundary conditions, such as Dirichlet, Neumann, and Robin-type boundary conditions, primarily in multidimensional cases, because of its physical significance and significant applications.

As a result, several mathematical models for multidimensional cases having physical relevances and practical uses might be presented in the near future. It is important to take into account how nonlinearity affected the solution profiles in both the reaction and dispersion/diffusion terms. It is possible to expand the solute transport equations (porous medium equations) for heterogeneous media from those studied in homogeneous media.

Numerous physical and chemical processes are explained by fractional as well as variable-order nonlinear differential equations in a wide range of scientific and engineering domains. Specifically, mathematical models for groundwater hydrology that are specifically intended for modeling the movement of passive tracers carried by fluid flow in a porous medium are typically found to be nonlinear RADEs. Groundwater contamination-related industrial challenges are found to increase in the use of complex nonlinear RADEs that forecast the behavior and mobility of pollutant concentrations. There is plenty of opportunity to expand the physical difficulties of current integer-order system to fractional- and variable-order systems.

Recently, many researchers are interested in the field of aerosol transportation due to its direct connection with the climate change. Many models on aerosol transport have been developed viz., a model in one-dimension for simulation of aerosol transport and deposition in human lung, aerosol model for atomic spectrometry, aerosol transport in sequentially bifurcating airways, motion of inertial spheroidal particles in a shear flow near a solid wall with special application to aerosol transport in microgravity, a model for Micro-particle transport and deposition in a human oral airway, a model to validate the size-resolved particle dry depositions scheme for application in aerosol transport model, a model of aerosol microscopic module for large scale etc. But to best of my knowledge the multi-dimensional space-time fractional-order reaction-advection–dispersion equation in the finite domain with arbitrary initial and boundary conditions have not been studied by any researcher. Therefore an attempt can be taken to solve the model in two/three dimensional cases using shifted Legendre collocation method or through developing more efficient and appropriate numerical tools.

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