Chapter 4

Approximating fixed point results for pseudo-contractive map in the Banach space

This chapter presents a few convergence results for the Krasnoselskii-Mann iteration and also the modified Krasnoselskii-Mann iteration used for approximating the fixed points of k-strictly pseudo contractive map in the Banach space. Also, using the enrichment techniques, we identify that pseudo-contractive is an unsaturated class of mappings in the Banach space.

4.1 Introduction

In 1967, Browder and Petryshyn [32] introduced the notion of strictly pseudo-contractive map given by the following definition.

Definition 4.1. A self-map T on the normed linear space $(X, \|.\|)$ is said to be k-strictly pseudo-contractive map of Browder-Petryshyn type if there exists $k \in (0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - y - Tx + Ty\|^2 \quad \text{for all} \quad x, y \in X. \tag{4.1}$$

Iterative schemes can approximate the fixed points of demi-contractive mappings, one of the most general types of non-expansive mappings. In 1977, Hicks and Kubicek [47] introduced the notion of demi-contractive map given below in the definition (4.2). However, Măruşter introduced the idea of the demi-contractive map in a different way, first in the space $\mathbb{R}^n$ and then in 1977 [66] in real Hilbert space.

Definition 4.2. A self-map T on the Banach space $(X, \|.\|)$ is called k-demi-contractive map if there exists $k \in (0, 1)$ and $y \in Fix(T)$ such that

$$\|Tx - y\|^2 \leq \|x - y\|^2 + k\|x - Tx\|^2 \quad \text{for all} \quad x \in X. \tag{4.2}$$

It is obvious that any non-expansive map with $Fix(T) \neq \emptyset$ is a quasi non-expansive map and that any quasi non-expansive mapping is demi-contractive, but the converse may not be true. See [14] for various types of examples.

Again, a non-expansive map is a k-strictly pseudo-contractive of Browder-Petryshyn type and hence pseudo-contractive, but the converse is not true in general. Moreover, if we take $y \in Fix(T)$ in (4.1), we see that any k-strictly pseudo-contractive map of Browder-Petryshyn type is k-demi-contractive, but the converse is not valid in general.

Lemma 4.3. [16] *Let T be a self-map on convex subset E of Banach space $(X, \|.\|)$. Then, the mapping T_λ given by $T_\lambda(x) = (1 - \lambda)x + \lambda Tx$ for all $x \in E$ where $\lambda \in (0, 1)$ has the following property:*

$$Fix(T_\lambda) = Fix(T).$$

4.1.1 Delineation

The current chapter is structured as follows: Section 4.2 represents the existence and approximation of fixed point results using the Krasnoselskii-Mann iteration in the Banach space and in Subsection 4.2.1, we have presented the same result using the modified Krasnoselskii-Mann iteration. In Subsection 4.2.2 we compare the rate of convergence of these two iterations for the pseudo-contractive map in the Banach space. In Section 4.3, utilizing the concept of enriching techniques, we deduce that the class of strictly pseudo-contractive maps is an unsaturated class of mappings in the Banach space.

4.2 Convergence results for Krasnoselskii-Mann iteration

Let E be a convex subset of the Banach space X and a map $T : E \to E$ with $Fix(T) \neq \emptyset$ (according to [91]), is said to satisfy condition **A**, if there exists a increasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $f(0) = 0$ and $f(s) > s$, for $s > 0$, such that

$$\|x - Tx\| \geq f(d(x, Fix(T))) \quad \text{for all} \quad x \in E, \tag{4.3}$$

where

$$d(x, Fix(T)) = \inf\{\|x - y\| : y \in Fix(T)\}.$$

Remark 4.4. It should be noted that (4.3) is usually called a retraction-displacement condition; see [23, 24, 83] for more details. The authors have developed iterative algorithms such as Picard, Krasnoselskii-Mann, and Halpern based on a retraction-displacement condition and have given various examples.

Lemma 4.5. [91] *Let $T : E \to E$ be a map with $Fix(T) \neq \emptyset$, where E is a bounded and closed subset of the Banach space X. If $I - T$ maps a closed bounded subset of E onto a closed subset of E, where I is an identity operator, then T satisfies Condition **A**.*

A simple form of Condition **A** is said to be Condition **B** [91], which corresponds to the particular case $f(t) = \alpha t$, with $\alpha > 0$ in Condition **A**. Also, for the various fixed point algorithms and in conjunction with some specific contractive conditions, retraction-displacement condition [23] is equivalent to **B**.

In this subsection, the following lemma, which basically relates to the strictly pseudo-contractive map and non-expansive map, is crucial to the proof of our main results.

Lemma 4.6. *Let X be a uniformly convex Banach space, E be a closed and convex subset of X, and T be k-strictly pseudo-contractive mapping of E into E with $k \in (0, \frac{1}{2})$. Then, for any $\lambda \in (0, 1)$ satisfying the relation*

$$|4\lambda^2 - 4(k + \lambda) + 4k\lambda + 2| \leq 1 - 2k,$$

the averaged operator T_λ is non-expansive.

Proof. By hypothesis there exists $k \in (0, \frac{1}{2})$ such that for all $x, y \in X$ we have

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - y - Tx + Ty\|^2. \tag{4.4}$$

Now, using the triangle inequality we have $\|x - y - Tx + Ty\| \leq \|x - y\| + \|Tx - Ty\|$. Therefore, by squaring both sides, we obtain

$$\|x - y - Tx + Ty\|^2 \leq \Big(\|x - y\| + \|Tx - Ty\|\Big)^2. \tag{4.5}$$

Now, applying Cauchy-Schwarz inequality for the sums, we have

$$\Big(\|x - y\|\,\|1\| + \|Tx - Ty\|\,\|1\|\Big)^2 \leq \Big(\|x - y\|^2 + \|Tx - Ty\|^2\Big)\,\Big(\|1\|^2 + \|1\|^2\Big).$$

$$\text{Therefore,} \quad \Big(\|x - y\| + \|Tx - Ty\|\Big)^2 \leq 2\Big(\|x - y\|^2 + \|Tx - Ty\|^2\Big). \tag{4.6}$$

Now, from (4.5) and (4.6) we have

$$\|x - y - Tx + Ty\|^2 \leq 2\Big(\|x - y\|^2 + \|Tx - Ty\|^2\Big).$$

Thereby, from (4.4) we obtain

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2k\Big(\|x - y\|^2 + \|Tx - Ty\|^2\Big), \tag{4.7}$$

which implies,

$$\|Tx - Ty\|^2 \leq \frac{1 + 2k}{1 - 2k}\|x - y\|^2. \tag{4.8}$$

Again,

$$\begin{aligned}
\|T_\lambda x - T_\lambda y\|^2 &= \|\lambda(Tx - Ty) + (1 - \lambda)(x - y)\|^2 \\
&\leq 2\Big(\lambda^2\|Tx - Ty\|^2 + (1 - \lambda)^2\|x - y\|^2\Big) \\
&\leq 2\Big(\lambda^2\frac{1 + 2k}{1 - 2k} + (1 - \lambda)^2\Big)\|x - y\|^2. \qquad \text{[by using (4.8)]}
\end{aligned}$$

Therefore, we have

$$\|T_\lambda x - T_\lambda y\|^2 \leq \frac{4\lambda^2 - 4(k+\lambda) + 4k\lambda + 2}{1-2k}\|x-y\|^2. \tag{4.9}$$

Thereby,

$$\|T_\lambda x - T_\lambda y\| \leq \frac{\sqrt{|4\lambda^2 - 4(k+\lambda) + 4k\lambda + 2|}}{\sqrt{1-2k}}\|x-y\|. \tag{4.10}$$

Now if we choose λ such that $|4\lambda^2 - 4(k+\lambda) + 4k\lambda + 2| \leq 1 - 2k$ holds then, $\|T_\lambda x - T_\lambda y\| \leq \|x-y\|$ for all $x, y \in X$.

Hence the desired result. □

Theorem 4.7. (Senter and Dotson [91]) *Let E be a closed and convex subset of uniformly convex Banach space X, and T be a non-expansive mapping of E into E. If T satisfies Condition $\boldsymbol{A}$, then for any $x_1 \in E$, the iterative sequence*

$$x_{n+1} = (1-t_n)x_n + t_n T x_n, \quad n \geq 1, \tag{4.11}$$

where $\{t_n\} \subseteq [a,b]$, and $0 < a < b < 1$, converges strongly to $Fix(T)$.

The following is the convergence theorem for the Krasnoselskii-Mann iteration used for approximating the fixed point of k- strictly pseudo-contractive map in the Banach space.

Theorem 4.8. *Suppose E be a closed and convex subset of uniformly convex Banach space X, and T is k-strictly pseudo-contractive on E with $k \in (0, \frac{1}{2})$. Suppose T satisfies Condition $\boldsymbol{A}$. Then $Fix(T) \neq \emptyset$ and for arbitrary $x_1 \in E$, the sequence*

$$x_{n+1} = (1-t_n)x_n + t_n T x_n, \quad n \geq 1, \tag{4.12}$$

where $\{t_n\} \subseteq [a,b]$, and $0 < a < b < 1$, converges strongly to $Fix(T)$.

Proof. Since, T is k-strictly pseudo-contractive map with $k \in (0, \frac{1}{2})$, by Lemma 4.6 it follows that T_λ is non-expansive map for any λ satisfying $|4\lambda^2 - 4(k+\lambda) + 4k\lambda + 2| \leq 1 - 2k$.

Now, by Theorem 4.7 it shows that, for any $\{t_n\} \subseteq [a,b]$, where $0 < a < b < 1$, the iterative sequence

$$x_{n+1} = (1-t_n)x_n + t_n T x_n, \quad n \geq 1,$$

converges strongly to the fixed point of T_λ. Also, by the Lemma 4.3, we have $Fix(T) = Fix(T_\lambda)$. Hence, the desired assertion. □

4.2.1 Convergence results for modified Krasnoselskii-Mann iteration

Let T be a mapping on the Hilbert space H and $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0, 1]$. In 2009, Yao et al. [99] introduced the Modified Krasnoselskii-Mann iteration as follows:

$$y_n = (1 - \alpha_n)x_n,$$
$$x_{n+1} = (1 - \beta_n)y_n + \beta_n T_n y_n, \quad n \geq 1,$$

where $x_1 \in H$ is given, and proved that $\{x_n\}$ converges strongly to a fixed point of the non-expansive map T. Note that if $\alpha_n = 0$ and $\beta_n = \lambda$, then the Modified Krasnoselskii-Mann iteration reduces to the Krasnoselskii-Mann iteration. Recently, in 2015, Shehu and Ugwunnadi [92] have approximated the fixed point of non-expansive mapping in the Banach space by the modified Krasnoselskii-Mann iteration as follows:

Theorem 4.9. (Shehu and Ugwunnadi [92]) *Let X be a real uniformly convex Banach space and uniformly smooth. For each $n = 1, 2, \ldots,$ let $T_n : X \to X$ be a non-expansive map such that $\cap_{n=1}^{\infty} Fix(T_n) \neq \emptyset$. Let the sequence $\{x_n\}$ and $\{y_n\}$ be generated iteratively by $x_1 \in X$,*

$$y_n = (1 - \alpha_n)x_n,$$
$$x_{n+1} = (1 - \beta_n)y_n + \beta_n T_n y_n, \quad n \geq 1,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0, 1]$ are satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0,$

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty,$

(iii) $\beta_n \in [a, b] \subset (0, 1).$

Suppose that $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}x - T_n x\| : x \in B\} < \infty$ for any bounded subset B of X. Let T be a mapping on closed and convex set E of X and defined by $Ty = \lim_{n\to\infty} T_n y$, for all $y \in X$ and *suppose that* $Fix(T) = \cap_{n=1}^{\infty} Fix(T_n)$. *Then the sequence* $\{x_n\}$ *converges strongly to a point in* $\cap_{n=1}^{\infty} Fix(T_n)$.

The following is the convergence theorem for the modified Krasnoselskii-Mann iteration used for approximating the fixed point of k- strictly pseudo-contractive map in the Banach space.

Theorem 4.10. *Let* X *be a real uniformly convex Banach space and uniformly smooth and for each* $n = 1, 2, \dots$, *let* $T_n : X \to X$ *be* k*-strictly pseudo-contractive map with* $k \in (0, \frac{1}{2})$ *and* $(T_n)_\lambda$ *be its averaged operator such that* $\cap_{n=1}^{\infty} Fix(T_n)_\lambda \neq \emptyset$. *Suppose that* $\sum_{n=1}^{\infty} \sup\{\|(T_{n+1})_\lambda x - (T_n)_\lambda x\| : x \in B\} < \infty$ *for any bounded subset* B *of* X.
Let T_λ *be a mapping on closed and convex set* E *of* X *and defined by*

$$T_\lambda y = \lim_{n\to\infty} (T_n)_\lambda\, y, \quad \textit{for all} \;\; y \in E \quad \textit{and} \quad Fix(T_\lambda) = \cap_{n=1}^{\infty} Fix(T_n)_\lambda.$$

Then, the modified Krasnoselskii-Mann iteration converges strongly to $Fix(T)$.

Proof. Since, $\{T_n\}$ is a k-strictly pseudo-contractive sequence with $k \in (0, \frac{1}{2})$, by Lemma 4.6 it follows that each $(T_n)_\lambda$ is non-expansive map for any λ satisfying $|4\lambda^2 - 4(k+\lambda) + 4k\lambda + 2| \leq 1 - 2k$. Now, by Theorem 4.9, it shows that the modified Krasnoselskii-Mann iteration converges strongly to the fixed point of T_λ. Also, by the Lemma 4.3, we have $Fix(T) = Fix(T_\lambda)$. Hence, the desired assertion. □

Example 4.11. Let X be the real line with usual norm, $E = [\frac{1}{2}, 2]$ and $T : E \to E$ defined by $Tx = \frac{1}{2x}$, for all $x \in E$. If T is k-strictly pseudo-contractive map then there exists $k \in (0, 1)$ such that, for all $x, y \in E$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - y - Tx + Ty\|^2, \tag{4.13}$$

which reduces to

$$\left|\frac{1}{2x} - \frac{1}{2y}\right|^2 \leq |x - y||^2 + k\left|x - y - \frac{1}{2x} + \frac{1}{2y}\right|^2.$$

Therefore, $\frac{1}{4x^2y^2} \leq 1 + k\Big(1 + \frac{1}{2xy}\Big)^2$ for $x \neq y$. (Note that for $x = y$, (4.13) is satisfied trivially) By denoting $t = xy$, it follows that $t \in [\frac{1}{4}, 4]$ and hence we require to prove that there exists $k > 0$ such that

$$\frac{1 - 4t^2}{(1 + 2t)^2} \leq k < \frac{1}{2}, \quad \text{for all} \quad t \in [\frac{1}{4}, 4].$$

Let us consider the function $f(t) = \frac{1-4t^2}{(1+2t)^2}, \quad t \in [\frac{1}{4}, 4]$. Since $f'(t) = \frac{-4}{(1+2t)^2} < 0$, it follows that f is strictly decreasing on $[\frac{1}{4}, 4]$, which implies $f(t) \leq f(\frac{1}{4}) = \frac{1}{3}$ for all $t \in [\frac{1}{4}, 4]$. This shows that one can choose $k = \frac{1}{3}$ and so, T is $\frac{1}{3}$- strictly pseudo-contractive.

Assume that T is non-expansive map, i.e., $|Tx - Ty| \leq |x - y|$, for all $x, y \in E = [\frac{1}{2}, 2]$ and take $x = \frac{1}{8}$ and $y = \frac{1}{4}$ to get $|4 - 2| \leq \left|\frac{1}{8} - \frac{1}{4}\right| \iff 2 \leq \frac{1}{8}$, which is a contradiction.

Now for any $\lambda \in (0, 1)$ with $|4\lambda^2 - 4(k + \lambda) + 4k\lambda + 2| \leq 1 - 2k$, we have

$$\begin{aligned}
|T_\lambda x - T_\lambda y| &= |(1-\lambda)x + \lambda Tx - (1-\lambda)y - \lambda Ty| \\
&= \left|(1-\lambda)(x-y) - \frac{\lambda}{2xy}(x-y)\right| \\
&= |x-y|\left|(1-\lambda) - \frac{\lambda}{2xy}\right|.
\end{aligned}$$

Since $\lambda \in (0, 1)$ and $x, y \in [\frac{1}{2}, 2]$ thereby, $\left|(1-\lambda) - \frac{\lambda}{2xy}\right| \leq 1)$. Therefore we have

$$|T_\lambda x - T_\lambda y| \leq |x - y|.$$

Since T is k-strictly pseudo-contractive (with $k = \frac{1}{3}$), therefore T_λ is a non-expansive operator on $[\frac{1}{2}, 2]$ for any $\lambda \in (0, 1)$ with $|4\lambda^2 - 4(k + \lambda) + 4k\lambda + 2| \leq 1 - 2k$.

Now, from the Theorem 4.8 we can conclude that $Fix(T) \neq \emptyset$ and for arbitrary $x_1 \in E$, the sequence

$$x_{n+1} = (1 - t_n)x_n + t_n Tx_n, \quad n \geq 1,$$

where $\{t_n\} \subseteq [a, b]$, and $0 < a < b < 1$, converges strongly to $Fix(T)$.

4.2.2 Numerical illustration

Let us consider the map $Tx = \frac{1}{2x}$ on $[\frac{1}{2}, 2]$ as in Example 4.11. Here the Krasnoselskii-Mann iteration is $x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n$, for all $x \in [\frac{1}{2}, 2]$ and $n \geq 0$.

Therefore, the iterative scheme is

$$x_{n+1} = (1 - \lambda)x_n + \frac{\lambda}{2x_n}, \quad n \geq 0.$$

Table 4.1: Numerical experiment for the Krasnoselskii-Mann iteration:

n	$\lambda = 0.2$	$\lambda = 0.3$	$\lambda = 0.4$	$\lambda = 0.48$
0	0.25	0.25	0.25	0.25
1	0.6000	0.7750	0.9500	1.0900
2	0.6467	0.7360	0.7805	0.7870
3	0.6720	0.7190	0.7246	0.7142
4	0.6864	0.7119	0.7108	0.7074
5	0.6948	0.7090	0.7078	0.7071
6	0.6998	0.7079	0.7073	0.7071
7	0.7027	0.7074	0.7071	0.7071
8	0.7045	0.7072	0.7071	0.7071
9	0.7055	0.7072	0.7071	0.7071
10	0.7062	0.7071	0.7071	0.7071
11	0.7065	0.7071	0.7071	0.7071
12	0.7068	0.7071	0.7071	0.7071
13	0.7069	0.7071	0.7071	0.7071

Now, for the modified Krasnoselskii-Mann iteration let us choose $\alpha_n = \frac{1}{2n+1}$, $\beta_n = \frac{n}{3n+2}$. Therefore, the iterative scheme converts to the following:

$$
\begin{aligned}
y_n &= (1 - \frac{1}{2n+1})y_n, \\
x_{n+1} &= (1 - \frac{\lambda n}{3n+2})x_n + \frac{n\lambda}{2(3n+2)y_n}, \quad n \geq 0.
\end{aligned}
$$

As shown in the following Table 4.1, Table 4.2, Figure 4.1, and Figure 4.2, we obtain numerical results for different values of λ.

By analyzing the results, following conclusions about the rate of convergence to the fixed point could be drawn:

(i) The rate of convergence of the Krasnoselskii-Mann iteration for the considered strictly pseudo-contractive mapping depends on both parameter λ and initial point x_0.

(ii) Comparing the modified Krasnoselskii-Mann iteration with the original Krasnoselskii-Mann iteration, the modified one converges very slowly. Here, the modified Krasnoselskii-Mann

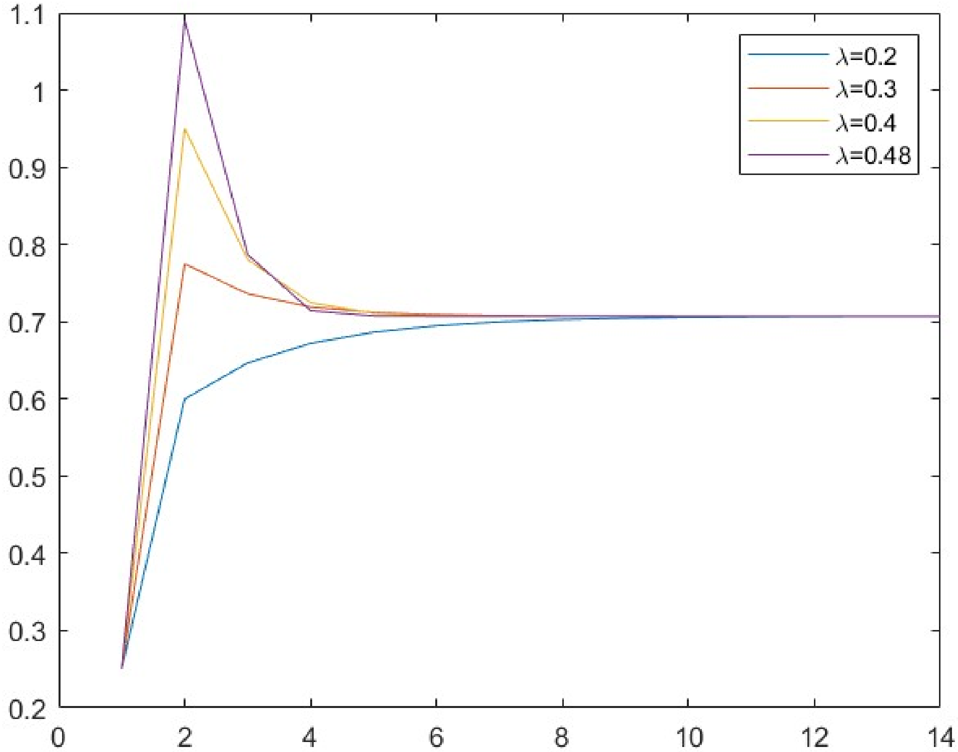

FIGURE 4.1: Graph using the Krasnoselskii-Mann iteration

TABLE 4.2: Numerical experiment for the modified Krasnoselskii-Mann iteration:

n	$\lambda = 0.2$	$\lambda = 0.3$	$\lambda = 0.4$	$\lambda = 0.48$
0	0.25	0.25	0.25	0.25
1	0.3618	0.4201	0.4784	0.5250
2	0.4235	0.4903	0.5468	0.5862
3	0.4667	0.5355	0.5879	0.6210
4	0.4991	0.5669	0.6145	0.6423
5	0.5242	0.5896	0.6324	0.6556
6	0.5441	0.6064	0.6445	0.6642
7	0.5600	0.6189	0.6530	0.6697
8	0.5729	0.6283	0.6588	0.6733
9	0.5834	0.6354	0.6629	0.6756
10	0.5919	0.6408	0.6658	0.6771
11	0.5990	0.6450	0.6678	0.6781
12	0.6048	0.6481	0.6692	0.6788
13	0.6069	0.6506	0.6702	0.6792

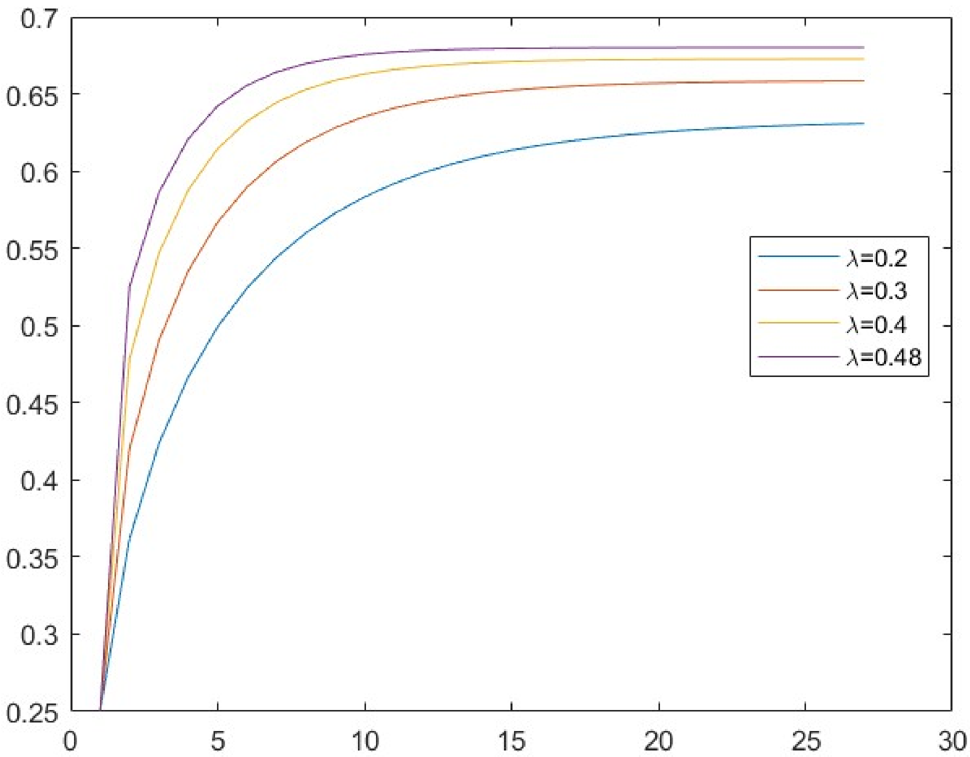

FIGURE 4.2: Graph using the modified Krasnoselskii-Mann iteration

iteration converges to the fixed point 0.7071 (correct up to 4 decimal places) after more than 1000 iterations, whereas the Krasnoselskii-Mann iteration converges to the fixed point upto 4 decimal place just after 15 iterations.

(iii) The numerical experiments reported here indicate that the Krasnoselskii-Mann iteration is more convenient than the modified Krasnoselskii-Mann iterations when approximating fixed points of pseudo-contractive maps.

4.3 Unsaturated class of contractive mappings

The following theorem shows that the class of k-strictly pseudo-contractive maps is an unsaturated class of mappings in the Banach space.

Theorem 4.12. *Let X be a Banach space. Then, the class of k-strictly pseudo-contractive maps (C_{SPC}) with $k \in (0, \frac{1}{2})$ is an unsaturated class of mappings.*

Proof. If possible, let the class of k-strictly pseudo-contractive map (C_{SPC}) coincide with the class of enriched non-expansive mappings (C^e_{NE}) in the Banach space. Let T be a strictly pseudo-contractive map in the Banach space then for all $x, y \in X$ we have

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - y - Tx + Ty\|^2. \tag{4.14}$$

Therefore, from (4.8) we obtain,

$$\|Tx - Ty\|^2 \leq \frac{1 + 2k}{1 - 2k}\|x - y\|^2. \tag{4.15}$$

Now,

$$\begin{aligned} \|b(x - y) + Tx - Ty\|^2 &\leq 2\Big(b^2\|x - y\|^2 + \|Tx - Ty\|^2\Big) \\ &\leq 2\Big(b^2 + \frac{1 + 2k}{1 - 2k}\Big)\|x - y\|^2. \qquad \text{(by using (4.15))} \end{aligned} \tag{4.16}$$

Let us claim that the relation (4.16) satisfies the enriched non-expansive map condition for some $b \in [0, \infty)$, i.e.,

$$\|b(x - y) + Tx - Ty\| \leq (b + 1)\|x - y\|, \quad \text{for all } x, y \in X. \tag{4.17}$$

Therefore, from (4.16) and (4.17) we have

$$2\Big(b^2 + \frac{1 + 2k}{1 - 2k}\Big) = (b + 1)^2,$$

which implies,

$$b^2 - 2b + \frac{6k + 1}{1 - 2k} = 0. \tag{4.18}$$

Now, the equation (4.18) has a solution if $4 - 4\frac{6k+1}{1-2k} \geq 0$, which implies $k \leq 0$, which is a contradiction.

Therefore there does not exist any b such that the following holds

$$\|b(x - y) + Tx - Ty\| \leq (b + 1)\|x - y\| \quad \text{for all } x, y \in X,$$

which shows that T is not an enriched non-expansive map in the Banach space.

Hence the desired assertion. □

4.4 Concluding remarks

The iterative algorithms can approximate the fixed points of demi-contractive mappings, which is one of the largest discontinuous class of non-expansive mappings, and include other important classes of mappings, e.g., non-expansive, quasi non-expansive, k-strictly pseudo-contractive, etc., which play a fundamental role in nonlinear analysis to solve various fixed point problems, common fixed point problems, convex problems, split minimization problems, split common fixed point problems, etc.

In this chapter, convergence theorems have been derived for the Krasnoselskii-Mann iteration and the modified Krasnoselskii-Mann iteration with fixed points of strictly pseudo-contractive map in the context of Banach space, and these can be derived from corresponding convergence theorems that are found in the setting of non-expansive maps in the Banach space.

Utilizing the concept of enriching contractive type map T, we have deduced that the class of k-strictly pseudo-contractive maps is an unsaturated class of mappings in the Banach space. Therefore, this class can be enlarged by the technique of enriching contractive mappings in the Banach space.

We conclude from the numerical experiments presented here that for approximating the fixed points of some pseudo-contractive mappings, the Krasnoselskii-Mann iteration is more convenient than the modified Krasnoselskii-Mann iteration.

It will be interesting to see if one can transpose similar fixed point theorems for the convergence of the Krasnoselskii-Mann iteration and also the modified Krasnoselskii-Mann iteration for the strictly pseudo-contractive map from Banach spaces to quasi-Banach spaces [5, 54].

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