

Chapter 6

DEM based Discrete-Time Twisting-like Algorithm and Discrete-Time Super-Twisting-like Algorithm

6.1 Introduction

In this chapter, a new approach based on the difference equation with minima is introduced for discrete-time twisting-like and super-twisting-like algorithms. These algorithms, distinct from their counterparts that commonly rely on Euler-discretization methods, are designed for both unperturbed and perturbed scenarios. Furthermore, the discrete-time super-twisting-like algorithm is explored for an essential application, the development of a discrete-time super-twisting observer (STO) aiming to achieve finite-time estimation of system states. The chapter includes an illustrative example involving a perturbed pendulum system with both known and unknown inputs, showcasing the efficacy of the proposed algorithm through simulation results.

The structure of this chapter is outlined as follows. Section 6.2 presents the discrete-time twisting-like algorithm along with the simulation results. In Section 6.3, the results pertaining to discrete-time super-twisting-like algorithm are presented along with the simulation results. Section 6.4 focuses on the design of discrete-time super-twisting observer which is used for the state estimation problem of a perturbed pendulum system. Finally,

Section 6.5 concludes the chapter.

6.2 DEM based Twisting-like Algorithm for Discrete-Time System

Within this section, a new type of discrete-time twisting-like algorithm designed for discrete-time systems is presented. This algorithm compellingly guides the system states to decay to zero within finite time steps. Notably, the algorithm avoids chattering altogether, thereby eliminating the requirement for unnecessary control effort.

Consider the following dynamical system

$$\begin{aligned} z_1(k+1) &= \mu_1 z_2(k), \\ z_2(k+1) &= \mu_2 z_1(k) - \text{sign}(z_1(k)) \min\{\mu_2 |z_1(k)|, l_1\} + \mu_3 z_2(k) \\ &\quad - \text{sign}(z_2(k)) \min\{\mu_3 |z_2(k)|, l_2\} \end{aligned} \quad (6.1)$$

where $z = [z_1, z_2]^\top \in D \subset \mathbb{R}^2$ is the system state vector, $k \in \mathbb{Z}_{\geq 0}$, $l_1 > 0$, $l_2 > 0$, $\mu_1, \mu_2, \mu_3 \in (0, 1)$.

We propose the following theorem to prove the stability of this algorithm.

Theorem 6.1 *Consider the dynamical system (6.1) with $\mu_1 + \mu_3 \leq 1$. Let $z = 0$ be the equilibrium point of (6.1), which is contained in domain D . Assume that there exist a continuous function $V : D \rightarrow \mathbb{R}_+$, which satisfies the following*

$$V(z) > 0, \quad \text{for all } z \in D \setminus \{0\}, \quad (6.2)$$

$$V(z = 0) = 0, \quad (6.3)$$

$$\Delta V(z) \leq -\min\{V(z), l\}, \quad (6.4)$$

where $l = \min\{l_1, l_2\}$. Then the solution $z(k) \equiv 0$, is finite-time stable. Moreover, the settling time function is given as, $\mathcal{K}(z_0) \leq \lceil \frac{V(z_0)}{l} \rceil$, $z_0 \in D$, where $\mathcal{K}(z_0)$ is lower semicontinuous on D .

Proof. Let us consider the Lyapunov candidate functional as $V(z(k)) = |z_1(k)| + |z_2(k)|$. Then, $V(z(k+1)) = |z_1(k+1)| + |z_2(k+1)|$. We obtain the change in Lyapunov energy

as

$$\begin{aligned}\Delta V(z) &= V(z(k+1)) - V(z(k)) \\ &= |\mu_1 z_2(k)| + |\mu_2 z_1(k) - \text{sign}(z_1(k)) \min\{\mu_2 |z_1(k)|, l_1\} + \mu_3 z_2(k) \\ &\quad - \text{sign}(z_2(k)) \min\{\mu_3 |z_2(k)|, l_2\} - |z_1(k)| - |z_2(k)|.\end{aligned}$$

Since, $\mu_2 |z_1(k)| \geq \min\{\mu_2 |z_1(k)|, l_1\}$, and $\mu_3 |z_2(k)| \geq \min\{\mu_3 |z_2(k)|, l_2\}$, then

$$\begin{aligned}\Delta V(z) &\leq (\mu_2 - 1)|z_1(k)| + (\mu_1 + \mu_3 - 1)|z_2(k)| \\ &\quad - |\min\{\mu_2 |z_1(k)|, l_1\}| - |\min\{\mu_3 |z_2(k)|, l_2\}|.\end{aligned}\tag{6.5}$$

Inequality (6.5) is further deduced based on the outcome of the function $\min\{\cdot, \cdot\}$. This results in four possible scenarios, which cover the complete admissible state space.

Scenario I: $\mu_2 |z_1(k)| \leq l_1$ and $\mu_3 |z_2(k)| > l_2$

With this scenario, (6.5) can be further written as,

$$\Delta V(z) \leq (\mu_2 - 1)|z_1(k)| + (\mu_1 + \mu_3 - 1)|z_2(k)| - \mu_2 |z_1(k)| - l_2$$

Since, $\mu_1 + \mu_3 \leq 1$, it further follows

$$\Delta V(z) \leq -|z_1(k)| - l_2 \leq -l_2.\tag{6.6}$$

Scenario II: $\mu_2 |z_1(k)| > l_1$ and $\mu_3 |z_2(k)| \leq l_2$

This scenario further solves the Inequality (6.5) to

$$\Delta V(z) \leq (\mu_2 - 1)|z_1(k)| + (\mu_1 + \mu_3 - 1)|z_2(k)| - l_1 - \mu_3 |z_2(k)|$$

which further simplifies as

$$\begin{aligned}\Delta V(z) &\leq - (1 - \mu_2)|z_1(k)| - (1 - \mu_1)|z_2(k)| - l_1 \\ &\leq -l_1.\end{aligned}\tag{6.7}$$

Scenario III: $\mu_2 |z_1(k)| > l_1$ and $\mu_3 |z_2(k)| > l_2$

Here, inequality (6.5) becomes,

$$\Delta V(z) \leq (\mu_2 - 1)|z_1(k)| + (\mu_1 + \mu_3 - 1)|z_2(k)| - l_1 - l_2$$

and it solves further as

$$\Delta V(z) \leq -l_1 - l_2 \leq -l.\tag{6.8}$$

Scenario IV: $\mu_2|z_1(k)| \leq l_1$ and $\mu_3|z_2(k)| \leq l_2$

With this scenario, inequality (6.5) solves to,

$$\Delta V(z) \leq (\mu_2 - 1)|z_1(k)| + (\mu_1 + \mu_3 - 1)|z_2(k)| - \mu_2|z_1(k)| - \mu_3|z_2(k)|$$

which further simplifies as,

$$\Delta V(z) \leq -(V(z(k)) - \mu_1|z_2(k)|)$$

Additionally, if $z_2(0) = 0$, then from (6.1), the inequality further solves to

$$\Delta V(z) \leq -V(z(k)). \quad (6.9)$$

Note that the condition in (6.9) will yield dead-beat response, which will collapse whole system dynamics. Moreover, if $z_2(0) \neq 0$, then the governing dynamics (6.1) will transform *Scenario IV* to either of the other scenarios. Finally, combining all four conditions from (6.6), (6.7), (6.8) and (6.9), we obtain the following

$$\Delta V(z) \leq -\min\{V(z), l\}. \quad (6.10)$$

Further, the calculation of the settling time function from (6.10) is simple and follows the similar steps as demonstrated in the proof of Theorem 1. Hence, the expression for this can be obtained as $\mathcal{K}(z_0) \leq \lceil \frac{V(z_0)}{l} \rceil$. \square

Remark 6.2 *The twisting-like algorithm works well even in the presence of bounded perturbation. However, the states would remain in an invariant set instead of going to zero. Further, the Lyapunov based condition for that case would be, $\Delta V(z) \leq -\min\{V(z(k)), l\} + 2d_0$, with settling time function expressed as, $\mathcal{K}(d_0, z_0) \leq \lceil \frac{V(z_0) - 2d_0}{l - 2d_0} \rceil$, $z_0 \in D \setminus \mathcal{B}_{2d_0}(0)$, $l > 2d_0$.*

6.2.1 Illustrative Example

Consider the dynamical system as in (6.1). The initial condition for the system is taken as $z_1(0) = 10$ and $z_2(0) = 15$ and the values of the design parameters are selected as $l_1 = 2$, $l_2 = 2$, $\mu_1 = 0.5$, $\mu_2 = 0.5$ and $\mu_3 = 0.4$. The trajectory of the system states is shown in Figure 6.1. Since there is no perturbation, the system states converge to the origin in finite time. Figure 6.2 depicts the phase portrait of the system, which shows the convergence of both the states in finite-time steps.

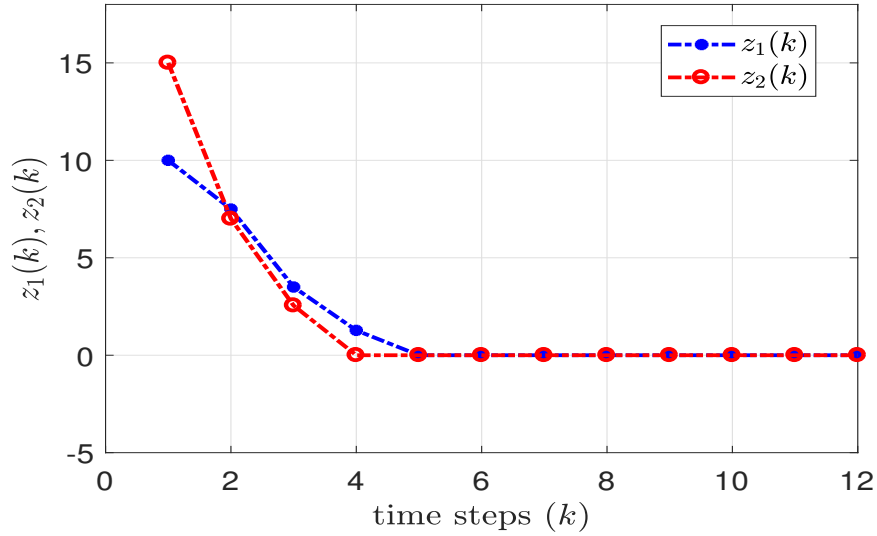


Figure 6.1: Evolution of states of the system

6.3 DEM based Discrete-Time Super-Twisting-like Algorithm

In this section, we introduce a new variant of a super-twisting-like algorithm designed in the framework of difference equation with minima specifically tailored for discrete-time systems. Distinguishing itself from the Euler-discretized super-twisting algorithms found in existing literature, this proposed algorithm stands out for its notable capability to entirely eliminate chattering.

6.3.1 Super-Twisting-like Algorithm for Unperturbed DTS

Consider the following nonlinear discrete-time dynamics

$$\begin{aligned}
 z_1(k+1) &= \wp_1 z_1(k) - l_1 \text{sign}(z_1(k)) \min \left\{ \frac{\wp_1 |z_1(k)|}{l_1}, |z_1(k)|^{\frac{1}{2}} \right\} + z_2(k), \\
 z_2(k+1) &= \wp_2 z_1(k) - \text{sign}(z_1(k)) \min \{ \wp_2 |z_1(k)|, l_2 \}
 \end{aligned} \tag{6.11}$$

where $z = [z_1, z_2]^\top \in D \subseteq \mathbb{R}^2$ denotes the system states, $k \in \mathbb{Z}_{\geq 0}$, $\wp_1, \wp_2 \in (0, 1)$, $l_1, l_2 \in \mathbb{R}_+$.

Remark 6.3 A paramount observation is that, the algorithm proposed is not the Euler-discretization of the continuous-time STA. In particular, it is to be noted that the sec-

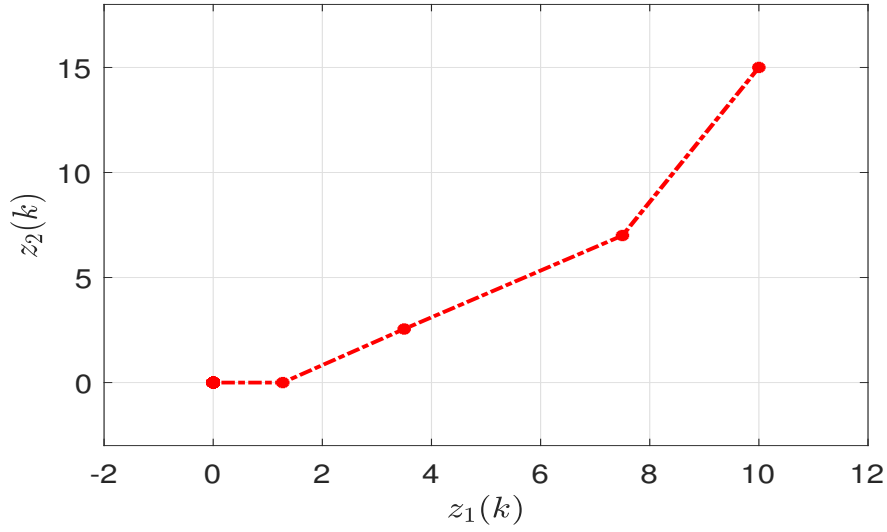


Figure 6.2: Phase portrait of the system

ond equation in (6.11) has $z_1(k)$ instead of $z_2(k)$, which is different from the Euler-discretization based STAs (see [68, 70, 72, 73]).

In the following theorem, we establish the FTS of the zero solution of (6.11).

Theorem 6.4 Consider the system (6.11) with $\wp_1 + \wp_2 \leq 1$ and $l_1 > \left(1 + \left|\frac{z_2(k)}{z_1(k)}\right|\right)^{\frac{1}{2}}$, $z_1(k) \neq 0$. Let $z = 0$ be an equilibrium point for (6.11) and $D \subset \mathbb{R}^2$ be a domain containing $z = 0$. Assume that there exist a continuous function $V : D \rightarrow \mathbb{R}_{\geq 0}$, which satisfies:

$$V(z) > 0, \forall z \in D \setminus \{0\}, \quad (6.12)$$

$$V(z=0) = 0 \text{ and} \quad (6.13)$$

$$\Delta V(z) \leq -\min\{V(z), (V(z))^{\frac{1}{2}}, l_2\}, \forall z \in D \setminus \{0\}. \quad (6.14)$$

Then the solution $z(k) \equiv 0$ exhibits finite-time stability. Moreover, if $D = \mathbb{R}^2$, $V(\cdot)$ is radially unbounded and $\Delta V(z) \leq -\min\{V(z), (V(z))^{\frac{1}{2}}, l_2\}$, $\forall z \in \mathbb{R}^2 \setminus \{0\}$, then the solution $z(k) \equiv 0$ exhibits global finite-time stability.

Proof: We consider the scenario when $z_1(k) \neq 0$, since otherwise for $z_1(k) = 0$, (6.11) would result in $z_1(k+1) = z_2(k)$; $z_2(k+1) = 0$, thereby evolving further and resulting in $z_1(k) \neq 0$ until $z_1(k) = z_2(k) = 0$. Let us construct a Lyapunov functional candidate as

$V(z(k)) = |z_1(k)| + |z_2(k)|$. Then, $V(z(k+1)) = |z_1(k+1)| + |z_2(k+1)|$. Further, the change in the Lyapunov energy is

$$\begin{aligned}\Delta V(z) &= V(z(k+1)) - V(z(k)) \\ &= \left| \wp_1 z_1(k) - l_1 \text{sign}(z_1(k)) \min \left\{ \frac{\wp_1 |z_1(k)|}{l_1}, |z_1(k)|^{\frac{1}{2}} \right\} \right| + |z_2(k)| \\ &\quad + \left| \wp_2 z_1(k) - \text{sign}(z_1(k)) \min \{ \wp_2 |z_1(k)|, l_2 \} \right| - |z_1(k)| - |z_2(k)|\end{aligned}$$

Since, $\wp_1 |z_1(k)| \geq l_1 \min \left\{ \frac{\wp_1 |z_1(k)|}{l_1}, |z_1(k)|^{\frac{1}{2}} \right\}$ and $\wp_2 |z_1(k)| \geq \min \{ \wp_2 |z_1(k)|, l_2 \}$, we can further write

$$\begin{aligned}\Delta V(z) &\leq - \left| l_1 \min \left\{ \frac{\wp_1 |z_1(k)|}{l_1}, |z_1(k)|^{\frac{1}{2}} \right\} \right| \\ &\quad - \left| \min \{ \wp_2 |z_1(k)|, l_2 \} \right| + (\wp_1 + \wp_2 - 1) |z_1(k)|\end{aligned}\tag{6.15}$$

We discuss four possible cases which arise out of the system (6.11) and cover the whole state space, based on the function obtained as the output of the minimum function.

The four cases are:

- (i) If $\frac{\wp_1 |z_1(k)|}{l_1} \leq |z_1(k)|^{\frac{1}{2}}$ and $\wp_2 |z_1(k)| > l_2$,
- (ii) If $\frac{\wp_1 |z_1(k)|}{l_1} > |z_1(k)|^{\frac{1}{2}}$ and $\wp_2 |z_1(k)| \leq l_2$,
- (iii) If $\frac{\wp_1 |z_1(k)|}{l_1} > |z_1(k)|^{\frac{1}{2}}$ and $\wp_2 |z_1(k)| > l_2$,
- (iv) If $\frac{\wp_1 |z_1(k)|}{l_1} \leq |z_1(k)|^{\frac{1}{2}}$ and $\wp_2 |z_1(k)| \leq l_2$.

These cases are analysed and difference inequality (6.15) is further deduced.

Case I: If $\frac{\wp_1 |z_1(k)|}{l_1} \leq |z_1(k)|^{\frac{1}{2}}$ and $\wp_2 |z_1(k)| > l_2$

With $\wp_1 + \wp_2 \leq 1$, one can further write,

$$\Delta V(z) \leq -l_1 \min \left\{ \frac{\wp_1 |z_1(k)|}{l_1}, |z_1(k)|^{\frac{1}{2}} \right\} - \min \{ \wp_2 |z_1(k)|, l_2 \}$$

Further it follows,

$$\Delta V(z) \leq -\wp_1 |z_1(k)| - l_2 \leq -l_2\tag{6.16}$$

Case II: If $\frac{\wp_1 |z_1(k)|}{l_1} > |z_1(k)|^{\frac{1}{2}}$ and $\wp_2 |z_1(k)| \leq l_2$

With this case, along with $\wp_1 + \wp_2 \leq 1$, one can further write (6.15) as

$$\Delta V(z) \leq -l_1 |z_1(k)|^{\frac{1}{2}} - \wp_2 |z_1(k)| \leq -l_1 |z_1(k)|^{\frac{1}{2}}$$

Since, l_1 is selected such that condition $l_1 > \left(1 + \left|\frac{z_2(k)}{z_1(k)}\right|\right)^{\frac{1}{2}}$, then

$$\Delta V(z) \leq - (V(z(k)))^{\frac{1}{2}} \quad (6.17)$$

Case III: If $\frac{\wp_1|z_1(k)|}{l_1} > |z_1(k)|^{\frac{1}{2}}$ and $\wp_2|z_1(k)| > l_2$

Considering (6.15) along with $\wp_1 + \wp_2 \leq 1$, it follows that,

$$\Delta V(z) \leq -l_1|z_1(k)|^{\frac{1}{2}} - l_2 \leq -\min\{l_1|z_1(k)|^{\frac{1}{2}}, l_2\}$$

Additionally, with $l_1 > \left(1 + \left|\frac{z_2(k)}{z_1(k)}\right|\right)^{\frac{1}{2}}$, one can further write,

$$\Delta V(z) \leq -\min\{(V(z(k)))^{\frac{1}{2}}, l_2\} \quad (6.18)$$

Case IV: If $\frac{\wp_1|z_1(k)|}{l_1} \leq |z_1(k)|^{\frac{1}{2}}$ and $\wp_2|z_1(k)| \leq l_2$

With this case, (6.15) can be further written as

$$\Delta V(z) \leq -(V(z(k)) - |z_2(k)|)$$

Moreover, if $z_2(0) = 0$, then from (6.11) it directly follows that

$$\Delta V(z) \leq -V(z(k)) \quad (6.19)$$

Note from the equation (6.19) that, $\Delta V(z) \leq -V(z(k)) \implies V(z(k+1)) \leq 0$ in just one sample instant showing the dead-beat type response. Further, with *Case IV*, and if $z_2(0) \neq 0$, then with the evolution of (6.11), *Case IV* transforms into either *Case I*, *Case II* or *Case III*. Furthermore, by taking into consideration the conditions obtained in (6.16), (6.17), (6.18) and (6.19), it follows that,

$$\Delta V(z) \leq -\min\{V(z(k)), (V(z(k)))^{\frac{1}{2}}, l_2\} \quad (6.20)$$

Hence, the solution $z(k) \equiv 0$ is FTS. Furthermore, since, $V(\cdot)$ is radially unbounded, and if $D = \mathbb{R}^2$, then the global FTS follows in accordance with the similar arguments presented above. \square

Remark 6.5 *Note that, with the conditions given in Case IV, system (6.11) behaves as a linear system in the neighbourhood of origin. Additionally, if (6.19) satisfies, then system exhibits a dead-beat type response and hence converging to the origin in just one sample instant.*

Remark 6.6 In (6.20), with the trajectory $V(z(k))$ entering the unit ball, it comes out to be minimum of all the arguments and hence stimulating a dead-beat type response, resulting in convergence to the origin.

Remark 6.7 It may seem that when $z_1 \rightarrow 0$, $l_1 \rightarrow \infty$, but this is not the case, since at this instant $z_2 \rightarrow 0$ too. Thus, l_1 will always satisfy the gain condition mentioned in the theorem.

6.3.2 Super-Twisting-like Algorithm for Perturbed DTS

Consider the following nonlinear DTS

$$\begin{aligned} z_1(k+1) &= \wp_1 z_1(k) - l_1 \text{sign}(z_1(k)) \min \left\{ \frac{\wp_1 |z_1(k)|}{l_1}, |z_1(k)|^{\frac{1}{2}} \right\} + z_2(k), \\ z_2(k+1) &= \wp_2 z_1(k) - \text{sign}(z_1(k)) \min \{ \wp_2 |z_1(k)|, l_2 \} + \delta(k) \end{aligned} \quad (6.21)$$

where system variables and design parameters are defined as earlier. $\delta(k)$ represents the bounded type perturbation such that $\delta : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}$, and satisfies $|\delta(k)| \leq \delta_0$. We propose the following theorem to establish the FTISS of the system (6.21).

Theorem 6.8 Consider the system (6.21) with $\wp_1 + \wp_2 \leq 1$ and $l_1 > \left(1 + \left| \frac{z_2(k)}{z_1(k)} \right| \right)^{\frac{1}{2}}$, $z_1(k) \neq 0$ and $l_2 > \delta_0$. Assume that there exist a continuous function $V : \mathbb{Z}_{\geq 0} \times D \rightarrow \mathbb{R}_{\geq 0}$, which satisfies:

$$\alpha_1(\|z(k)\|) \leq V(k, z) \leq \alpha_2(\|z(k)\|), \forall k \in \mathbb{Z}_{\geq 0}, \forall z \in D \setminus \{0\}, \quad (6.22)$$

$$V(k, 0) = 0, \forall k \in \mathbb{Z}_{\geq 0} \text{ and} \quad (6.23)$$

$$\Delta V(k, z) \leq -\min\{V(k, z), (V(k, z))^{\frac{1}{2}}, l_2\} + \alpha_3(|\delta(k)|), \forall k \in \mathbb{Z}_{\geq 0}, \forall z \in D \setminus \{0\}, \quad (6.24)$$

where α_1, α_2 and $\alpha_3 \in$ class \mathcal{K} functions. Then the system (6.21) is FTISS. Moreover, if $D = \mathbb{R}^2$, α_1 and $\alpha_2 \in$ class \mathcal{K}_{∞} functions and $\Delta V(k, z) \leq -\min\{V(k, z), (V(k, z))^{\frac{1}{2}}, l_2\} + \alpha_3(|\delta(k)|), \forall k \in \mathbb{Z}_{\geq 0}, \forall z \in \mathbb{R}^2 \setminus \{0\}$, then the system (6.21) is global FTISS.

Proof: As mentioned previously, we take the scenario when $z_1(k) \neq 0$. Let us construct a Lyapunov functional candidate as $V(z(k)) = |z_1(k)| + |z_2(k)|$. Then, $V(z(k+1)) =$

$|z_1(k+1)| + |z_2(k+1)|$. The change in the Lyapunov energy is

$$\begin{aligned}\Delta V(k, z) &= V(z(k+1)) - V(z(k)) \\ &= |\wp_1 z_1(k) - l_1 \text{sign}(z_1(k)) \min \left\{ \frac{\wp_1 |z_1(k)|}{l_1}, |z_1(k)|^{\frac{1}{2}} \right\} + z_2(k)| \\ &\quad + |\wp_2 z_1(k) - \text{sign}(z_1(k)) \min \{ \wp_2 |z_1(k)|, l_2 \} + \delta(k)| \\ &\quad - |z_1(k)| - |z_2(k)|\end{aligned}$$

Since, $\wp_1 |z_1(k)| \geq l_1 \min \left\{ \frac{\wp_1 |z_1(k)|}{l_1}, |z_1(k)|^{\frac{1}{2}} \right\}$ and $\wp_2 |z_1(k)| \geq \min \{ \wp_2 |z_1(k)|, l_2 \}$, we can further write

$$\begin{aligned}\Delta V(k, z) &\leq - \left| l_1 \min \left\{ \frac{\wp_1 |z_1(k)|}{l_1}, |z_1(k)|^{\frac{1}{2}} \right\} \right| + |\delta(k)| \\ &\quad - \left| \min \{ \wp_2 |z_1(k)|, l_2 \} \right| + (\wp_1 + \wp_2 - 1) |z_1(k)|.\end{aligned}\tag{6.25}$$

Different cases are analysed from here and difference inequality (6.25) is deduced further.

Case I: If $\frac{\wp_1 |z_1(k)|}{l_1} \leq |z_1(k)|^{\frac{1}{2}}$ and $\wp_2 |z_1(k)| > l_2$

With $\wp_1 + \wp_2 \leq 1$, one can further write,

$$\Delta V(k, z) \leq -l_1 \min \left\{ \frac{\wp_1 |z_1(k)|}{l_1}, |z_1(k)|^{\frac{1}{2}} \right\} - \min \{ \wp_2 |z_1(k)|, l_2 \} + |\delta(k)|.$$

Further, it follows that,

$$\begin{aligned}\Delta V(k, z) &\leq -\wp_1 |z_1(k)| - l_2 + |\delta(k)| \\ &\leq -l_2 + |\delta(k)|.\end{aligned}\tag{6.26}$$

Case II: If $\frac{\wp_1 |z_1(k)|}{l_1} > |z_1(k)|^{\frac{1}{2}}$ and $\wp_2 |z_1(k)| \leq l_2$

With $\wp_1 + \wp_2 \leq 1$, one can further write (6.25) as,

$$\begin{aligned}\Delta V(k, z) &\leq -l_1 |z_1(k)|^{\frac{1}{2}} - \wp_2 |z_1(k)| + |\delta(k)| \\ &\leq -l_1 |z_1(k)|^{\frac{1}{2}} + |\delta(k)|.\end{aligned}\tag{6.27}$$

Further, with $l_1 > \left(1 + \left| \frac{z_2(k)}{z_1(k)} \right| \right)^{\frac{1}{2}}$

$$\Delta V(k, z) \leq - (V(z(k)))^{\frac{1}{2}} + |\delta(k)|.\tag{6.28}$$

Case III: If $\frac{\wp_1|z_1(k)|}{l_1} > |z_1(k)|^{\frac{1}{2}}$ and $\wp_2|z_1(k)| > l_2$

Considering (6.25) and with $\wp_1 + \wp_2 \leq 1$, it follows that,

$$\begin{aligned}\Delta V(k, z) &\leq -l_1|z_1(k)|^{\frac{1}{2}} - l_2 + |\delta(k)| \\ &\leq -\min\{l_1|z_1(k)|^{\frac{1}{2}}, l_2\} + |\delta(k)|.\end{aligned}\quad (6.29)$$

Moreover, with $l_1 > \left(1 + \left|\frac{z_2(k)}{z_1(k)}\right|\right)^{\frac{1}{2}}$, one can further write,

$$\Delta V(k, z) \leq -\min\{(V(z(k)))^{\frac{1}{2}}, l_2\} + |\delta(k)|. \quad (6.30)$$

Case IV: If $\frac{\wp_1|z_1(k)|}{l_1} \leq |z_1(k)|^{\frac{1}{2}}$ and $\wp_2|z_1(k)| \leq l_2$

With this case, (6.25) can be further written as,

$$\Delta V(k, z) \leq -(V(z(k)) - |z_2(k)|) + |\delta(k)|$$

Further, if $z_2(0) = 0$, then from (6.21) it directly follows that

$$\Delta V(k, z) \leq -V(z(k)) + |\delta(k)|. \quad (6.31)$$

Further, with *Case IV*, and if $z_2(0) \neq 0$, then with the evolution of (6.21), *Case IV* transforms into either *Case I*, *Case II* or *Case III*. Furthermore, by taking into consideration the conditions obtained in (6.26), (6.28), (6.30) and (6.31), it follows that

$$\Delta V(k, z) \leq -\min\{V(z(k)), (V(z(k)))^{\frac{1}{2}}, l_2\} + \alpha_3(|\delta(k)|) \quad (6.32)$$

Hence, the system (6.21) is FTISS. Furthermore, since, $V(\cdot)$ is radially unbounded, and if $D = \mathbb{R}^2$, then the global FTISS follows as per the similar arguments presented above. This completes the proof. \square

6.3.3 Simulation Result for Super-Twisting-like Algorithm

The MATLAB simulation is done for three different sets of initial conditions to validate the effectiveness of the proposed algorithm. The system parameters were chosen as $l_1 = 3$, $l_2 = 5$, $\wp_1 = 0.6$ and $\wp_2 = 0.4$. The trajectories of the system states without perturbation are shown in Fig. 6.3. The trajectory $z_1(k)$ is shown in Fig. 6.3(a) and trajectory $z_2(k)$ can be seen in Fig. 6.3(b). One can clearly validate from the Fig. 6.3 that, the super-twisting-like algorithm makes the system states converge to the origin in finite-time steps.

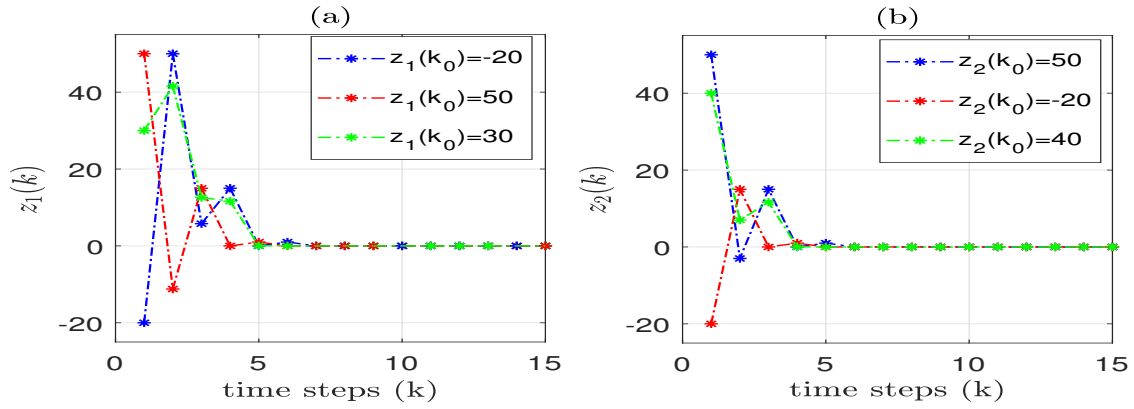


Figure 6.3: Evolution of states of the unperturbed DTS

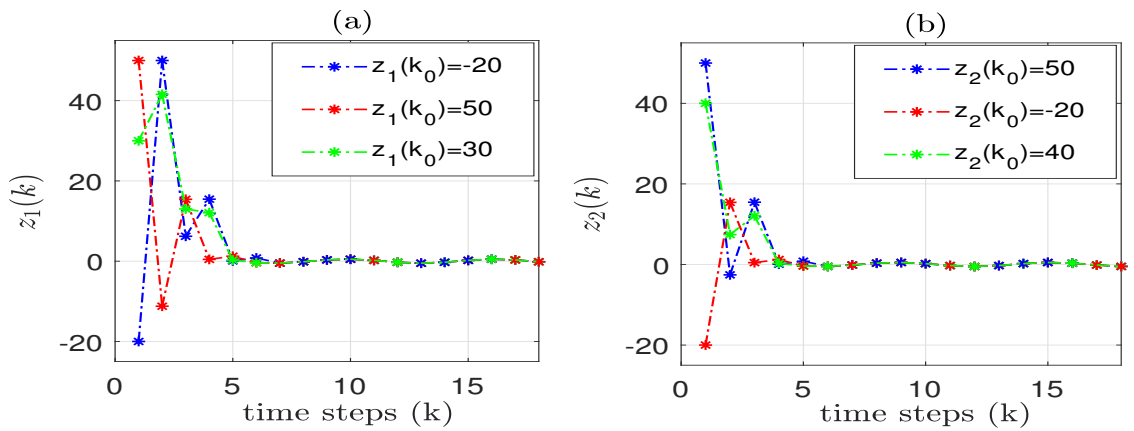


Figure 6.4: Evolution of states of the perturbed DTS

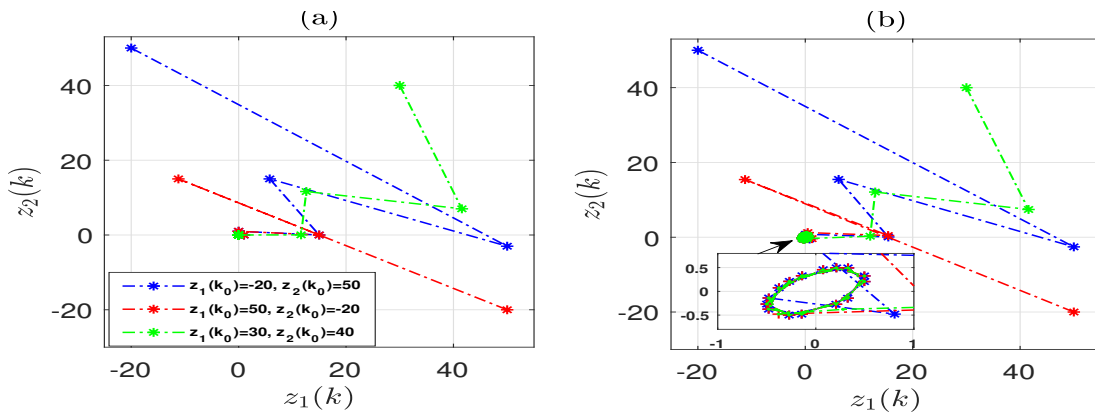


Figure 6.5: Phase portrait: (a) unperturbed and (b) perturbed, DTS

The corresponding phase portrait of the system is shown in Fig. 6.5(a). It can be seen that the algorithm drives the states to zero in finite-time steps. Further, with MTBP as $0.5 \sin(k)$, the states are shown in Fig. 6.4. It can be seen from Fig. 6.4(a) and Fig.

6.4(b) that, in presence of MTBP the states reside in the vicinity of the origin, that can be seen in Fig. 6.5(b).

Remark 6.9 *Algorithm proposed for unperturbed DTS in (6.11) drives the system states to origin rather than keeping them in the neighbourhood of the origin. In most of the Euler-discretization based STAs ([72, 74, 75]), the system states ultimately reside in an invariant set containing origin but not able to converge to the origin.*

6.4 Difference Equation with Minima based STO

In this section, the construction of a new type of observer derived from the proposed algorithm is discussed. This observer estimates the system states in finite time. Consider the following dynamical system

$$\begin{aligned} z_1(k+1) &= z_2(k) \\ z_2(k+1) &= f(z_1(k), z_2(k), u(k)) + \delta(k) \end{aligned} \quad (6.33)$$

with system output given as $y(k) = z_1(k)$, where $z = [z_1, z_2]^\top \in D \subseteq \mathbb{R}^2$ are system states, $k \in \mathbb{Z}_{\geq 0}$, $u \in \mathbb{R}$ is the input and $\delta(k)$ represents the MTBP and satisfies $|\delta(k)| \leq \delta_0$.

We construct the difference equation with minima based STO as

$$\begin{aligned} \hat{z}_1(k+1) &= \varphi_1(\hat{z}_1(k) - z_1(k)) - l_1 \text{sign}(\hat{z}_1(k) - z_1(k)) \\ &\quad \min \left\{ \frac{\varphi_1 |\hat{z}_1(k) - z_1(k)|}{l_1}, |\hat{z}_1(k) - z_1(k)|^{\frac{1}{2}} \right\} + \hat{z}_2(k), \\ \hat{z}_2(k+1) &= \varphi_2(\hat{z}_2(k) - z_2(k)) - \text{sign}(\hat{z}_2(k) - z_2(k)) \\ &\quad \min\{\varphi_2 |\hat{z}_2(k) - z_2(k)|, l_2\} + \hat{f}(z_1(k), \hat{z}_2(k), u(k)) \end{aligned} \quad (6.34)$$

where $l_1, l_2 \in \mathbb{R}_+$ and $\varphi_1, \varphi_2 \in (0, 1)$ are positive constants chosen appropriately. We define the error variables as $e_1 := \hat{z}_1(k) - z_1(k)$ and $e_2 := \hat{z}_2(k) - z_2(k)$. The error dynamics attains the form as that of the discussed super-twisting-like algorithm in (6.11):

$$\begin{aligned} e_1(k+1) &= \varphi_1 e_1(k) - l_1 \text{sign}(e_1(k)) \min \left\{ \frac{\varphi_1 |e_1(k)|}{l_1}, |e_1(k)|^{\frac{1}{2}} \right\} + e_2(k), \\ e_2(k+1) &= \varphi_2 e_2(k) - \text{sign}(e_2(k)) \min\{\varphi_2 |e_2(k)|, l_2\} + \tilde{f}(z_1(k), \hat{z}_2(k), u(k)) - \delta(k) \end{aligned} \quad (6.35)$$

where $\tilde{f}(z_1(k), \hat{z}_2(k), u(k)) := \hat{f}(z_1(k), \hat{z}_2(k), u(k)) - f(z_1(k), z_2(k), u(k))$. Moreover, we

can further write

$$\begin{aligned} e_1(k+1) &= \wp_1 e_1(k) - l_1 \text{sign}(e_1(k)) \min \left\{ \frac{\wp_1 |e_1(k)|}{l_1}, |e_1(k)|^{\frac{1}{2}} \right\} + e_2(k), \\ e_2(k+1) &= \wp_2 e_1(k) - \text{sign}(e_1(k)) \min \{ \wp_2 |e_1(k)|, l_2 \} + \phi(k) \end{aligned} \quad (6.36)$$

where $\phi(k) := \tilde{f}(z_1(k), \hat{z}_2(k), u(k)) - \delta(k)$, and satisfies $|\phi(k)| \leq \phi_0$.

The following corollary is a consequence of Theorem 6.8.

Corollary 6.10 *Consider the dynamics in (6.36) with $\wp_1 + \wp_2 \leq 1$ and $l_1 > \left(1 + \left| \frac{e_2(k)}{e_1(k)} \right| \right)^{\frac{1}{2}}$, $e_1(k) \neq 0$ and $l_2 > \phi_0$. Assume that there exist a continuous function $V : \mathbb{Z}_{\geq 0} \times D \rightarrow \mathbb{R}_{\geq 0}$, which satisfies:*

$$\alpha_4(\|e\|) \leq V(k, e) \leq \alpha_5(\|e\|), \forall k \in \mathbb{Z}_{\geq 0}, \forall e \in D \setminus \{0\}, \quad (6.37)$$

$$V(k, 0) = 0, \forall k \in \mathbb{Z}_{\geq 0} \text{ and} \quad (6.38)$$

$$\Delta V(k, e) \leq -\min\{V(k, e), (V(k, e))^{\frac{1}{2}}, l_2\} + \alpha_6(|\phi(k)|), \forall k \in \mathbb{Z}_{\geq 0}, \forall e \in D \setminus \{0\} \quad (6.39)$$

where α_4, α_5 and $\alpha_6 \in$ class \mathcal{K} functions. Then the system (6.36) is FTISS. Moreover, if $D = \mathbb{R}^2$, α_4 and $\alpha_5 \in$ class \mathcal{K}_{∞} functions and $\Delta V(k, e) \leq -\min\{V(k, e), (V(k, e))^{\frac{1}{2}}, l_2\} + \alpha_6(|\phi(k)|), \forall k \in \mathbb{Z}_{\geq 0}, \forall e \in \mathbb{R}^2 \setminus \{0\}$, then the system (6.36) is global FTISS.

Proof: We take similar Lyapunov candidate as $V(e(k)) = |e_1(k)| + |e_2(k)|$. Then, $V(e(k+1)) = |e_1(k+1)| + |e_2(k+1)|$. The change in the Lyapunov energy is

$$\begin{aligned} \Delta V(k, e) &= \left| \wp_1 e_1(k) - l_1 \text{sign}(e_1(k)) \min \left\{ \frac{\wp_1 |e_1(k)|}{l_1}, |e_1(k)|^{\frac{1}{2}} \right\} \right. \\ &\quad \left. + e_2(k) \right| + \left| \wp_2 e_1(k) - \text{sign}(e_1(k)) \min \{ \wp_2 |e_1(k)|, l_2 \} - \phi(k) \right| \\ &\quad - |e_1(k)| - |e_2(k)| \end{aligned}$$

Further, with similar analysis as carried out in obtaining (6.26), (6.28), (6.30) and (6.31) for the super-twisting-like algorithm for perturbed DTS, we can further write,

$$\Delta V(k, e) \leq -\min\{V(k, e), (V(k, e))^{\frac{1}{2}}, l_2\} + \alpha_6(|\phi(k)|) \quad (6.40)$$

Hence, the error dynamical system (6.36) exhibits FTISS. Furthermore, since, $V(\cdot)$ is radially unbounded, and if $D = \mathbb{R}^2$, global FTISS follows as per the similar arguments presented in previous section. This completes the proof. \square

6.4.1 Illustrative Example

In this section, a state observation problem for perturbed pendulum system is discussed. We design an STO for this system and show that the estimation of states is achieved in finite time. Consider the following dynamical system

$$\begin{aligned} z_1(k+1) &= z_1(k) + z_2(k) \\ z_2(k+1) &= z_2(k) - \frac{mgl}{2J}\sin(z_1(k)) - \frac{B}{J}z_2(k) + \frac{1}{J}u(k) + \delta(k) \end{aligned} \quad (6.41)$$

with $y(k) = z_1(k)$ as an output of the system, where $z_1(k)$ is the pendulum angle from the mean position, $z_2(k)$ is the angular velocity, m is the mass of the pendulum, g is the acceleration due to gravity, l is the length of the pendulum, J is the inertia of the pendulum arm, B is the friction coefficient and $u(k)$ is the input which is taken as $0.5 \sin(0.6k)$. The bounded perturbation $\delta(k)$ is considered as $0.2 \sin(0.3k)$. The difference equation with minima based STO for the pendulum is designed as per the (6.34). The error dynamics of the observer takes the form as described in (6.35) and hence follows the FTISS. For simulation purpose, the system parameters are taken as follows: $m = 1.1\text{kg}$, $l = 1\text{m}$, $g = 9.81 \frac{\text{m}}{\text{s}^2}$, $B = 0.18 \frac{\text{kg}\cdot\text{m}}{\text{s}^2}$. The system initial conditions are taken as $z_1(0) = 5$ and $z_2(0) = 2$. The initial conditions of the STO are taken as $\hat{z}_1(0) = 0$ and $\hat{z}_2(0) = 0$. The parameters of the super-twisting observer are chosen as $l_1 = 4$, $l_2 = 6$, $\wp_1 = 0.5$ and $\wp_2 = 0.3$. The trajectories of the system states and observer states are shown in Fig. 6.6.

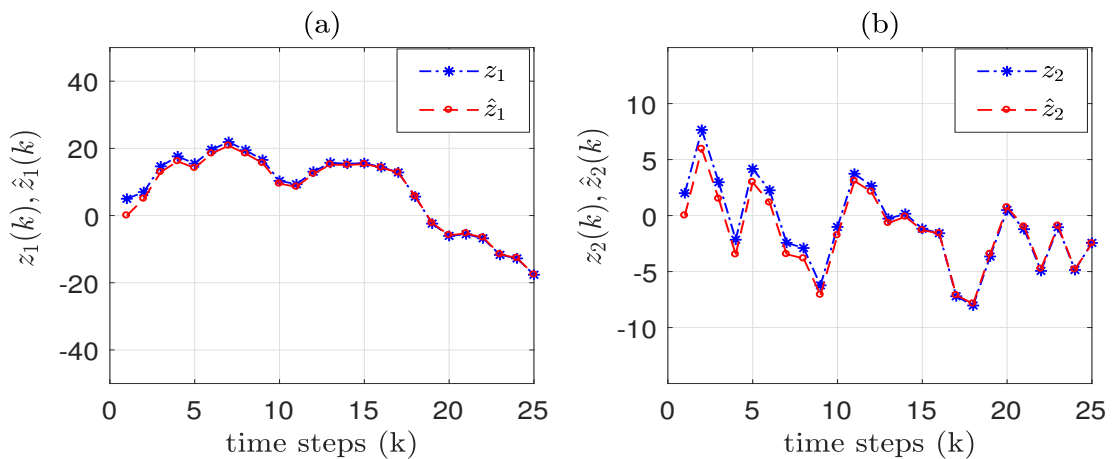


Figure 6.6: Evolution of states of the system and super-twisting observer

The estimation from the STO are very close to the actual states of the perturbed system, achieving this in finite-time steps. The phase portrait of the system errors is shown in

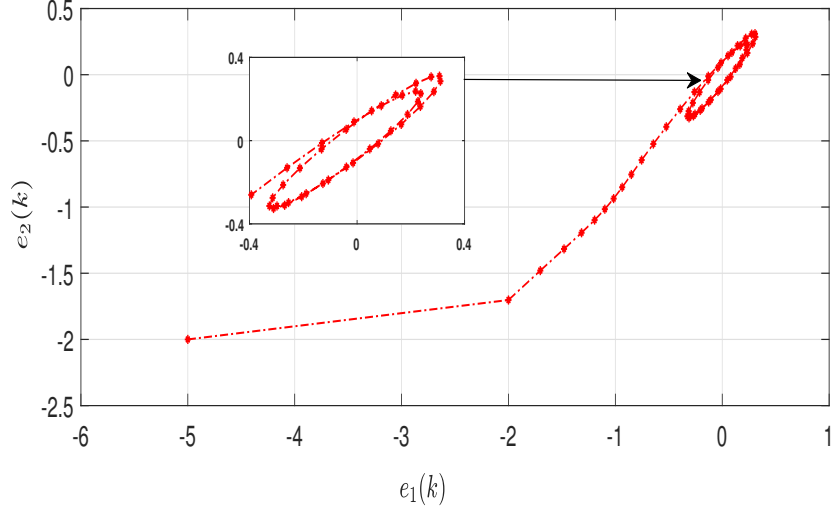


Figure 6.7: Phase portrait between error variables of the system

Fig. 6.7. It can be seen that the algorithm proposed drive the system error variables in an ultimately bounded region containing origin in finite-time steps.

6.5 Conclusion

This chapter delved into the DEM-based discrete-time twisting-like algorithm and discrete-time super-twisting-like algorithm, providing a departure from the conventional Euler-discretization-based approaches. The algorithms were comprehensively explored for both unperturbed and perturbed scenarios, accompanied by proofs for finite-time stability and finite-time input-to-state stability. Additionally, the chapter discussed a crucial application of the discrete-time super-twisting-like algorithm in designing a discrete-time super-twisting observer. An illustrative example featuring a pendulum system, incorporating both known and unknown inputs for the STO, was presented. Simulation results vividly illustrated the effectiveness of the DEM-based discrete-time super-twisting-like algorithm.

In the upcoming chapter, the emphasis shifts towards the practical application of the difference equation with minima-based discrete sliding mode control.