Chapter 3

Predefined-Time Convergent Gradient Flows to Solve Convex Optimization Problems

3.1 Introduction

This chapter first investigates unconstrained optimization problems through dynamical GF systems. For strongly convex objective functions, a GF with predefined time convergence is proposed, which is suitable for relaxing the objective function to only satisfy the Polyak Łojasiewicz (P-Ł) condition. In our schemes, the upper bound of convergence time can be chosen a priori, uniform with respect to initial conditions. Additionally, this approach does not involve complex parameter calculations. The convergence of proposed dynamics is rigorously proved by using the Lyapunov method. Further, in this chapter, we utilize the prospect of integrating the idea of a predefined-time stable gradient flow dynamics and projected gradient techniques to solve convex optimization problems with linear equality constraints (LEC). We propose a projected gradient flow algorithm with a predefined-time convergence property, where the upper bound of settling time can be selected a priori. A perturbed projected gradient flow is also studied within the framework of ISS. Examples based on linear least squares and resource allocation problems (RAP) are solved using the proposed dynamics and simulation results illustrate the effectiveness of the proposed methods.

The present chapter is structured as follows: Section 3.2 presents the problem statement. Section 3.3 introduces modified GF dynamics for solving unconstrained optimization problems. Section 3.5 focuses on solving convex optimization problems with LEC using predefined-time gradient flow dynamics. The efficacy of the methods proposed in Sections 3.3 and 3.5 is further validated in Sections 3.4 and 3.6, respectively, using illustrative examples like least squares.

3.2 Problem Statement

Consider the following unconstrained optimization problem

$$\min_{x \in \mathbb{R}^n} \mathcal{G}(x)$$

where $\mathcal{G} : D \to \mathbb{R}$ is a continuously differentiable function and $x \in \mathbb{R}^n$ is the decision variable. Suppose that $\mathcal{G}(x)$ satisfies that the gradient is locally Lipschitz continuous, and the optimal value of $\mathcal{G}(x)$ is attained, i.e., there exists x^* such that $\infty < \mathcal{G}^* = \mathcal{G}(x^*)$. Additionally, if $\mathcal{G}$ is a convex function, then $\|\nabla \mathcal{G}(x^*)\| = 0$ is both necessary and sufficient condition for the global minimum.

Lemma 3.1 *[10](Lojasiewicz's Inequality) Let $\mathcal{G} : \mathbb{R}^n \to \mathbb{R}$ be a real analytic function on a neighbourhood of $y \in \mathbb{R}^n$. Then there exists a constant $d > 0$ and $\kappa \in [0, 1)$ such that $\|\nabla \mathcal{G}(x)\| \geq d|\mathcal{G}(x) - \mathcal{G}(y)|^\kappa$ holds in some neighbourhood of y.*

Assumption 3.1 *(PL-Inequality (Gradient Dominance)) The scalar function $\mathcal{G}$ is a continuously differentiable function. It has a unique minimizer $x = x^*$ and is gradient dominated, i.e., $\exists\ \beta > 0$, such that*

$$\frac{1}{2}\|\nabla \mathcal{G}(x)\|^2 \geq \beta(\mathcal{G}(x) - \mathcal{G}^\star) \qquad \forall x \in \mathbb{R}^n \tag{3.1}$$

In what follows, we demonstrate that the solution to the proposed predefined-time convergent GF system corresponds to the solution of the previously stated optimization problem.

Remark 3.2 *In this chapter, we adhere to the notions of stability aligned with the arbitrary-time stability as discussed in [48]. To ensure consistency with existing literature [15, 49], we adopt the term "Predefined-time stability" instead of arbitrary time stability.*

3.3 Predefined-Time Convergent Gradient Flow Systems

In this section, we study the predefined-time gradient systems, whose convergence time can be explicitly given as a parameter in advance.
Consider the following non-autonomous GF system

$$\dot{x}(t) = \begin{cases} -\vartheta\varphi(t,x), & \text{for } \ t_0 \le t < t_f, \\ -\vartheta\nabla\mathcal{G}(x(t))^\top, & \text{for } \ t \ge t_f, \end{cases} \tag{3.2}$$

where $x(t) \in \mathbb{R}^n$, $\vartheta \in \mathbb{R}$ and satisfies $\vartheta \ge 1$, $\nabla\mathcal{G}(x(t)) = (\nabla\mathcal{G}_{x1}(x), \nabla\mathcal{G}_{x2}(x), \cdots, \nabla\mathcal{G}_{xn}(x)) \in \mathbb{R}^n$, $\varphi(t,x) = (\varphi_1, \varphi_2, ..., \varphi_n)^\top$, $\varphi_i := \frac{(e^{\nabla\mathcal{G}_{x_i}(x)}-1)}{e^{\nabla\mathcal{G}_{x_i}(x)}(t_f-t)}$, if $t_0 \le t < t_f$, for $1 \le i \le n$.

Definition 3.3 *(Predefined-time stable gradient flow system) The origin of the non-autonomous GF system 3.2 is called the predefined-time stable GF system if*

- *it is fixed-time stable,*
- *there exists an explicit scalar t_f , which is independent of any initial conditions and can be chosen a priori, and the condition $t_f > t_a$ follows for all $x(0) \in \mathbb{R}^n$, where t_a is the actual convergence time.*

Theorem 3.4 *Consider the GF system* (3.2) *and the objective function $\mathcal{G}$ satisfies Assumption 3.1. Assume a continuously differentiable proper loss function $V : \mathbb{R}^n \to \mathbb{R}$ for* (3.2)*, then the system* (3.2) *is a predefined-time stable gradient flow system.*

Proof: Consider the following proper loss function as

$$V(x) = \frac{1}{2}(\mathcal{G}(x) - \mathcal{G}^*)^2, \tag{3.3}$$

We begin the analysis for the time interval $t_0 \le t < t_f$. The time derivative of the appropriate loss function is given by

$$\begin{aligned} \dot{V}(x) =& (\mathcal{G}(x) - \mathcal{G}^*)\nabla\mathcal{G}(x(t))\big(\dot{x}\big) \\ =& (\mathcal{G}(x) - \mathcal{G}^*)\nabla\mathcal{G}(x(t))\big(-\vartheta\varphi(t,x)\big) \\ \le& -\vartheta(\mathcal{G}(x) - \mathcal{G}^*)\nabla\mathcal{G}_{x_i}(x)\varphi_i(t,x) \quad \text{for any } \ i \\ \le& -\vartheta\sqrt{2}V^{\frac{1}{2}}\frac{|\nabla\mathcal{G}_{x_i}(x)|(e^{|\nabla\mathcal{G}_{x_i}(x)|}-1)}{e^{|\nabla\mathcal{G}_{x_i}(x)|}(t_f-t)} \end{aligned} \tag{3.4}$$

Using Lojasiewicz's Inequality given in Lemma 3.1, note that there exists at least one i such that

$$|\nabla\mathcal{G}_{x_i}(x)| \geq \frac{d}{\sqrt{n}}|\mathcal{G}(x) - \mathcal{G}^*|^\kappa \tag{3.5}$$

since otherwise

$$\begin{aligned}\|\nabla\mathcal{G}(x)\| =& \sqrt{(\nabla\mathcal{G}_{x_1})^2 + (\nabla\mathcal{G}_{x_2})^2 + \cdots + (\nabla\mathcal{G}_{x_n})^2} \\ <& d|\mathcal{G}(x) - \mathcal{G}^*|^\kappa\end{aligned}$$

Then, equation (3.4) and (3.5) gives

$$\begin{aligned}\dot{V}(x) \leq& -\vartheta\sqrt{2}V^{\frac{1}{2}}\frac{d}{\sqrt{n}}|\mathcal{G}(x) - \mathcal{G}^*|^\kappa \frac{(e^{\frac{d}{\sqrt{n}}|\mathcal{G}(x)-\mathcal{G}^*|^\kappa} - 1)}{e^{\frac{d}{\sqrt{n}}|\mathcal{G}(x)-\mathcal{G}^*|^\kappa}(t_f - t)} \\ \leq& -\lambda\frac{\alpha(V^\sigma(x))(e^{\alpha(V^\sigma(x))} - 1)}{e^{\alpha(V^\sigma(x))}(t_f - t)}\end{aligned}$$

where $d > 0$, $0 < \kappa < 1$, $\lambda = 2^{\frac{1+\kappa}{2}}\theta\frac{d}{c}$, $\sigma = \frac{\kappa}{2}$, $\alpha \in \mathcal{K}_\infty$.That means the trajectories of the GF systems converge to the minima of $\mathcal{G}(x)$ within a priori chosen time.

Further, we carry out the analysis for the system (3.2) for $t \geq t_f$. Taking the time derivative of the considered loss function along the system trajectories (3.2) for $t \geq t_f$, we get

$$\begin{aligned}\dot{V}(x) =& (\mathcal{G}(x) - \mathcal{G}^*)\nabla\mathcal{G}(x(t))\big(-\vartheta\nabla\mathcal{G}(x(t))^\top \\ =& -\vartheta(\mathcal{G}(x) - \mathcal{G}^*)\|\nabla\mathcal{G}(x)\|^2\end{aligned}$$

Using the relation $(\mathcal{G}(x) - \mathcal{G}^*) = \sqrt{2}V^{\frac{1}{2}}$ we can write,

$$\begin{aligned}\dot{V}(x) \leq& -\vartheta\sqrt{2}V^{\frac{1}{2}}\frac{1}{2}(\mathcal{G}(x) - \mathcal{G}^*)^2 \\ \leq& -\vartheta\sqrt{2}\alpha(V^{\sigma_1})\end{aligned}$$

where $\sigma_1 > 1$, $\alpha \in \mathcal{K}_\infty$. Hence, for $t \geq t_f$, the GF system (3.2) is asymptotically stable.

3.4 Numerical Example: Least Squares

In this section, we explore a least squares problem, which involves finding the best possible fit for a given dataset.

Least Squares

Consider the following least squares problem

$$\min_{x\in\mathbb{R}^2} \|Cx - d\|^2 \tag{3.6}$$

where $x \in \mathbb{R}^2$, $C \in \mathbb{R}^{3\times 2}$, $d \in \mathbb{R}^3$ and $\mathcal{G}(x) := \|Cx - d\|^2$. Matrix C is given by $C = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ and vector d is given by $d = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Find the best possible solution by minimizing the objective function (3.6).

Note that the objective function (3.6) is a strongly convex problem, and hence the obtained solution will be valid globally. While simulating for the predefined-time convergent GF

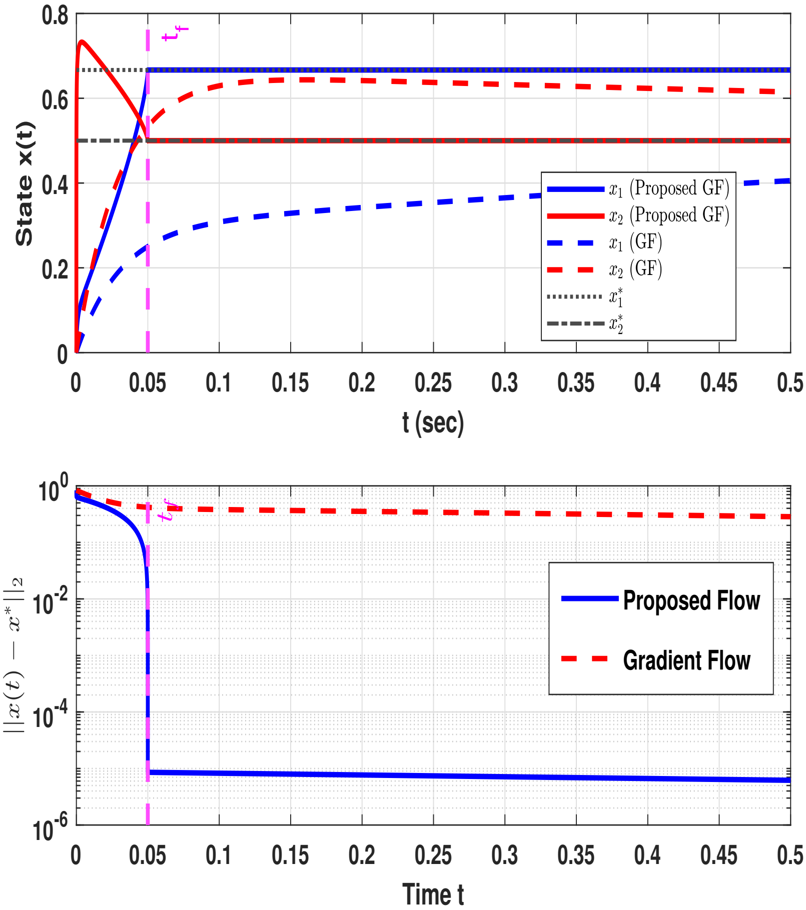

Figure 3.1: Comparison of proposed flow with gradient flow showing convergence of states to the optimal points within $t_f = 0.05$ sec and evolution of norm of the gradient..

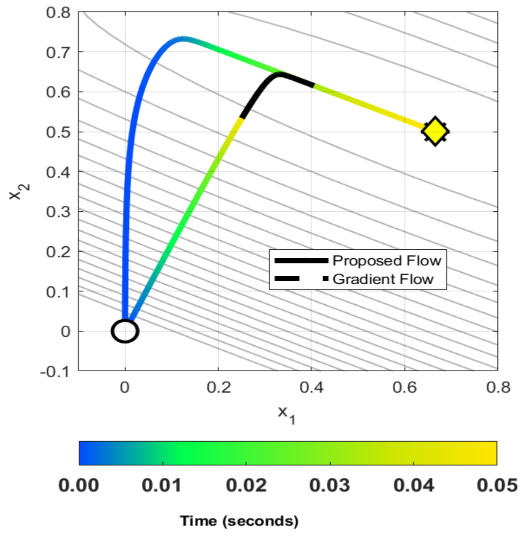

Figure 3.2: Contour plot.

scheme, the initial conditions were taken as $x(0) = [0 \;\; 0]^\top$. Figure 3.1 shows the evolution of the states, showing that the states reach optimal points within a predefined time $t_f = 0.05$ sec which is chosen in advance, and comparison with gradient flow dynamics. In Figure 3.1 also presents the corresponding curve of the gradient norm, clearly illustrating that the gradient becomes zero within the predefined time $t_f = 0.05$ sec and the proposed dynamics is much more accurate than gradient flow dynamics. Figure 3.2 shows the contour plot for the proposed and standard GF dynamics.

3.5 Convex Optimization Problem with Linear Equality Constraints

Consider the following optimization problem

$$\begin{aligned} &\min_{x\in\mathbb{R}^n} \mathcal{F}(x) \\ &\text{subject to } \mathcal{C}^\top x = d, \end{aligned} \tag{3.7}$$

where, $x \in \mathbb{R}^n$, $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}$ is an objective function, $\mathcal{C} \in \mathbb{R}^{n\times m}$, $d \in \mathbb{R}^m$. Let $\mathbb{X}_{feas} := \{x \in \mathbb{R}^n : \mathcal{C}^\top x = d\}$ is the set of feasible points. The optimum of (3.7) is $\mathcal{F}^* = \inf_{x\in\mathbb{R}^n} \{\mathcal{F}(x)|\mathcal{C}^\top x = d\}$. If $\mathcal{F}(x^*) = \mathcal{F}^*$ and $x^* \in \mathbb{X}_{feas}$, then we say x^* is an optimal point.

Assumption 3.2 *$\mathcal{F} \in C^2(\mathcal{D}, \mathbb{R})$ and $\mathcal{F}$ is convex.*

Assumption 3.3 *The columns of* $\mathcal{C}$ *are linearly independent.*

Lemma 3.5 *[13] Suppose Assumption 3.2 and 3.3 holds then* x^* *is the optimal solution to the* (3.7) *iff,*

$$\mathcal{C}^\top x^* = d$$
$$\nabla \mathcal{F}(x^*) \in \mathcal{R}(\mathcal{C}). \tag{3.8}$$

3.5.1 Predefined-Time Convergent Projected Gradient Flow

In this subsection, we propose a dynamics that is capable of finding the optimal solution of (3.7) in a priori chosen time. The equilibrium point of the proposed dynamics coincides with the optimal point x^* of (3.7). We propose the following non-autonomous predefined-time convergent Projected Gradient Flow (PGF) dynamics:

$$\dot{x}(t) = \begin{cases} -\theta P^\top \text{Sign}(\wp(x))\varphi(t, \wp(x)), & \text{for } t \in [t_0, t_f), \\ -\theta \wp(x), & \text{for } \ t \geq t_f, \end{cases} \tag{3.9}$$

where $x(t) \in \mathbb{R}^n$, $\theta \in \mathbb{R}$, $\theta > 1$, $\wp(x) = P\nabla\mathcal{F}(x)$, $\wp = (\wp_1, \ldots, \wp_n) \in \mathbb{R}^n$, $P \in \mathbb{R}^{n\times n}$ is projection matrix, $\varphi(t, \wp) = (\varphi_1, \varphi_2, \ldots, \varphi_n)^\top$, $\varphi_i(t, x) := \frac{(e^{|\wp_i(x)|}-1)}{e^{|\wp_i(x)|}(t_f - t)}$, for $1 \leq i \leq n$ and $\text{Sign}(\wp) := \text{diag}(\text{sign}(\wp_1), \ldots, \text{sign}(\wp_n)) \in \mathbb{R}^{n\times n}$.

Remark 3.6 *The projection matrix (P) is structured to follow* $\mathcal{N}(P) = \mathcal{R}(\mathcal{C})$

Proposition 3.7 *Assume that* $x(t_0) \in \mathbb{X}_{feas}$*, and Assumption 3.2,3.3 and 3.6 holds. Then,* $\mathbb{X}_{feas}$ *is positively invariant under the dynamics* (3.9) *and the unique equilibrium point of* (3.9) *is the optimal solution of* (3.7)*.*

Proof Since for a matrix S, $\mathcal{R}(S) \perp \mathcal{N}(S^\top)$ (see [50]), we get that $\mathcal{R}(P^\top) = \mathcal{N}(\mathcal{C}^\top)$, which follows $\mathcal{C}^\top P^\top = \mathbf{0}$. We consider the following Filippov time-varying set-valued map of the vector field in (3.9) for time interval $[t_0, t_f)$

$$\mathfrak{K}(\dot{x}) = -\theta P^\top l \varphi(t, \wp) \tag{3.10}$$

where $P \in \mathbb{R}^{n\times n}, \varphi \in \mathbb{R}^n$, $l := \text{diag}(l_1, \cdots, l_n) \in \mathbb{R}^{n\times n}$

and $l_i = \begin{cases} \{-1\}, & \text{sign}(\wp_i) < 0 \\ [-1, 1], & \text{sign}(\wp_i) = 0 \\ \{1\}, & \text{sign}(\wp_i) > 0 \end{cases}$. By carrying out multiplication of $\mathcal{C}^\top$ on both sides of

(3.10), we get $\mathcal{C}^\top \mathfrak{K}(\dot{x}) = \mathbf{0}$. Hence, $\mathbb{X}_{feas}$ is positively invariant.
Furthermore, we demonstrate that the origin of (3.9) is the optimal solution of (3.7). The origin of (3.9) is calculated by equating $\dot{x} = 0$. For the interval $[t_0, t_f)$,

$$P^\top \text{Sign}(\wp)\varphi(t, \wp) = \mathbf{0} \tag{3.11}$$

By performing multiplication on both sides of (3.11) from the left by $(\nabla\mathcal{F})^\top$, we get $P\nabla\mathcal{F} = \mathbf{0}$. Similarly for $t > t_f$ also we get $P\nabla\mathcal{F} = \mathbf{0}$ So, $\nabla\mathcal{F}(x) \in \mathcal{N}(P) = \mathcal{R}(\mathcal{C})$.
Hence, with the help of Lemma 3.5 and positive invariance of $\mathbb{X}_{feas}$, we can say that the equilibrium point of (3.9) is the optimal point of (3.7).

3.5.2 Convergence Analysis

In the present section, we perform predefined-time convergence analysis of (3.9) using Lyapunov's direct method.

Theorem 3.8 *For any $t_f > 0$, consider the dynamics* (3.9). *Let Assumptions 3.2, 3.3 and 3.6 are satisfied and $x(0) \in \mathbb{X}_{feas}$. Assume that $\exists$ a $\mathcal{V} \in C^1(\mathbb{R}^n, \mathbb{R})$ that adheres to $\psi_1(x) \leq \mathcal{V}(x) \leq \psi_2(x)$, $\forall x \in \mathbb{R}^n \backslash \{0\}$ where ψ_1 and ψ_2 are continuous positive definite functions, $\mathcal{V}(0) = 0$ and for $\mathcal{V} \neq 0$, $\dot{\mathcal{V}} \leq \frac{-\theta(e^{\mathcal{V}}-1)}{e^{\mathcal{V}}(t_f - t)}, \forall t \in [t, t_f), \theta > 1$ and $\dot{\mathcal{V}} \leq 0, \forall t \geq t_f$. Then, the equilibrium point of the projected GF system* (3.9) *is predefined-time stable.*

Proof Consider a candidate Lyapunov function as $\mathcal{V}(x) = (\mathcal{F} - \mathcal{F}^*)$, and when $\mathcal{V}(x) = 0$, $\mathcal{F}(x) = \mathcal{F}^*$ and optimal solution is reached.
First, we perform the analysis for $t_0 \leq t < t_f$. By calculating the time derivative of considered Lyapunov function $\mathcal{V}(x)$ along the dynamics of (3.9), we get

$$\begin{aligned}
\dot{\mathcal{V}} &= (\nabla\mathcal{F})^\top \dot{x} \\
&= -\theta(\nabla\mathcal{F})^\top P^\top \text{Sign}(\wp)\varphi(t, \wp) \\
&= -\theta\wp^\top \text{Sign}(\wp)\varphi(t, \wp) \\
&= -\theta|\wp|^\top \varphi(t, \wp)
\end{aligned}$$

where $|\wp| = (|\wp_1|, ..., |\wp_n|) \in \mathbb{R}^n$. Next, for any $i \in (1, .., n)$

$$\dot{\mathcal{V}} \leq -\theta|\wp_i|\varphi_i \tag{3.12}$$

As $|\wp_i| \geq p_i |\nabla_i \mathcal{F}|$, where, $p_i > 0$. Next, from PL inequality in Lemma 2.20, there exists i, for which $|\nabla_i \mathcal{F}| \geq \sqrt{\frac{\alpha}{n}}(\mathcal{F} - \mathcal{F}^*)^{\frac{1}{2}}$. Therefore, we can write

$$\dot{\mathcal{V}} \leq -\theta\delta\sqrt{\mathcal{V}}\frac{(e^{\delta\sqrt{\mathcal{V}}} - 1)}{e^{\delta\sqrt{\mathcal{V}}}(t_f - t)},$$

where, $\delta := p_i\sqrt{\frac{\alpha}{n}}$. Let $\hat{\mathcal{V}} = \delta\sqrt{\mathcal{V}}$. Then $\dot{\hat{\mathcal{V}}} = \frac{\delta^2}{2}\frac{\dot{\mathcal{V}}}{\hat{\mathcal{V}}}$. Therefore,

$$\dot{\hat{\mathcal{V}}} \leq -\hat{\theta}\frac{(e^{\hat{\mathcal{V}}} - 1)}{e^{\hat{\mathcal{V}}}(t_f - t)}, \tag{3.13}$$

where $\hat{\theta} = \frac{\theta\delta^2}{2} > 1$. Hence, according to Lemma 2.9 inequality (3.13) shows that the equilibrium point of (3.9) is predefined-time stable. Next, we perform the analysis for $t \geq t_f$. By computing the time derivative of $\mathcal{V}(x)$ along the dynamics of (3.9) for $t \in [t_f, \infty)$, we get

$$\dot{\mathcal{V}} = -\theta(\nabla\mathcal{F})^\top \wp(x) \tag{3.14}$$

Since, $P^\top P = P$, it follows that

$$\dot{\mathcal{V}} \leq -\theta\wp^\top\wp \tag{3.15}$$

which represents that the origin of (3.9) is asymptotically stable for $t \geq t_f$. Hence, the trajectory of (3.9) will remain at its equilibrium point. Hence, the equilibrium point of PGF system is predefined-time stable.

Remark 3.9 *The selection of P may not be unique. One of the choice is $P = I_n - A(A^\top A)^{-1}A^\top$ which satisfies Assumption 3.6. (see [1])*

Computational Cost: We can discuss the convergence rate of Euler discretization of (3.9) to calculate complexity, considering the assumption of Lipschitz smoothness. Let us consider the Euler discretized version of the proposed PUBST-based PGF dynamics. To simplify the computation of complexity, we can write $\varphi(t, x) := \frac{P\nabla\mathcal{F}}{t_f - t}$. Let $k_f = t_f/h$, then the forward Euler discretized version of the proposed PUBST-based PGF dynamics, taking h as a practical step size, can be written as:

$$x(k+1) = \begin{cases} x(k) - \frac{h\theta P^\top P\nabla\mathcal{F}}{k_f - k}, & for\ 0 \leq k < k_f, \\ x(k) - h\theta P\nabla\mathcal{F}, & for\ k \geq k_f \end{cases} \tag{3.16}$$

We need to introduce an extra assumption of L-Lipschitz smooth, as given below:

Assumption: ((Lipschitz Smoothness of Order q) We assume the function $\mathcal{F}$ is L-Lipschitz smooth of order $q \in (1,2]$, i.e., for any $z, y \in \mathbb{R}^n$,

$$\|\nabla\mathcal{F}(y) - \nabla\mathcal{F}(x)\| \leq L\|y - x\|^{q-1}$$

The smoothness assumption is a prevalent setting in optimization algorithm literature. This assumption is also called (L, q) Holder continuity, it will lead to the following property:

$$\mathcal{F}(y) \leq \mathcal{F}(x) + \langle\nabla\mathcal{F}(x), y - x\rangle + \frac{L}{q}\|y - x\|^q,$$

When $q = 2$, the function will be Lipschitz smooth. Considering the following assumptions, for our case, we can calculate iteration complexity as follows:

$$\mathcal{F}(x(k+1)) \leq \mathcal{F}(x(k)) + \langle\nabla\mathcal{F}(x(k)), x(k+1) - x(k)\rangle + \frac{L}{q}\|x(k+1) - x(k)\|^q,$$

Using algorithm (3.16), for $0 \leq k < k_f$, considering the Lipschitz continuity of the gradient and $q = 2$,

$$\mathcal{F}(x(k+1)) \leq \mathcal{F}(x(k)) - \frac{h\theta}{k_f - k}\|P\nabla\mathcal{F}\|^2 + \frac{Lh^2\theta^2}{2(k_f - k)^2}\|P\nabla\mathcal{F}\|^2,$$

Let $\eta_k := \frac{\theta}{k_f - k}$, then we can write $\mathcal{F}(x(k+1)) \leq \mathcal{F}(x(k)) - \left(\eta_k - \frac{Lh\eta_k^2}{2}\right)h\|P\nabla\mathcal{F}\|^2$,. Choose a step size such that $h < \frac{2k_f}{L\theta}$, and Let $\|P\nabla\mathcal{F}\|^2 > p_m\|\nabla\mathcal{F}\|^2$, where, $p_m > 0$. Hence, we can further write $\mathcal{F}(x(k+1)) \leq \mathcal{F}(x(k)) - \left(\eta_k - \frac{Lh\eta_k^2}{2}\right)hp_m\|\nabla\mathcal{F}\|^2$,. Let us define $\sigma_k := \left(\eta_k - \frac{Lh\eta_k^2}{2}\right)$ and using assumption of PL inequality: $\|\nabla\mathcal{F}\|^2 \geq \alpha(\mathcal{F}(x) - \mathcal{F}(x^*))$, then further

$$\mathcal{F}(x(k+1)) \leq \mathcal{F}(x(k)) - \sigma_k hp_m\alpha(\mathcal{F}(x(k)) - \mathcal{F}(x^*)),$$

Subtracting $\mathcal{F}(x^*)$ on both the sides, we get

$$\mathcal{F}(x(k+1)) - \mathcal{F}(x^*) \leq (1 - \sigma_k hp_m\alpha)(\mathcal{F}(x(k)) - \mathcal{F}(x^*)),$$

The for k_f iterations with fixed step size h,

$$\mathcal{F}(x(k_f)) - \mathcal{F}^* \leq (1 - \sigma_k hp_m\alpha)^{k_f}(\mathcal{F}(x(0)) - \mathcal{F}^*).$$

Using inequality $e^{-x} \geq 1 - x$, we can get that the corresponding iteration complexity $O\left((\sigma_k hp_m\alpha)^{-1}\ln\frac{1}{\epsilon}\right)$ for ϵ closeness of $x(k)$ to x^*. Similarly, if we run the algorithm for K iterations, then for $k_f \leq k \leq K$, the corresponding iteration complexity is $O\left((\sigma hp_m\alpha)^{-1}\ln\frac{1}{\epsilon}\right)$, where $\sigma := \left(1 - \frac{L}{2}h\theta\right)$.

3.5.3 Predefined-Time Convergent Projected Newton Flow

We propose the following non-autonomous, Predefined-Time Convergent Projected Newton Flow (PTPNF) dynamics:

$$\dot{x}(t) = \begin{cases} -\theta(\nabla^2\mathcal{F})^{-1}P^\top \text{Sign}(\wp(x))\varphi(t,\wp(x)), & \text{for } t \in [t_0, t_f), \\ -\theta(\nabla^2\mathcal{F})^{-1}\wp(x), & \text{for } \ t \in [t_f, \infty), \end{cases} \tag{3.17}$$

where $x(t) \in \mathbb{R}^n$, $\theta > 1$, $\wp(x) = P\nabla\mathcal{F}(x)$, $\nabla^2\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ denotes Hessian of the function $\mathcal{F}$, $\wp = (\wp_1, \dots, \wp_n) \in \mathbb{R}^n$, $P \in \mathbb{R}^{n\times n}$ is projection matrix, $\varphi(t,\wp) = (\varphi_1, \varphi_2, \dots, \varphi_n)^\top$, $\varphi_i(t,x) := \frac{(e^{|\wp_i(x)|}-1)}{e^{|\wp_i(x)|}(t_f-t)}$, for $1 \le i \le n$ and $\text{Sign}(\wp) := \text{diag}(\text{sign}(\wp_1), \dots, \text{sign}(\wp_n)) \in \mathbb{R}^{n\times n}$.

Theorem 3.10 *For any $t_f > 0$, consider the dynamics* (3.17)*. Let Assumptions 3.2, 3.3 and 3.6 are satisfied and $x(0) \in \mathbb{X}_{feas}$. Assume that $\exists$ a function $\mathcal{V} \in C^1(\mathbb{R}^n, \mathbb{R})$ which follows $\psi_1(x) \le \mathcal{V}(x) \le \psi_2(x)$, $\forall x \in \mathbb{R}^n\backslash\{0\}$ where ψ_1 and ψ_2 are continuous positive definite functions, $\mathcal{V}(0) = 0$ and for $\mathcal{V} \neq 0$, $\dot{\mathcal{V}} \le \frac{-\theta(e^{\mathcal{V}}-1)}{e^{\mathcal{V}}(t_f-t)}, \forall t \in [t, t_f), \theta > 1$ and $\dot{\mathcal{V}} \le 0, \forall t \ge t_f$. Then, the equilibrium point of the PTPNF system* (3.17) *is predefined-time stable.*

Proof Consider the candidate Lyapunov function as $\mathcal{V}(x) = \frac{1}{2}\|P\nabla\mathcal{F}\|^2$. Using Assumptions 3.2, 3.3 and 3.6, we can say that if $\mathcal{V}(x) = 0$, the conditions in Lemma 3.5 are satisfied and optimal solution is reached. First, we perform the analysis for $t \in [t_0, t_f)$. By calculating the time derivative of $\mathcal{V}(x)$ along the dynamics of (3.17), we get

$$\begin{aligned} \dot{\mathcal{V}} &= (P\nabla\mathcal{F})^\top P\nabla^2\mathcal{F}\dot{x} \\ &= -\theta\nabla\mathcal{F}^\top P^\top PP^\top \text{Sign}(\wp(x))\varphi(t,\wp(x)) \end{aligned}$$

Note that the Projection matrix has the properties $P^\top = P$, $P^2 = P$. Next,

$$\dot{\mathcal{V}} = -\theta|\wp|^\top\varphi(t,\wp(x))$$

where $|\wp| = (|\wp_1|, ..., |\wp_n|) \in \mathbb{R}^n$. Next, for some $i \in (1, 2, .., n)$

$$\begin{aligned} \dot{\mathcal{V}} &\le -\theta|\wp_i|\varphi_i \\ &\le -\theta\delta_1\sqrt{\mathcal{V}}\frac{(e^{\delta_1\sqrt{\mathcal{V}}}-1)}{e^{\delta_1\sqrt{\mathcal{V}}}(t_f-t)}, \end{aligned} \tag{3.18}$$

where, $\delta_1 = \frac{2}{\sqrt{n}}$. Let $\hat{\mathcal{V}} = \delta_1 \sqrt{\mathcal{V}}$. Then $\dot{\hat{\mathcal{V}}} = \frac{\delta_1^2}{2} \frac{\dot{\mathcal{V}}}{\hat{\mathcal{V}}}$. Therefore,

$$\dot{\hat{\mathcal{V}}} \leq -\hat{\theta} \frac{(e^{\hat{\mathcal{V}}} - 1)}{e^{\hat{\mathcal{V}}}(t_f - t)}, \tag{3.19}$$

where $\hat{\theta} = \frac{\theta \delta_1^2}{2} > 1$. Hence, according to Lemma 2.9, inequality (3.19) shows that the equilibrium point of (3.17) is predefined-time stable. Next, we perform the analysis for $t \geq t_f$. By computing the time derivative of $\mathcal{V}(x)$ along the dynamics of (3.17) for $t \geq t_f$, we get

$$\begin{aligned} \dot{\mathcal{V}} &= -\theta \nabla \mathcal{F}^\top P^\top \wp(x) \\ &= -\theta \mathcal{V}, \end{aligned}$$

which represents that the origin of (3.17) is asymptotically stable for $t \geq t_f$. Hence, the trajectory of (3.17) will remain at its equilibrium point. Hence, the equilibrium point of the PTPNF system is predefined-time stable.

3.5.4 Robustness Analysis of Perturbed Projected Gradient Flow

In this subsection, we discuss the effect of perturbations on PGF employing the formalism of Input to State Stability (ISS).

Proper Loss Functions (PLFs)

Definition 3.11 *(see [44]) A function $\mathcal{F} \in C^1(\mathbb{Y}, \mathbb{R})$ with locally Lipschitz gradient $\nabla \mathcal{F}$ is PLF with respect to $(\mathbb{Y}, \mathcal{M})$ if $\mathcal{F}(x) = \mathcal{F}^*$ for $x \in \mathcal{M}$ and satisfies: (i) $\mathcal{F} - \mathcal{F}^*$ is a size function for $(\mathbb{Y}, \mathcal{M})$, (ii) $|P \nabla \mathcal{F}(x)|$ is a size function for $(\mathbb{Y}, \mathcal{M})$.*

Lemma 3.12 *(see [44]) Given a $\mathcal{F} \in C^1(\mathbb{Y}, \mathbb{R})$ with a locally Lipschitz gradient $\nabla \mathcal{F}$ such that $\mathcal{F}(x) - \mathcal{F}^*$ is a size function for $(\mathbb{Y}, \mathcal{M})$. Then, the following pair of characteristics are equivalent:*

- *$\mathcal{F}$ is a PLF,*
- *$\alpha(\zeta(x)) \leq \|P \nabla \mathcal{F}(x)\|^2$, $\forall x \in \mathbb{Y}$, where $\alpha \in \mathcal{K}_\infty$ and ζ denotes size function.*

Consider the following perturbed projected GF system:

$$\dot{x}(t) = \begin{cases} -\theta_1 P^\top \mathrm{Sign}(\wp(x))\varphi(t,\wp(x)) - \theta_1 \nabla\mathcal{F} + \Delta x(t), & \text{for } t_0 \le t < t_f, \\ -\theta_1 \wp(x) - \theta_1 \nabla\mathcal{F} + \Delta x(t), & \text{for } t \in [t_f, \infty), \end{cases} \tag{3.20}$$

where $\Delta x \in \mathbb{R}^m$ represents disturbance inputs, $x(t) \in \mathbb{R}^n$, $\theta_1 \in \mathbb{R}$, $\theta_1 > 1$, $\wp(x) = P\nabla\mathcal{F}(x)$, $\wp = (\wp_1, \ldots, \wp_n) \in \mathbb{R}^n$, $P \in \mathbb{R}^{n\times n}$ is projection matrix, $\varphi(t,\wp) = (\varphi_1, \ldots, \varphi_n)^\top$, $\varphi_i(t,x) := \frac{(e^{|\wp_i(x)|}-1)}{e^{|\wp_i(x)|}(t_f-t)}$, for $1 \le i \le n$ and $\mathrm{Sign}(\wp) := \mathrm{diag}(\mathrm{sign}(\wp_1), \ldots, \mathrm{sign}(\wp_n)) \in \mathbb{R}^{n\times n}$, and $\nabla\mathcal{F} \in \mathbb{R}^n$ as gradient vector.

Theorem 3.13 *Let $\mathcal{F}$ be a proper loss function. Then, system* (3.20) *is a small disturbance Predefined Upper Bound of Settling Time Based -Input to state Stable (PUBST-ISS).*

Proof: We show that $\mathcal{F} - \mathcal{F}^*$ is a small disturbance PUBST-ISS Lyapunov function. Consider $\mathcal{V} = \mathcal{F} - \mathcal{F}^*$ as a candidate PUBST-ISS-Lyapunov function. As we can see, $\mathcal{F} - \mathcal{F}^*$ is a size function with respect to x^*. Further, for interval $[t_0, t_f)$, By taking the time derivative of $\mathcal{V}$ along the dynamics of (3.20), we get

$$\begin{aligned} \dot{\mathcal{V}} &= (\nabla\mathcal{F})^\top \dot{x} \\ &= -\theta_1 (\nabla\mathcal{F})^\top P^\top \mathrm{Sign}(\wp)\varphi(t,\wp) - \theta_1 \|\nabla\mathcal{F}\|^2 + \\ &\quad (\nabla\mathcal{F})^\top \Delta x(t), \end{aligned}$$

Using Cauchy-Schwarz inequality, we can further write,

$$\dot{\mathcal{V}} \le -\theta_1 |\wp|^\top \varphi(t,\wp) - \theta_1 \|\nabla\mathcal{F}\|^2 + \|\nabla\mathcal{F}\|\|\Delta x\|,$$

where $|\wp| = (|\wp_1|, ..., |\wp_n|) \in \mathbb{R}^n$. Next, for any $i \in (1, .., n)$

$$\dot{\mathcal{V}} \le -\theta_1 |\wp_i| \varphi_i - \theta_1 \|\nabla\mathcal{F}\|^2 + \|\nabla\mathcal{F}\|\|\Delta x\|,$$

Using Young's inequality, we get,

$$\begin{aligned} \dot{\mathcal{V}} &\le -\theta_1 |\wp_i| \varphi_i - \theta_1 \|\nabla\mathcal{F}\|^2 + \frac{\|\nabla\mathcal{F}\|^2}{2} + \frac{\|\Delta x\|^2}{2} \\ &\le -\theta_1 |\wp_i| \varphi_i + \frac{\|\Delta x\|^2}{2} \end{aligned}$$

Using Lemma 3.12, we have, $|\wp_i|^2 \geq \frac{\mu}{n}(\mathcal{F} - \mathcal{F}^*)$, and define, $\delta_d := \sqrt{\frac{\mu}{n}}$, where, $\mu > 0$, we can further write,

$$\dot{\mathcal{V}} \leq -\theta_1 \delta_d \sqrt{\mathcal{V}} \frac{(e^{\delta_d \sqrt{\mathcal{V}}} - 1)}{e^{\delta_d \sqrt{\mathcal{V}}}(t_f - t)} + \frac{\|\Delta x\|^2}{2},$$

Let $\hat{\mathcal{V}} = \delta_d \sqrt{\mathcal{V}}$. Then $\dot{\hat{\mathcal{V}}} = \frac{\delta_d^2}{2} \frac{\dot{\mathcal{V}}}{\hat{\mathcal{V}}}$. Therefore,

$$\dot{\hat{\mathcal{V}}} \leq -\hat{\theta} \frac{(e^{\hat{\mathcal{V}}} - 1)}{e^{\hat{\mathcal{V}}}(t_f - t)} + \frac{\|\Delta x\|^2}{2},$$

where $\hat{\theta} = \frac{\theta_1 \delta_d^2}{2} > 1$. Let $\frac{\|\Delta x\|^2}{2} \leq \gamma(\|\Delta x\|)$, where, $\gamma(\cdot)$ is a class $\mathcal{K}$ function. Then we can further write using Lemma 2.16

$$\dot{\hat{\mathcal{V}}} \leq -\beta(\hat{\mathcal{V}}, t_f - t) + \gamma(\|\Delta x\|) \tag{3.21}$$

where, $\beta(\cdot, \cdot)$ is a $\mathcal{PGKL}$ class function and $\gamma(\cdot)$ is a class $\mathcal{K}$ function. Next, we perform the analysis for $t \geq t_f$. By computing the time derivative of $\mathcal{V}$ along the dynamics of (3.9) for $t \geq t_f$, we get

$$\dot{\mathcal{V}} = -\theta_1 (\nabla \mathcal{F})^\top \wp(x) + (\nabla \mathcal{F})^\top \Delta x(t)$$

Using the property of P i.e. $P^\top P = P,\ P^\top = P$, Cauchy-Schwarz inequality, and Young's inequality, we get

$$\dot{\mathcal{V}} \leq -\theta_1 \|\wp\|^2 + \frac{\|\Delta x\|^2}{2}$$

using Lemma 3.12, we get

$$\dot{\mathcal{V}} \leq -\alpha_d(\mathcal{V}(x)) + \gamma(\|\Delta x\|) \tag{3.22}$$

where, $\alpha_d(\cdot)$ is a class $\mathcal{K}$ function. Combining (3.21) and (3.22), we get

$$\dot{\hat{\mathcal{V}}} \leq -\beta(\hat{\mathcal{V}}, t_f - t) - \alpha_d(\hat{\mathcal{V}}(x)) + \gamma(\|\Delta x\|),\ \forall\, t \geq 0.$$

If $\alpha_d(\hat{\mathcal{V}}(x)) \geq \gamma(\|\Delta x\|)$, then we get

$$\dot{\hat{\mathcal{V}}} \leq -\beta(\hat{\mathcal{V}}, t_f - t) - \alpha_d(\hat{\mathcal{V}}(x)), \quad \forall\, t \geq 0. \tag{3.23}$$

Hence, according to Lemma 2.16, $\mathcal{V}$ is a small disturbance PUBST-ISS Lyapunov function. Further, according to Lemma 2.17, dynamics (3.20), is a small disturbance PUBST-ISS.

3.6 Illustrative Examples

In the current section, we discuss problems based on constrained linear least squares and resource allocation using the proposed predefined-time convergent PGF techniques. Here the results are also compared with the FTS stable signed projected GF system proposed in [1].

3.6.1 Linear Least Squares

Consider the following problem:

$$\min_{x\in\mathbb{R}^4} \ \frac{1}{2}||\mathcal{C}x - d||^2$$
$$\text{subject to} \ \ Ax = b \tag{3.24}$$

$$\text{where } \mathcal{C} = \begin{bmatrix} 0.9501 & 0.7620 & 0.6153 & 0.4057 \\ 0.2311 & 0.4564 & 0.7919 & 0.9354 \\ 0.6068 & 0.0185 & 0.9218 & 0.9169 \\ 0.4859 & 0.8214 & 0.7382 & 0.4102 \\ 0.8912 & 0.4447 & 0.1762 & 0.8936 \end{bmatrix} \in \mathbb{R}^{4\times5},\ d = \begin{bmatrix} 0.0578 \\ 0.3528 \\ 0.8131 \\ 0.0098 \\ 0.1388 \end{bmatrix} \in \mathbb{R}^{5\times1},$$

$A = \begin{bmatrix} 3 & 5 & 7 & 9 \end{bmatrix} \in \mathbb{R}^{1\times4}$, $b = 4$. The optimal points of (3.24) is: $x^* = [0.017 \quad -$

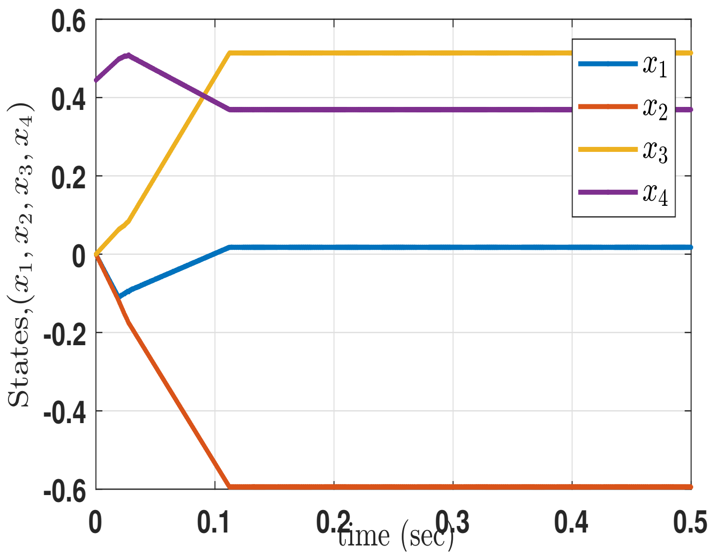

Figure 3.3: States with finite-time convergence (FTC) using dynamics in [1] with **convergence time around 0.12 sec**.

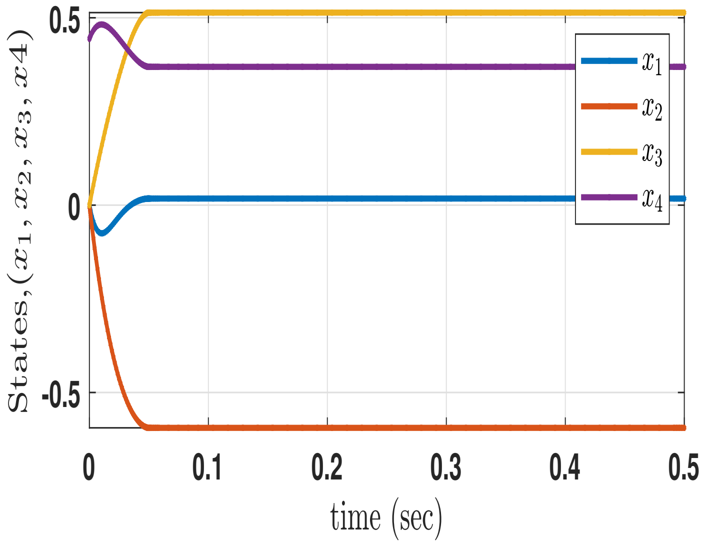

Figure 3.4: States with PUBST convergence as per our proposed dynamics (3.9) with prior chosen **convergence time around 0.05 sec**.

$0.594 \quad 0.513 \quad 0.369]^\top$. We first solve the Linear Least Squares (LLS) problem using the dynamics presented in [1] and compare its results from algorithm (3.9). For both the techniques, we take initial time as $t(0) = 0$ sec and set $x(0) = [0 \quad 0 \quad 0 \quad 0.4444] \in \mathbb{X}_{feas}$. Figure 3.3 shows that state trajectories reach x^* around 0.12 sec using the algorithm proposed in [1], whereas Figure 3.4 shows the evolutions of states and an optimal solution is reached in predefined-time $t_f = 0.05$ sec using dynamics (3.9).

3.7 Resource Allocation Problem (RAP)

Consider the economic dispatch problem with the following quadratic cost function: $f(\mathcal{P}_j) = a_j + b_j\mathcal{P}_j + c_j\mathcal{P}_j^2$, where a_j, b_j, c_j are cost coefficients. The layout of the Microgrid

Table 3.1: Generators Cost Parameters

Generator	a_j	b_j	c_j
1	0.5	3	2
2	1.4	4	1
3	3	5	0.5
4	1	2	1.5

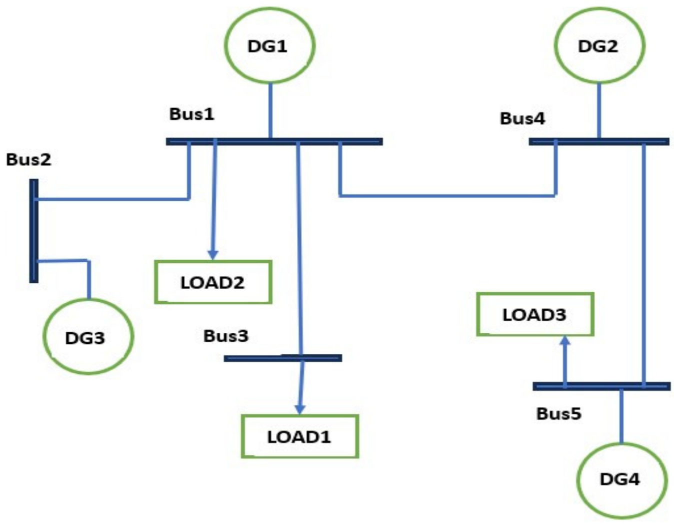

Figure 3.5: Microgrid system with four generators

system is shown in Figure 3.5. There are three loads and four generators in the system. The parameters of the generators are shown in Table 3.1. The total demand of the system is fixed as $d_0 = 145$. We take initial conditions as: $\mathcal{P}_1(0) = 40, \mathcal{P}_2(0) = 40, \mathcal{P}_3(0) = 40, \mathcal{P}_4(0) = 25$ which is inside the feasible set according to Proposition 3.7. The optimal power outputs of each generator are: $\mathcal{P}_1^* = 17.66, \mathcal{P}_2^* = 34.82, \mathcal{P}_3^* = 68.64, \mathcal{P}_4^* = 23.88$. Similar to the LLS example, here also, we first solve the RAP problem using the algorithm given in [1] and compare its results from algorithm (3.9). In Figure 3.6, state trajectories reach their optimal value around 5 sec, using the algorithm proposed in [1]. Here, we cannot choose the convergence time, whereas in Figure 3.7, states converge to their optimal points within predefined-time $t_f = 0.2$ second, which is selected a priori, using proposed algorithm (3.9), with $\theta = 4$.

3.8 Conclusion

In this chapter, the predefined-time convergent gradient flow systems are proposed to solve continuous time optimization problems. Under the condition that the objective function is strongly convex or satisfies the P-Ł inequality. A predefined-time convergent projected gradient flow algorithm is proposed to solve convex optimization problems with linear equality constraints. Further, a robustness analysis using the idea of ISS is also

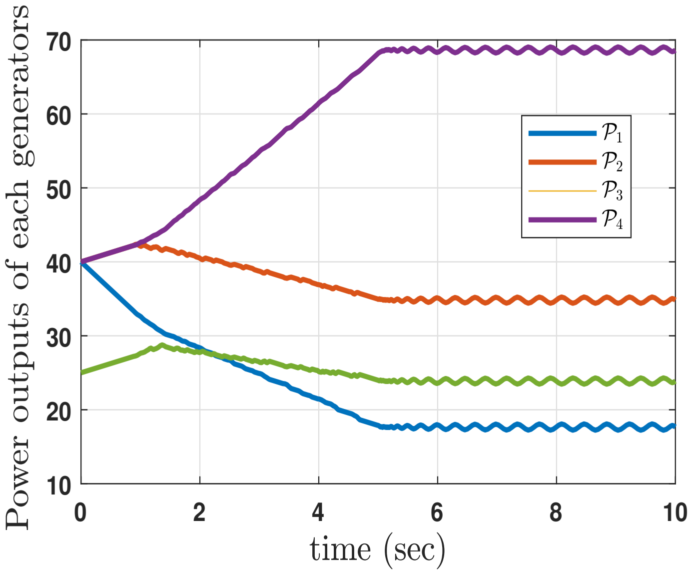

Figure 3.6: Power outputs of generators with FTC as per the dynamics in [1] with **convergence time 5 sec**.

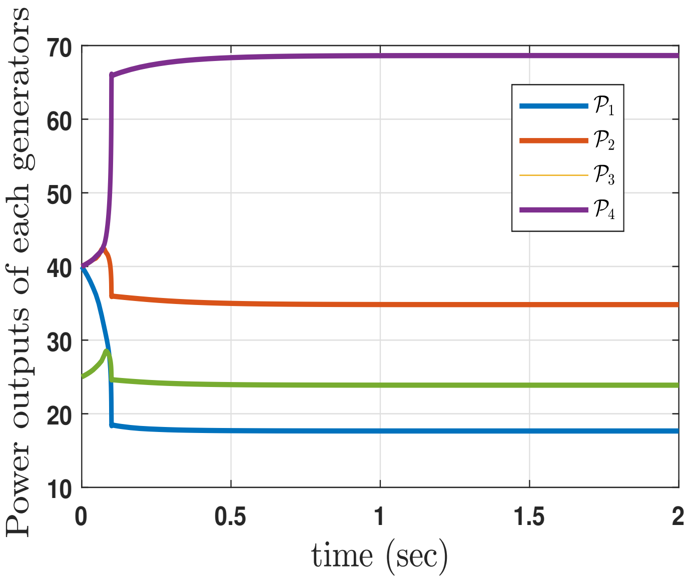

Figure 3.7: Power outputs of generators with PUBST convergence as per our proposed dynamics (3.9) within prior chosen **convergence time 0.2 sec**.

discussed, considering perturbed PUBST based PGF dynamics. Next, the convergence analysis of the proposed dynamics are discussed using Lyapunov stability theory. The convergence time of proposed algorithms can be specified in advance and does not involve the initial value of the system and any parameters.

In the upcoming chapter, we propose modified GF dynamics to solve constrained time-varying optimization problems with a predefined-time convergence property.