Chapter 1

Introduction

1.1 Dynamical Systems

Dynamics is the investigation of progress and dynamical systems are the formula for depicting how a system of variables collaborate and moves with time. The idea is that normally anything that develops after some time may be known as a dynamical system. So, let us begin with portraying mathematical dynamical systems. Dynamical system is a fixed rule that depicts the dependency of time of a point in a geometrical space. In mathematical language, the dynamical rule is depending on a function that takes as its input the state of the system at one time and gives as its yield the state of the system at the next time. A dynamical system is said to be deterministic if for a given time segment a unique future state follows from the current state, otherwise it is said to be stochastic. Nonlinear dynamics began with work by Henry Poincare in the late 1800's while he tried to solve the three body problem. Later this approach had been utilized as a part of numerous different fields. After the advent of modern computers, Mathematicians and Scientists have tackled numerous nonlinear dynamical problems.

1.1.1 Mathematical definition of dynamical systems

Let S is a state space and T is a set of time. A dynamical system is a mapping R defined as $R : S \times T \to S$ that gives the consequents to a state $s \in S$.

1.2 Types of Dynamical Systems

There are mainly two types of dynamical systems those are expressed in the form of Difference equations and Differential equations. Difference equations are also known as iterated maps, recursion relation or simply maps. In difference equations time should be discrete while in differential equations it should be continuous. The rule $x_{n+1} = \cos x_n$ is a map which is one dimensional, in light of the fact that the points x_n belongs to the one-dimensional space of real numbers and the sequence $x_0, x_1, x_2, \ldots$ is known as *orbit* beginning from x_0. Moreover, differential equations describe the evolution of dynamical system in continuous time. Further we will classify this in linear/nonlinear and autonomous/non-autonomous according to their nature and are discussed as follows.

1.2.1 Linear System

A dynamical system is said to be linear system if it satisfies two properties

- Superposition
- Homogeneity.

According to the principle of superposition, the function ϕ must satisfies

$\phi(u+v) = \phi(u) + \phi(v)$,

and the property of homogeneity states that for the function ϕ and for any real number k,

$\phi(ku) = k\phi(u)$.

Any function that does not follow any of the superposition and homogeneity properties is said to be nonlinear. It is also important that there is no unifying characteristic of nonlinear systems, except for not satisfying the two above mentioned properties. The dynamics of linear systems can be expressed as

$$u' = \alpha(t)u . \tag{1.1}$$

where $\alpha(t)$ is a $m \times m$ matrix.

1.2.2 Nonlinear System

A dynamical system is said to be nonlinear if there exist a set of nonlinear differential equations such that

$$u' = \phi(u,t), \tag{1.2}$$

where ϕ is the $m \times 1$ vector function which is nonlinear and u is the $m \times 1$ state vector. Here m is known as order of system.

1.2.3 Autonomous and Non-Autonomous systems

The nonlinear system (1.2) is known as an autonomous system if ϕ does not depend on time explicitly. Therefore, the system is expressed as $u' = \phi(u)$. Further if ϕ depends on time explicitly then the system (1.2) is called non-autonomous.

1.3 Stability Theory

Stability analysis has important place in the study of dynamical system. Stability analysis states whether the dynamical system is stable or not or will be stable with perturbation. This analysis has a pivotal part in an extensive variety of uses as the vast majority of the wonders saw in reality can be portrayed utilizing differential equations. The stability analysis also depicts about behaviours in trajectories of dynamical systems under little changes in initial conditions and the stability of solutions of a system of differential equations.

1.3.1 Equilibrium points of a system

Let us take coupled differential equations given as

$$\begin{cases} \dfrac{du}{dt} = \phi(u,v), \\ \dfrac{dv}{dt} = \psi(u,v), \end{cases} \tag{1.3}$$

where ϕ and ψ are nonlinear functions of u, v. System is an autonomous system because ϕ and ψ are independent from t. From Newton's second law in mechanics we obtaine

$$\frac{d^2u}{dt^2} = G\left(u, \frac{du}{dt}\right), \tag{1.4}$$

where G is the force. This equation can be written as

$$\begin{aligned} \frac{du}{dt} &= v = \phi, \\ \frac{dv}{dt} &= G = \psi. \end{aligned} \tag{1.5}$$

The equilibrium, fixed, critical or stationary points of the equation (1.3) correspond to the points in the phase plane ($uv-$plane) where $\frac{du}{dt} = 0$ and $\frac{dv}{dt} = 0$. The number and positions of the fixed points in the $uv-$plane are found by solving the simultaneous nonlinear equations

$$\phi(u, v) = 0, \psi(u, v) = 0. \tag{1.6}$$

But if ϕ and ψ are nonlinear functions then more than one fixed point can be possible.

1.3.2 Stability of the system

Let us take an autonomous system

$$\frac{du}{dt} = \phi(u) . \tag{1.7}$$

A critical point u_e of the system (1.7) is known as stable if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that each solution $u = \tau(t)$ at $t = 0$ satisfies

$$\|\tau(0) - u_e\| < \delta , \tag{1.8}$$

and

$\|\tau(t)-u_e\|<\varepsilon$, for all $t \ge 0$. (1.9)

A critical point u_e is said to be asymptotic stable if u_e is stable and if there exists a δ_0, with $0<\delta_0<\delta$, such that if a solution $x=\tau(t)$ satisfies

$\|\tau(0)-u_e\|<\delta_0$, (1.10)

then

$\lim_{t\to\infty}\tau(t)=u_e$. (1.11)

Geometrically, we can say that trajectories that begin "sufficiently close" to u_e must stay "close" as well as must in the eventually approach u_e as $t\to\infty$.

1.3.3 Lyapunov's First Method

Theorem 1.1 Suppose $u=0$ be an equilibrium point of an autonomous nonlinear system

$u'=\phi(u(t))$, (1.12)

where $\phi: D\to R^n$ a continuously differentiable function and D is the neighbourhood of the equilibrium point. Suppose $\xi_j\,(j=1,2,...,n)$ are the latent roots of the matrix

$J=\frac{\partial\phi}{\partial u}\Big|_{u=0}$.

(i) If $\mathrm{Re}(\xi_j)<0$ for every j then the equilibrium point $u=0$ is asymptotically stable for the system (1. 12).

(ii) If $\mathrm{Re}(\xi_j)>0$ for at least one or more j then the equilibrium point $u=0$ is unstable for the system (1. 12).

(iii) If $\mathrm{Re}(\xi_j)<0$ for all j and at least one $\mathrm{Re}(\xi_j)=0$ then the equilibrium point $u=0$ is stable, asymptotically stable or unstable for the system (1. 12).

1.3.4 Lyapunov's Second Method (Lyapunov's direct method)

In 1892, A.M. Lyapunov, a Russian mathematician, defined the theory of stability and analysed the theory of stability for ordinary differential equations. The use of Lyapunov function to prove stability has become common and called Lyapunov's second method. The method involves determining a family of closed curves or closed surfaces in state space such that the general behaviour of nearby trajectories of a dynamical system can be investigated. This technique is widely used to examining the global stability of nonlinear systems.

Theorem 1.2: Consider $u = 0$ be an equilibrium point of an autonomous nonlinear system $u' = \phi(u(t)), \qquad u(0) = u_0$,

where $u(t) \in D \not\subset R^n$ and D is an open set containing origin and $\phi : D \to R^n$ is a continuous function. Let $V : R \to R$ be a positive definite continuously differentiable function in a neighbourhood D of $u = 0$, such that $V'(u) \leq 0$ in D along the path of the system. Then, the equilibrium point $u = 0$ is stable. In addition, if $V'(u) < 0$ in $D - \{0\}$ then $u = 0$ is said to be asymptotically stable.

Theorem 1.3: Let $u = 0$ be an equilibrium point of a nonlinear system $u' = \phi(u)$. Suppose that $V : R^n \to R$ be a positive definite continuously differentiable function such that $V(u) \to \infty$ as $\|u\| \to \infty$ and $V'(u) \leq 0$ for all $u \neq 0$. Then $u = 0$ is globally asymptotically stable. It should be noticed that Lyapunov functions are not unique for a particular system.

1.4 History of Chaos

"Chaos" is evolved from the Greek word '*Χάος*', which signifies a state without order or predictability. According to Greek mythology, chaos is the "*primeval emptiness preceding the genesis of the universe, turbulent and disordered, mixing of all the elements*".

In the modelling of physical systems differential equations are very useful. If the solutions of differential equations are bounded then the solutions are either settle down to a fixed state or oscillate in a periodic state and sometimes in a quasi-periodic state. There are a few systems whose solutions do not lies in any of these categories. These solutions exhibit aperiodic (or irregular) motion forever and never settle. In addition, these solutions are highly sensitive to initial conditions. In this way it is hard to foresee the behaviour of the solution for a long time. Such systems are called chaotic dynamical systems.

In 1860, James Clerk Maxwell was likely saw the chaos while contemplating the movement of two impacting gas particles in a box which was unpredictable for long duration. Henry Poincare was the first person to see the likelihood of chaos in 1890 while studying the famous three-body problem. In which Poincare discovered orbits were non-periodic but not everlastingly expanding nor drawing closer towards a fixed point which is these days known as chaos. The significant leap forward in chaos theory and nonlinear dynamics was after the revelation of fast PCs in 1950.

In 1963, the meteorologist Edward Norton Lorenz presented the strange attractor notion and coined the term "butterfly effect". The model analysed by Lorenz emerging in climate forecast was comprising of autonomous system of three ordinary differential equations' conditions containing nonlinear terms. "Strange attractors" (Ruelle and Takens (1971)) was coined by D. Ruelle and F. Takens during a phenomenon in which they tried to explain turbulence in fluid dynamics. Li and Yorke (1975) demonstrated the sustained aperiodic and random behaviors emerging in deterministic nonlinear maps. Their exploration exhibited 'chaos' for the different phenomena that explained aperiodicity which sensitively dependence on initial conditions. May (1976) studied one dimensional maps in a simple electronic circuit resulting chaotic attractor (difference equations) applied in modelling of population dynamics. He observed that an exceptionally straightforward model can generate extremely complicated dynamics. This was the spearheading work in the investigation of chaos in maps. In this manner, the chaos can occur in (i) one dimensional maps, (ii) nonlinear, autonomous system of differential equations of order three and higher and (iii) nonlinear, non-autonomous system of differential equations of order two and higher. One of the premier supporters of this area of research was Benoit Mandelbrot. Utilizing a PC, Mandelbrot (1982) initiated the fractals. His fractals (the geometry of fractional dimensions) served to explain the chaos, rather than clarify it.

These days, chaos theory attracted the attention of the several scientists and adds to a lot of the continuous research concerning various fields, such as message encryption, change of the weather, evolution of the solar system, behavior of the stock markets, control of chemical reactions, information theory, etc. Before going for additionally think about in the area of chaos, give us a chance to make an endeavor to portray it.

1.4.1 Definition of chaos

In nonlinear sciences, chaos is a functioning region of research since most recent couple of decades. There is no unified, universally accepted, rigorous meaning of chaos in the current scientific literature. A regularly utilized definition of chaos which accompanying idea of this has mentioned by Steven H. Strogatz in his monograph (Strogatz (1994)).

"*Chaos is aperiodic long-term behaviour, in a deterministic system that exhibits sensitive dependence on initial conditions*".

“Aperiodic long-term behaviour" depicts that there are the trajectories which do not settle down to fixed points, periodic orbits, or quasi-periodic orbits as time becomes large.

"Deterministic" implies that the system has no irregular or noisy inputs or parameters. Nonlinear terms in the systems are more accountable for irregular behaviour, rather than from noisy driving forces.

"Sensitive dependence on initial conditions" explains that small change in initial conditions will creates dynamically bigger changes in further states or we can state as arbitrary small variation of current trajectory may prompt to large different future behaviour or trajectories separate exponentially fast. In other word, we can say that the system has a positive Lyapunov exponent.

1.4.2 Attractor and strange attractor

An attractor is nothing but a set whose all neighbouring trajectories converge. The examples of attractors are stable limit cycles and stable fixed points. More precisely by, an attractor to be a closed set A satisfying the following properties (Strogatz (1994)):

(i) A is an invariant set, i.e., for any trajectory $x(t)$ that starts in A stays in A for all time.

(ii) There exits an open set U containing A such that if $x(0) \in U$, then the distance from $x(t)$ to A tends to zero as $t \to \infty$. We can also say that A attracts an open set of initial conditions, i.e., A attracts all trajectories that start sufficiently close to it. The largest such set U is called the basis of attraction of A.

(iii) A is minimal, if there is no proper subset of A that satisfies conditions (i) and (ii).

At last, a strange attractor is an attractor that shows sensitive dependence on initial conditions and since it is often fractal sets so this is knew as strange. These days this geometric property is viewed as less imperative than the dynamical property of sensitive dependence on initial conditions.

1.4.3 Lyapunov Exponent

The Lyapunov exponent gives a determination of convergence or divergence of two neighbouring trajectories. This quantity has evaluated the sensitive dependence on initial conditions. Suppose a continuous time dynamical system is defined as

$$\frac{dX}{dx} = \phi(X), X \in R^n. \tag{1.13}$$

Here $X(t)$ is its corresponding trajectory and $X(0)$ is an initial condition. Now let a small displacement occurs in the direction of the tangent vector $u(0)$ from the initial condition $X(0)$, then evolution of the tangent vector $u(t)$ can be expressed as

$$\frac{du(t)}{dx} = D\phi(X(t), u(t)), \tag{1.14}$$

where $D\phi$ denotes Jacobian matrix of ϕ. This decides the evolution of very small displacement $u(t)$ of the trajectory from the unperturbed trajectory $X(t)$. Presently it assumed that the exponential growth rate of $\|u(t)\|$ is a number λ which can be expressed as

$$\|u(t)\| = e^{\lambda t} \|u(0)\|. \tag{1.15}$$

This gives the value of λ as

$$\lambda = \frac{1}{t} \ln\left(\frac{\|u(t)\|}{\|u(0)\|}\right). \tag{1.16}$$

Here λ is called a Lyapunov exponent. Now three cases arise.

(i) If $\lambda > 0$, it depicts sensitive dependence on initial conditions and therefore chaotic in nature, i.e., the neighbouring trajectories separate exponentially fast.

(ii) If $\lambda < 0$, the trajectory attracts to a fixed stable point or stable to periodic orbit.

(iii) If $\lambda = 0$, the trajectories will be a neutral fixed point or an eventually fixed point.

A system may have many Lyapunov exponents as per the number of dimensions of the phase space i.e., n dynamical system has n Lyapunov exponents. Any system is known as chaotic system if it contains at least one positive Lyapunov exponent and hyper-chaotic system if it contains more than one positive Lyapunov exponent.

1.4.4 Chaos in fractional order systems

Fractional order chaotic dynamical systems are obtained by substituting fractional derivative in the place of the integer order derivative in the system. For every fractional order chaotic dynamical systems there exits a critical value below which the system is regular and for the higher values system is chaotic. Fractional derivatives made secure systems more powerful and reliable. There is one more motivation to consider fractional order derivative because it acts as additional parameter which works as a key.

1.5 Chaos Synchronization

Fujisaka and Yamada (1983a, 1983b) made ready with their spearheading contemplates on chaos synchronization, In 1990, Pecora and Carroll (1990) presented their technique for chaotic synchronization and recommended application to secure communication, which got significant consideration inside main stream researchers. First of all, L. M. Pecora and T. L. Carroll acquainted a technique to synchronize drive and response systems with different initial conditions.They wrote that:

"*Chaotic systems would seem to be dynamical systems that defy synchronization. Two identical autonomous chaotic systems started at nearly the same initial points in phase space have trajectories which quickly become uncorrelated, even though each map out the same attractor in phase space. It is thus practically impossible to construct, identical, chaotic, synchronized system in laboratory*".

In chaos synchronization two or more than two, non-identical or identical, chaotic systems alter a given property of their motion to common coupled behaviour due to forcing or coupling. It may appear that the synchronization of chaotic system is hard to accomplish because of their extremely sensitive dependence on initial conditions and system parameters. Because of this property synchronization of the complex chaotic systems has major applications in security purposes. This is the main reason synchronization attracted several researchers and scientists to do work in this field.

1.5.1 Types of synchronization

The investigation of Fujisaka and Yamada (1983a, 1983b) and Pecora and Carroll (1990) on synchronization of chaotic systems open a new door of research in the field of applied mathematics. Motivated by the works of these researchers, different types of synchronization have been analysed viz. hybrid synchronization, phase synchronization, lag synchronization, projective synchronization, generalised synchronization, dual synchronization, function projective synchronization, combination synchronization, combination-combination synchronization, dual combination synchronization etc. Few different types of synchronization are described in details in following sub-sections.

1.5.1.1 Complete synchronization

First, Pecora and Carroll (1990) analysed complete synchronization. In complete synchronization equality of the state variables shows while evolving in time. In complete synchronization the chaotic trajectories of the coupled systems remain in step with each other when time evolves.

Let two continuous time chaotic systems are given as

$$\dot{x}(t) = f\big(x(t)\big), \tag{1.17}$$

$$\dot{y}(t) = g\big(y(t)\big) + u\big(x(t),\ y(t)\big), \tag{1.18}$$

where $f,\ g : R^n \to R^n$ are nonlinear continuous functions and $x(t), y(t) \in R^n$ are the state vectors the systems (1.17) and (1.18) respectively, $u\big(x(t),\ y(t)\big)$ is the control function. The considered chaotic systems (1.17) and (1.18) will be synchronized if $\lim_{t\to\infty} \|y(t) - x(t)\| = 0$ for initial conditions $x(0)$ and $y(0)$. Complete synchronization is also known as convention synchronization, or simply synchronization, or identical synchronization.

1.5.1.2 Anti synchronization

Two chaotic systems are said to be anti synchronized, if the respective states of chaotic systems $x(t)$ and $y(t)$ have the same magnitude but opposite in sign. Mathematically, anti synchronization for equations (1.17) and (1.18) is achieved when $\lim_{t\to\infty} \|y(t) + x(t)\| = 0.$

1.5.1.3 Lag and Anticipating synchronization

In the lag synchronization $\tau < 0$ appears as the asymptotic boundedness of the contrast between the yield of one framework at time t and the yield of the other moved in time of a lag time. In the anticipating synchronization, the state is characterised by a time interval τ such that the dynamical variables of the chaotic systems are connected by $y(t) = x(t+\tau)$. This implies that the dynamics of one of the systems follows the dynamics of the other. In case of anticipating synchronization, the response anticipates the dynamics of the drive. Particularly, if the time delay become zero then anticipating synchronization and lag synchronization are become complete synchronization.

1.5.1.4 Phase synchronization

The phase synchronization occurs when their phase difference bounded by a constant between the coupled chaotic systems while their amplitudes are not correlated. This phenomenon is found in coupled non-identical systems. In this synchronization, if $\phi_1(t)$ and $\phi_2(t)$ depict the phases of the two coupled chaotic systems then synchronization is given by the relation $n\phi_1(t) = m\phi_2(t)$, where m and n are whole numbers.

1.5.1.5 Modified projective synchronization

First of all, G. H. Li (2007) gave the idea of modified projective synchronization. In this synchronization, error state is defined as $e(t) = y(t) - A\,x(t)$, where $A = diag\,[a_1\ , a_2\ , \cdots , a_n]$ is known as scaling constant matrix such that a_i's are called constant scaling factors $\forall i \in N$. The systems (1.17) and (1.18) are modified projective synchronized, if the constant matrix A exists such that $\lim_{t\to\infty} \| e(t) \| = 0$.

1.5.1.6 Dual synchronization

In this type of synchronization two drive systems are considered as

$$\dot{X} = F(X)\ , \tag{1.19}$$

$$\dot{Y} = G(Y)\ , \tag{1.20}$$

where X and Y are state vectors.

The linear combination of the drive systems are given as

$$V_m = \sum_{i=1}^{n} a_i X_i + \sum_{i=1}^{n} b_i Y_i$$

$$= [a_1,\ a_2,\cdots,\ a_n]X + [b_1,b_2,\cdots,b_n]Y$$

$$= A^T X + B^T Y$$

$$= [A^T\ B^T]\begin{bmatrix} X \\ Y \end{bmatrix} = C^T \xi,$$

where $A = [a_1,\ a_2,\cdots,\ a_n]^T$ and $B = [b_1,b_2,\cdots,b_n]^T$ are known and $C = [A^T\ \ B^T]^T$.

Now two response systems are considered as

$$\dot{x} = f(x) + u^{(1)} \ , \tag{1.21}$$

$$\dot{y} = g(y) + u^{(2)} \ , \tag{1.22}$$

where x and y are state vectors and $u^{(1)}(t)$ and $u^{(2)}(t)$ are control functions and $u^{(i)}(t) = [u_1^{(i)},\ u_2^{(i)},\cdots,u_n^{(i)}]^T,\ \ i = 1,\ 2$.

The linear combination of the both response systems are given as

$$V_s = \sum_{i=1}^{n} a_i x_i + \sum_{i=1}^{n} b_i y_i \ ,$$

$$= [a_1,a_2,\cdots,\ a_n]x + [b_1,b_2,\cdots,b_n]y$$

$$= A^T x + B^T y$$

$$= [A^T\ B^T]\begin{bmatrix} X_m \\ Y_m \end{bmatrix} = C^T \eta.$$

The objective is to acquire the dual synchronization among drive and response systems. Now we define the error function among the drive systems (1.19), (1.20) and response systems (1.21), (1.22) as $e = V_s - V_m$. The drive systems (1.6.3), (1.6.4) and response systems (1.21), (1.22) are said to be dual synchronized if $\lim_{t\to\infty}\| e \| = 0$, where $\|\cdot\|$ denotes matrix norm.

1.5.1.7 Combined or Combination synchronization

In this section, the drive systems are given as

$$\dot{x}_1 = f_1(x_1), \tag{1.23}$$

$$\dot{x}_2 = f_2(x_2), \tag{1.24}$$

and the response system is considered as

$$\dot{y} = f(y) + U(x_1, x_2, y), \tag{1.25}$$

where $x_1 = [x_1^1, x_2^1, \cdots, x_n^1]^T$, $x_2 = [x_1^2, x_2^2, \cdots, x_n^2]^T$ and $y = [y_1, y_2, \cdots, y_n]^T$ are the state vector variables of the chaotic systems and f_1, f_2, $f : R^n \to R^n$ are continuous functions while $U(x_1, x_2, y)$ is a control function.

Drive systems (1.23), (1.24) and response system (1.25) are called combination synchronized, if three constants matrices (scaling matrices) A_1, A_2, A_3 exist and $A_3 \neq 0$, such that $\lim_{t\to\infty} \| A_1x_1 + A_2x_2 - A_3y \| = 0$, where $\| \cdot \|$ is the matrix norm.

It should be noted that if $A_1 \neq 0$, $A_2 = 0$, $A_n = I$, then the whole problem will be converted into the projective synchronization, where I is a $n \times n$ unity matrix. Similarly, if the scaling matrix A_1 is taken as a function, then synchronization is reduced into function projective synchronization.

1.5.1.8 Dual combination synchronization

In the dual combination synchronization, synchronization is investigated among four drive and two response systems. Here, first two drive systems are defined by the equations (1.19) and (1.20) and next two drive systems are given as

third Drive system:

$$\dot{X}' = f(X'), \tag{1.26}$$

fourth Drive system:

$$\dot{Y}' = g(Y'), \tag{1.27}$$

where X' and Y' are state vectors.

The linear combination of the drive systems third and fourth are given as

$$V_m' = \sum_{i=1}^{n} a_i X_i' + \sum_{i=1}^{n} b_i Y_i'$$

$$= [a_1, a_2, \ldots. a_n] X' + [b_1, b_2, \ldots. b_n] Y'$$

$$= A^T X + B^T Y = [A^T \; B^T] \begin{bmatrix} X' \\ Y' \end{bmatrix} = C^T \xi'.$$

Two response systems with control functions are defined by systems (1.6.5) and (1.6.6). Among four drive systems (1.19), (1.20), (1.26), (1.27) and response systems (1.21), (1.22) the error function is defined as $e = V_s - V_m - V_m'$.

If $\lim_{t \to \infty} \| e \| = 0$, where $\| \cdot \|$ represents the matrix norm. Then drive systems (1.19), (1.20), (1.26), (1.27), and the response systems (1.21), (1.22) are said to be synchronized according to dual combination synchronization.

1.6 Methodology of chaos synchronization

Pecora and Carroll (1990) composed Pecora-Carroll scheme and examined synchronization between two identical chaotic systems with different initial conditions. They demonstrated hypothetically and tentatively demonstrated that it isn't difficult to synchronize chaotic systems by suitable couplings between these systems. After this, several researchers to developed many methods to synchronize the chaotic systems in which some methods are Adaptive control method, Pecora-Carroll method, Active control method, Tracking control method, Backstepping method, Nonlinear control method etc.

1.6.1 Active Control Method

This method was first designed by Bai and Lonngren (1997) in the study of synchronization of the identical Lorenz chaotic systems. Ho and Hung (2002) analysed the active control method for synchronization of easy periodic system and Rossler system. Yan and Li (2007) talked about synchronization of Lorenz, Rossler and Chen systems of fractional order. Vincent and Laoye (2007) investigated chaos synchronization between two nonlinear systems using active control and back stepping control methods as far as transient examination. Same year, Zhou and Cheng (2008) discussed synchronization between various chaotic systems viz., Chen & Rossler systems and Chua & Chen systems. Srivastava et al. (2014) also applied this method for anti-synchronization between identical and non-identical fractional order chaotic systems.

1.6.2 Nonlinear Control Method

First of all, Park (2005) used nonlinear control method to study the synchronization of chaotic systems. Dong et al. (2006) used this method to analyse synchronization of the hyper chaotic Rossler system with uncertain parameters. Further, Li and Ge (2011) introduced this method in the study of adaptive synchronization of different orders chaotic systems with uncertain parameters. Singh et al. (2014) also used the nonlinear control method during study of synchronization and anti-synchronization of chaotic systems.

To process the method for synchronization, the drive system is considered as

$$\dot{x}_i = P x_i + Q f(x_i) , \; i = 1, 2, \cdots, n , \qquad (1.28)$$

where the state vector $x_i = [x_1, x_2, \cdots, x_n]^T \in R^n$, P and Q are $n \times n$ matrices of the system parameters and $f : R^n \to R^n$ is a nonlinear function of the system.

Consider the response system as

$$\dot{y}_i = P_1\, y_i + Q_1\, g(y_i) + u_i \ , \ i = 1,\ 2,\ \cdots,\ n\ , \tag{1.29}$$

where the state vector $y_i = [y_1,\ y_2,\ \cdots,\ y_n]^T \in R^n$, P_1 and Q_1 are $n \times n$ parameter matrices, $g : R^n \to R^n$ is a nonlinear function. u_i's are known as control functions of the system.

The errors can be defined by $e_i = y_i - x_i$, $i = 1,\ 2,\ \cdots,\ n$ and error system becomes

$$\dot{e}_i = P_1\, e_i + Q_1\, g(y_i) + (P_1 - P)\, x_i - Q\, f(x_i) + u_i\ . \tag{1.30}$$

For synchronization our main aim is to find u_i's such that the error system (1.30) should be stabilized in order to get $\lim_{t \to \infty} \| \ e(t) \ \| = 0$ for every $e(0) \in R^n$. Here u_i are known as appropriate feedback controllers.

Lyapunov function is defined as

$V = \dfrac{1}{2} e_i^T\ e_i$,with $e_i(t) = [e_1(t), e_2(t),\ \cdots,\ e_n(t)]^T$.

Derivative of Lyapunov function $V(t)$ w. r.t. t is obtained as

$$\frac{dV}{dt} = \frac{1}{2}\frac{d(e_i^T e_i)}{dt} = \frac{1}{2}\frac{d}{dt}(e_1^2 + e_2^2 + \ldots\ldots + e_n^2) = \sum_{i=1}^{n} \frac{1}{2}\frac{de_i^2}{dt} = \sum_{i=1}^{n} e_i \frac{de_i}{dt}\ .$$

Now, control functions are chosen as $u_i = -(P_1 + 1)e_i - (P_1 - P)x_i - Q_1\, g(y_i) + Q\, f(x_i)$.

In this case $\dfrac{dV}{dt} = -\sum_{i=1}^{n} e_i^2$. (1.31)

Thus $V(t)$ becomes negative definite which is necessary to get the required synchronization of the systems (1.28) and (1.29).

1.7 Basic of Fractional Calculus

In 1695, Leibniz asked a question: "*Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders ?*" L. Hospital replied the answer of this question to Leibniz: "*What if the order will be 1/2?*" Further, Leibnitz answered: "*It will lead to a paradox, from which one day useful consequences will be drawn*" The question raised by Leibnitz for a fractional order derivative was an on-going theme for in excess of three hundred years. Several renowned mathematicians such as J. Liouville, H. Weyl, B. Riemann, J. Fourier, S. F. Lacroix, N. H. Abel, A. K. Grunwald, G. Leibniz and A. V. Letnikov contributed to this notion over the years. In 1893, Oliver Heaviside introduced fractional derivative in electromagnetic theory. H. Weyl and G. H. Hardy studied properties of fractional derivative/integral in 1917. A. Erdelyi and T. J. Oslar defined fractional derivatives and Leibniz rule in 1939 and 1971 respectively. In the book of Ross (1975) it is mentioned that Niels Henrik Abel used fractional derivative in 1823 in tautochrone problem. Further, in 1832, J. Liouville gave the first definition of fractional derivative. In 1844, G. Boole used fractional calculus during solving linear differential equations with constant coefficients. In 1847, Bernhard Riemann (Ross (1975)) gave the following definition of fractional integration as

$$D^{-\alpha}\varphi(x) = \frac{1}{\Gamma(\alpha)}\int_{c}^{x}(x-t)^{\alpha-1}\varphi(t)dt + \psi(x) \ , \tag{1.32}$$

where $\psi(x)$ is Riemann's complementary function. This new branch of mathematics called fractional calculus (Miller and Ross (1993)) handles arbitrary order derivatives and integrals developed. Currently arbitrary real and even complex numbers can be considered as order of differentiation (Kilbas (2006)).

1.7.1 Some results on fractional derivatives

1.7.1.1 Grunwald-Letnikov fractional derivative

Successive differentiations of function $\phi(t)$ are given as

$$\phi^{(1)}(t) = \lim_{h\to 0} \frac{\phi(t) - \phi(t-h)}{h} ,$$

$$\phi^{(2)}(t) = \lim_{h\to 0} \frac{\phi^{(1)}(t) - \phi^{(1)}(t-h)}{h} ,$$

$$= \lim_{h\to 0} \frac{\phi(t) - 2\phi(t-h) + \phi(t-2h)}{h^2} ,$$

Proceeding in similar way, we get

$$\phi^{(m)}(t) = D^m \phi(t) = \lim_{h\to 0} \frac{1}{h^m} \sum_{k=0}^{m} (-1)^k \binom{m}{k} \phi(t-kh) , \tag{1.33}$$

where $\binom{m}{k} = \frac{m!}{k!(m-k)!}$. For a non-integer $\alpha > 0$, it may be expressed as $\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)}$.

The Grunwald-Letnikov definition is nothing but the generalisation of the definition (1.33) to a non-integer $\alpha > 0$.

$${}^{GL}D_a^{\alpha} \phi(t) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\left[\frac{t-a}{h}\right]} (1)^k \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)} \phi(t-kh) . \tag{1.34}$$

The $\alpha > 0$ order fractional integral is defined as

$${}^{GL}D_a^{-\alpha} \phi(t) = \lim_{h\to 0} h^{\alpha} \sum_{k=0}^{\left[\frac{t-a}{h}\right]} \frac{\Gamma(\alpha+k)}{k!\Gamma(\alpha)} \phi(t-kh) . \tag{1.35}$$

1.7.1.2 Riemann-Liouville fractional derivative

Riemann-Liouville fractional operator has important place in fractional integral. It is generalization of the Cauchy's formula for an n-fold integral as (Podlubny, 1999)

$$\underbrace{\int_a^x dt \int_a^{x_1} dt \cdots \int_a^{x_{n-1}} \phi(t)\, dt}_{n-\text{times}} = \frac{1}{(n-1)!}\int_a^x \frac{\phi(t)}{(x-t)^{1-n}}\, dt \,. \qquad (1.36)$$

Definition 1.1: If $\phi(x) \in C[a,b]$ and $l > 0$, then

$$J_{a^+}^{\alpha}\phi(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\phi(t)}{(x-t)^{1-\alpha}}dt,\ x > a,$$

$$J_{b^-}^{\alpha}\phi(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \frac{\phi(t)}{(x-t)^{1-\alpha}}dt,\ x < b \ , \qquad (1.37)$$

are known as the right and left sided Riemann-Liouville fractional integrals of order α respectively.

Definition 1.2: Left side Riemann-Liouville fractional derivative of order α is expressed as

$${}^{RL}D_a^{\alpha}\phi(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{\phi(t)}{(x-t)^{\alpha}}dt = DI_a^{1-\alpha}\phi(x),\ 0 < \alpha < 1, \qquad (1.38)$$

whenever the RHS exists (Podlubny (1999)).

Definition 1.3: Let $n-1 < \alpha \leq n$, then the right and left sided Riemann-Liouville fractional derivatives of order α are defined as (Podlubny (1999))

$${}^{RL}D_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x \frac{f(t)}{(x-t)^{\alpha+1-n}}dt = D^n\, I_{a^+}^{n-\alpha} f(x)\ ,\ x > a\ ,$$

$${}^{RL}D_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_x^b \frac{f(t)}{(x-t)^{\alpha+1-n}}dt = D^n\, I_{b^-}^{n-\alpha} f(x)\ ,\ x < b, \qquad (1.39)$$

respectively, whenever the RHS's exist.

Unless mentioned otherwise, we denote ${}^{RL}D_{a^+}^{\alpha} f(x)$ by ${}^{RL}D_{a}^{\alpha} f(x)$ and $J_{a^+}^{\alpha} f(x)$ by $J_{a}^{\alpha} f(x)$, respectively. Also ${}^{RL}D^{\alpha} f(x)$ and $J^{\alpha} f(x)$ refer to ${}^{RL}D_{0^+}^{\alpha} f(x)$ and $J_{0^+}^{\alpha} f(x)$, respectively.

Properties: (i) The fractional derivative (Riemann-Liouville) of constant is not zero.

$$ {}^{RL}D^{\alpha} C = \frac{C t^{-\alpha}}{\Gamma(1-\alpha)} \neq 0 . \tag{1.40} $$

(ii) If $n-1 \leq \alpha < n$, then initial value problem containing Riemann-Liouville fractional derivative requires initial conditions of the form ${}^{RL}D^{\alpha-j} f(0)$ i.e.,

$$ J^{\alpha}\left({}^{RL}D^{\alpha} f(x)\right) = f(t) - \sum_{j=1}^{n} {}^{RL}D^{\alpha-j} f(0) \frac{t^{\alpha-j}}{\Gamma(\alpha-j+1)} , \tag{1.41} $$

It is not applicable in real world problem. Due to these shortcoming, Caputo and Mainardi (1971) investigated a new definition of derivatives which permits the formulation of initial conditions for fractional IVP's in another form which involves only the limit values of integer order derivatives at the lower terminal.

1.7.1.3 Caputo fractional derivative

Let us suppose that $n-1 < \alpha < n$ and $f \in C^{n}[a, b]$ then Caputo fractional derivative is defined as

$$ {}_{a}^{C}D_{x}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{x} \frac{f^{(n)}(t)}{(x-t)^{(\alpha-n+1)}} dt , \; a < x < b . \tag{1.42} $$

Properties:

(i) if C is a constant then ${}_{a}^{C}D^{\alpha} C = 0$, (1.43)

(ii) $\lim_{\alpha \to n} {}_{a}^{C}D^{\alpha} f(x) = f^{(n)}(x)$. (1.44)

Lemma 1.7: (Norelys et al. (2014)) Let us suppose that $f(t) \in R$ be a continuous and derivable function. Then for every $t \geq t_0$,

$$\frac{1}{2} D^{\alpha} f^2(t) \leq f(t) D^{\alpha} f(t), \ \forall \alpha \in (0,1). \tag{1.45}$$

1.7.1.4 Relation between Riemann-Liouville and Caputo derivatives

Theorem 1.8: Let $n-1 < \alpha < n$ and $f \in C^n[a, b]$ then Riemann-Liouville and Caputo fractional derivatives are expressed by the relation given as

$$^{RL}D_a^{\alpha} f(x) = D_a^{\alpha} f(x) + \sum_{k=0}^{n-1} \frac{f^{(k)}(a^+)}{\Gamma(1+k-\alpha)} (x-a)^{k-\alpha} \ . \tag{1.46}$$

Proof: $^{RL}D^{\alpha} f(x) = D^n J^{n-\alpha} f(x)$

$$= D^n \left[J^{n-\alpha} \left(J^n f^{(n)}(x) + \sum_{k=0}^{n-1} \frac{f^{(k)}(a^+)}{k!} (x-a)^k \right) \right]$$

$$= J^{n-\alpha} f^{(n)}(x) + D^n J^{n-\alpha} \sum_{k=0}^{n-1} \frac{f^{(k)}(a^+)}{k!} (x-a)^k \tag{1.47}$$

$$= D_a^{\alpha} f(x) + \sum_{k=0}^{n-1} \frac{f^{(k)}(a^+)}{\Gamma(1+k-\alpha)} (x-a)^k . \tag{1.48}$$

From the above theorem, the following results can be easily obtained:

(i) If $\alpha = n \in N$, then $^{RL}D_a^{\alpha} f(x) = D_a^{\alpha} f(x) = D^n f(x)$.

(ii) If $f^{(k)}(a) = 0$ for $k = 0, 1, \cdots, n-1$, then $^{RL}D_a^{\alpha} f(x) = D_a^{\alpha} f(x)$.

(iii) If $0 < \alpha < 1$, then $^{RL}D_a^{\alpha} f(x) = D_a^{\alpha} f(x) + \dfrac{f(a)}{\Gamma(1-\alpha)} (x-a)^{-\alpha}$.

Theorem 1.9: Let $n-1 < \alpha < n$ and $f \in C^n[a, b]$ then

$$J_a^{\alpha} D_a^{\alpha} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a^+)}{k!} (x-a)^k , \ x \geq a \ . \tag{1.49}$$

Proof: If $J_a^{\alpha}\, D_a^{\alpha} f(x) = J_a^{\alpha}\, J_a^{n-\alpha}\, f^{(n)}(x) = J^{(n)}\, f^{(n)}(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k$,

$f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k$, $x \geq a$, then equation (1.49) is a particular case of

$$J_a^{\alpha}\, D_a^{r} f(x) = J_a^{\alpha}\, J_a^{m-r}\, f^{(m)}(x) = J_a^{(\alpha-r)}\left(J^{(n)}\, f^{(n)}(x)\right)\ \alpha > r\,, \tag{1.50}$$

$$= J_a^{\alpha-r} f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(\alpha-r+k+1)}(x-a)^{\alpha-r+k},\ x \geq a\,,\ m-1 < r < m\,.$$

1.7.1.5 Leibniz rule

Let $\phi(x), g(x) \in C^{\infty}[a, t]$ then the Leibniz rule for fractional derivative is given by (Podlubny (2002))

$$_aD_a^{\alpha}(g(t)\phi(t)) = \sum_{k=0}^{\infty} \binom{\alpha}{k} g^{(k)}(t) D^{\alpha-k}\, \phi(t) - R_n^{\alpha}(t)\,, \tag{1.51}$$

where $R_n^{\alpha}(t) = \frac{1}{n!\Gamma(-\alpha)} \int_a^t \frac{\phi(\tau)}{(t-\tau)^{\alpha+1}} d\tau \int_{\tau}^{t} g^{(n+1)}(\xi)(\tau-\xi)^n d\xi$.

1.8 Numerical Methods

Numerical analysis is an import branch of mathematics to get numerical solution of a problem. Numerical methods are applied to solve the problems in mathematics, physics, medicine, social sciences, natural sciences and several branches of engineering. In the decade of 1980, computer had been started to use in the several fields for research purpose. Often mathematical models consist system of ordinary differential equations, partial differential equations, and integral equations in which most models cannot be solved exactly. To get probable result we approximate the derivatives or integrals contained in the equation of system. At a discrete set of points the approximating equation has a solution. In this section the numerical method for solving fractional order differential equation is discussed.

1.8.1 Adams-Bashforth-Moulton Method

Adams-Bashforth-Moulton method is a very useful method to solve fractional order differential equations. This method was proposed by Diethelm (Diethelm et al. (2004), Diethelm and Ford (2004)). This method is generalization of the one-step Adams-Bashforth-Moulton method which is used to solve first order differential equations. This algorithm generally used in nonlinear problems, and further can be extended to an equation containing multi-terms provided proper initial conditions are given. The solution of nonlinear differential equation of fractional order by using this method can be given as follows

Let fractional order differential equation is expressed as

$$D_t^{\beta} y(t) = \phi(t, y(t)),\ 0 \le t \le T, \tag{1.52}$$

with $\quad y^{(\alpha)}(0) = y_0^{(\alpha)}, \quad \alpha = 0,1,\dots,n-1, \qquad (1.53)$

where n is the smallest integer i.e., $n = \lceil \beta \rceil$, $n \ge \beta$ and Caputo derivative is used as differential operator. Since the initial value problem is equivalent to the Volterra integral equation so the initial value problem (1.52) is transformed into the following form

$$y(t) = \sum_{\alpha=0}^{\lceil \beta \rceil - 1} y_0^{(l)} \frac{t^{\alpha}}{\alpha!} + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \varphi(s, y(s))\, ds. \tag{1.54}$$

Let $t_m = mh, m = 0,1,........N \in Z^{+}, h = T/N$. Then (1.54) can be written as:

$$y_h(t_{m+1}) = \sum_{\alpha=0}^{\lceil \beta \rceil - 1} y_0^{(\alpha)} \frac{t_{m+1}^{\alpha}}{\alpha!} + \frac{h^{\alpha}}{\Gamma(\beta+2)} \phi(t_{m+1}, y_h^{p}(t_{m+1})) + \frac{h^{q}}{\Gamma(\beta+2)} \sum_{j=0}^{m} a_{j,m+1} \phi(t_h, y_h(t_j)), \tag{1.55}$$

$$a_{j,m+1} = \begin{cases} m^{\beta+1} - (m-\beta)(m+1)^{\beta}, & if \quad j = 0, \\ (m-j+2)^{\beta+1} + (m-j)^{\beta+1} - 2(m-j+1)^{\beta+1}, & if \quad 0 \le j \le m, \\ 1, & if \quad j = m+1, \end{cases} \tag{1.56}$$

where the value of $y_h^{p}(t_{m+1})$ is given by fractional Adams-Bashforth method as

$$y_h^p(t_{m+1}) = \sum_{\alpha=0}^{\lceil \beta \rceil - 1} y_0^{(\alpha)} \frac{t_{m+1}^{\alpha}}{\alpha!} + \frac{1}{\Gamma(\beta)} \sum_{j=0}^{m} b_{j,m+1} \phi(t_j, y_h(t_j)), \tag{1.57}$$

$$b_{j,m+1} = \frac{h^{\beta}}{\beta}((m+1-j)^{\beta} - (m-j)^{\beta}). \tag{1.58}$$

Here $a_{j,m+1}$ and $b_{j,m+1}$ are the weights functions and the equations (1.55) and (1.57) describe the fractional Adams-Bashforth-Moulton scheme.

Now error is estimated as

$$\max_{j=0,1,....N} \left| y(t_j) - y_h(t_j) \right| = O(h^p), \qquad p = \min(2, 1+\alpha). \tag{1.59}$$

1.9 Review of Literature

Chaotic dynamics had begun with worked by the French mathematician H. Poincare. In the 1880's, Poincare was mostly handled by the issue of the orbits of three celestial bodies encountering mutual gravitational attraction. By considering the behaviour of orbits emerging from sets of initial points, Poincare could exhibit that exceptionally entangled (chaotic) orbits were possible and he discovered sensitive dependence on initial conditions in a particular occasion of the three body problem and later recommended that such phenomena could be normal. J. Hadamard (1898) noted general divergence of trajectories in spaces of negative curvature.

Notable mathematical works on chaotic dynamics include G. Birkhoffin in the 1920's, J. E. Littlewood and M. L. Cartwright in the 1940's, S. Smale in the 1960's and A. N. Kolmogorov. Besides these works, the likelihood of chaos in real physical systems was not generally refreshing until moderately as of late. The reasons were that the scientific articles are hard to peruse for researchers in alternate fields, and furthermore the theorems demonstrated were frequently not sufficiently solid to persuade specialists regarding material science and building working in a similar zone of research. The greater part of the prior theory was created on the whole by mathematicians, known as ergodic theory. In spite of starting bits of knowledge in the primary portion of the twentieth century, chaos hypothesis wound up formalized after mid-century. The situation has changed definitely after the development of advanced PCs through which the broad numerical arrangement of dynamical systems can be performed. Utilizing such solutions, the chaotic character of the time evolution in circumstances of useful significance has turned out to be clear. Moreover, the multifaceted nature of the dynamics can't be blamed for obscure unessential exploratory impacts.

In 1950's, after the innovation of fast PCs, it becomes a turning point in the history of dynamics. The PC enabled the scientists to experiment with equations, which were impossible previously, and in this way to build up some instinct about nonlinear systems. Such investigations prompted Lorenz (1963) to find the chaotic motion on a strange attractor. He demonstrated the butterfly effect while trying to forecast the weather. Lorenz's revelation planted the seed for the new hypothesis of chaos science. He additionally demonstrated that it was structure in the chaos, the solutions to his equations fell onto a butterfly shaped set of points, when plotted in three dimensions. He explained that this set must be "an infinite complex of surfaces", which is currently viewed for instance of a fractal. Lorenz's work had little effect until the 1970's and later it became a boon for chaos. In 1971, D. Ruelle and F. Takens depicted a phenomenon which is known as the strange attractor. This strange phenomenon was called by them as phase space and a radical new component of chaos theory was conceived. They proposed another hypothesis for the beginning of turbulence in liquids, based on abstract considerations about strange attractors. A period is the time required for any cyclic system to come back to its original state.

Another pioneer worker of the new science was Mitchell Feigenbaum. His work, in the late 1970's, was revolutionary to the point that few of his first original copies were rejected for publication since they were so novel and considered irreverent (Gleick (1987)). He built up a technique to measure turbulence and found a structure installed in nonlinear systems. As per Gleick (1987), a scientific evidence of his thoughts was exhibited in 1979 by Oscar E. Lawford III. He found that there exist some fixed laws representing the change from regular to chaotic behaviour. His work built up a connection between chaos and phase transitions, which become fascinating to the physicists to investigate the dynamics. Li and Yorke (1975) demonstrated that sustained aperiodic behaviour could be found in one-dimensional maps that maintained aperiodic behaviour. Their article (Li and Yorke, (1975)) gave the term chaos for the different phenomena that showed aperiodicity along with sensitive dependence on initial conditions. One of the principal supporters of this area of research was Benoit Mandelbrot. Utilizing PC, Mandelbrot (1982) spearheaded the arithmetic of fractals, a term which he coined in 1975. Chaos could now be found in shading on computers. He indicated how they could be connected in an assortment of subjects and in the developing territory of mathematical biology. Winfree (1980) used the geometric techniques of dynamics to biological matter. At last, scientists for example, J, P. Gollub, H. L. Swinney, P. Linsay and F.C. Moon tried the new thoughts regarding chaos in experiments on semiconductors, fluids, electronic circuits, chemical reactions and mechanical oscillators.

Pikovsky et al. (2001) depict synchronization as an alteration of rhythms of oscillating objects because of a weak interaction. Any initial correlation introduce between identical chaotic systems, beginning from close initial conditions exponentially decrease to zero with time. Accordingly, for every single common sense reason, any initial synchronization between the systems is bound to disappear rapidly. As of late, in any case, a few techniques for accomplishing synchronized behaviour between chaotic systems have been proposed. The Dutch scientist Christiaan Huygens, most popular for his investigations in optics and the development of telescopes and clocks, was likely the principal researcher who watched and described the synchronization. The chaos synchronization of systems is a subject with starting in the ahead of schedule of 1980's. Debated issue over the most recent two decades because of its potential uses for secure correspondences, biological systems, nano oscillators and so on.

Yamada and Fujisaka (1983, 1984) did important work on synchronization. In 1990, Pecora and Carroll developed a new method of chaotic synchronization (Pecora and Carroll, 1990) and suggested applications in security. Pecora and Carroll (1997) showed that for chaotic identical systems if the chaotic outputs from a sufficient number of degrees of freedom in one system replace some of the degrees of freedom in a second one, the two systems may display similar chaotic dynamics. In more general terms, the chaos synchronization can be divided into two fields in the reference of coupled chaotic elements viz., synchronization of identical and non-identical chaotic systems.

Synchronization of chaotic systems has been investigated by several researchers (Ott et al. (1990), Chen and Dong (1993), Fuh and Tung (1995), Chen and Dong (1998), Zhang and Sun (2004)) due to its applications in ecology (Blasius et al. (1999)), chemical system (Han et al. (1995)), physical system (Lakshmanan and Murali (1996)), system identification, pattern recognition phenomena, modeling brain activity and secure communications (Kocarev and Parlitz (1995), Murali and Lakshmanan (2003)) and so on.

In the recent years several different types of synchronization schemes have been proposed. OGY method (Ott et al. (1990)), linear and nonlinear feedback synchronization (Chen and

Lu (2002)), adaptive feedback control (Zhu and Cao (2010), Zaid and Odibat (2010)), active control (Yassen (2005), Ho and Hung (2002)), sliding mode control (Yau (2004), Faieghi and Delavari (2012)), back-stepping design method (Wu and Lü (2003)) and so on. This concept can be extended to complete synchronization (Yu and Liu (2003)), generalized synchronization (Yang and Juan (1998), Yang (2000)), lag synchronization (Taherion and Lai (1999)), phase synchronization.

Ding (1999) discussed high dimensional implementation of the synchronization of chaos. Parlitz and Junge (1999) numerically analyzed different synchronization in coupled chaotic systems not only in case of low dimensional chaos but also in high dimensional. Zhou et al. (2002) developed mode decomposition method for study stability analysis three chaotic systems. Xie et al. (2002) investigated hybrid chaos synchronization and applied this in information processing. Synchronization of chaotic systems of third-order and second-order has discussed by Femat and Perales (2002). He explained that second-order driven oscillators is synchronize with the canonical projection of a third order system. Ge et al. (2004) studied synchronization of three time scales brushless DC motor system and depicts that it is not hard to obtained more chaotic phenomena of the system. Chaos synchronization of several dynamical systems have studied by many researchers (Agiza (2004), Lü et al.(2004), Deng and Li (2005a), Deng and Li (2005b), Boccaletti (2006)). Park (2006) discussed chaos synchronization between Genesio and Rossler systems. Bowong et al. (2006) analyzed chaos synchronization of new chaotic systems and constructed an augmented dynamical system from the error system. Wang and Zhou (2007) applied contraction principle in chaos synchronization. Yau and Yan (2008) analyzed chaos synchronization of different chaotic systems with nonlinear input and tried to get chaos in Liu–Lorenz, Lorenz–Chen and Chen–Liu systems. Jia (2008) investigated chaos synchronization of the Newton–Leipnik chaotic system and employing active control method. Chu et al. (2008) evaluated chaos synchronization for non-autonomous systems and by the phase portraits study the behavior of chaotic attractors. Wang et al. (2009) discussed a control law to realize finite-time synchronization for the unified chaotic system. Chen et al. (2009) used nonlinear feedback control, Lyapunov theorem and balanced gain scheme to achieve chaos synchronization for the unified chaotic systems. Sun (2009) gave the concept of partial synchronization and applied this on Genesio Tesi chaotic systems. Jia and Wang (2011) studied hybrid projective synchronization to control chaos for a class of new chaotic systems. Chai and Chen (2012) used sliding mode control method to investigate projective lag synchronization of spatiotemporal chaos. Karthikeyan and Sundarapandian (2014) introduced active control method to analyze four scroll systems.

Recently, chaos and synchronization have been discussed by several mathematicians. Aziz and Azzawi (2017) discussed hybrid chaos synchronization between modified hyperchaotic Pan system and hyperchaotic Liu system. They used nonlinear control method based on Gardano's method and Lyapunov's second method. Vishal and Agrawal (2017) developed a novel complex chaotic system and discuss about the dynamics and existence of chaos and chaos control. Kocamaz et al. (2017) studied synchronization of chaos and used sliding mode control method to control chaos. They also discussed dynamical behaviors of proposed chaotic system. Dual combination-combination synchronization has been studied by Khan et al. (2017). Zhang and Liao (2017) also discussed synchronization in coupled memristor-based FitzHugh-Nagumo circuits. Petereit and Pikovsky (2017) discussed chaos and synchronization by nonlinear coupling. Khan et al. (2018) analyzed multi-switching dual compound synchronization. Sliding mode control has applied by Mobayen (2018) in the study of synchronization of uncertain chaotic systems and by Chen et al. (2018) during study of adaptive synchronization of multiple uncertain coupled chaotic systems. Khan and Sikha (2018) discussed combination-combination synchronization in hyper-chaotic system without any equilibria. Fractional calculus has been applicable almost every field of applied mathematics. It has also more applications in the field of dynamical system. Li et al. (2006) studied synchronization of the two identical fractional Chua system and applied the active–passive decomposition method, Pecora Carroll method, one-way coupling method. Chaotic fractional Chua system also discussed by Zhu et al. (2009). Zhu et al. (2009) also investigated synchronization of the fractional Chen's system. Odibat et al. (2010) used linear control method in synchronization of fractional systems simultaneously Bhalekar and Gejji (2010) applied active control method in the synchronization of different chaotic system of fractional order. Faieghi and Delavari (2012) obtained some condition for occurrence of chaos in Genesio-Tesi system of fractional order. Zhang and Yang (2012) developed modified adaptive scheme and applied on fractional chaotic systems. Zhang et al. (2013) analyzed synchronization in fractional differential systems and obtained fabulous results. Fan et al. (2013) discussed functional synchronization of the fractional order chaotic system and used fractional order Rikitake chaotic System to validate their results. Hegazi et al. (2013) discussed commensurate fractional Liu system and analyzed chaos control of this system. Hammouch and Mekkaoui (2014) discussed chaos synchronization of a fractional nonautonomous system while Srivastava et al. (2014) studied fractional Rabinovich Fabrikant system. Jiang et al. (2015) proposed generalized combination complex synchronization. Liu et al. (2015) also studied complex modified generalized projective synchronization. Moreover, Weiwei and Dingyuan (2015) investigated generalized synchronization of fractional-order chaotic systems with unequal orders.

Rad et al. (2015) applied adaptive control scheme to studied synchronization of the fractional Genesio Tesi system. Fang and Peng (2017) investigated novel type of synchronization known as pinning synchronization and applied on of fractional complex networks by a single controller. Ouannas et al. (2017) developed a method to analyzed fractional hybrid chaos synchronization. Ouannas et al. (2016) discussed synchronization for different dimensional chaotic systems by using two scaling matrices and applied fractional order Lyapunov direct method to synchronize two chaotic systems of fractional order. Ouannas et al. (2017) have also investigated coexistence of chaos and discussed many synchronization in different fractional order chaotic systems. Xin et al. (2017) have used linear control method in discrete dynamical systems. Zhang et al. (2018) have analyzed chaos synchronization for differential systems of fractional order. Huang et al. (2018) have used feedback control method in synchronization a fractional order Lu chaotic system. Yang and Jiang (2018) have studied synchronization analysis of fractional order drive-response networks. Yadav et al. (2018) have discussed synchronization for different non-identical fractional order systems. Pham et al. (2018) have discussed about synchronization a chaotic system of fractional order without equilibrium. Andrieu et al. (2018) have discussed some results on exponential Synchronization of Nonlinear Systems. Luo et al. (2019) have also studied some new results of exponential synchronization of complex network with time-varying delays. Dai et al. (2018) have analyzed exponential synchronization for nonlinear systems in complex dynamical networks. Yadav et al. (2019) have investigated dual phase and dual anti-phase synchronization of fractional order chaotic systems in real and complex variables with uncertainties.

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