Chapter 4

Generalized Sobolev type spaces involving the Weinstein transform

4.1 Introduction

Sobolev spaces are a fundamental concept in functional analysis and the theory of partial differential equations. They provide a framework for studying the properties of functions that have derivatives that may not necessarily be defined in the classical sense. The norm in a Sobolev space is defined in terms of derivatives and integrals, and it measures the smoothness of a function. It is widely used for solving the problems of partial differential equations, as well as in nonlinear analysis, differential geometry, physics, and other areas of mathematics. In [40], Pahk and Kang extended the notion of Sobolev space to generalized distribution spaces of Beurling-Bjorck type [6, 8], and used the weight function ω to investigate the Sobolev imbedding theorem, the Rellich's compactness theorem, and others. Pathak and Pandey [44] introduced the Sobolev space of type $G^{p,s}_{\mu}$ and discussed some properties, such as completeness and inclusion, by using the definitions of the distributional Hankel transform and they introduced a new concept called the Hankel potential $\mathcal{H}^s_\mu$ and found that Hankel potential is a continuous linear mapping from the Zemanian space $\mathcal{H}^s_\mu$ into the itself . Later on, Pathak and Shrestha [45], defind the space of type $G^{p,s}_{\omega,\mu}$ and discussed many results. Pathak [43], considered the generalized Sobolev space $H^s_\omega(\mathbb{R}^n)$ which is a generalization of the Sobolev space $H^s(\mathbb{R}^n)$ and developed a multiresolution analysis for the generalised Sobolev space.

The Weinstein transform has a rich calculus and involves many problems in the mathematical sciences and their applications. Utilizing this theory, many researchers obtained observations in the theory of Sobolev-type spaces. In this connection, Salem and Dachraoui [4], studied the Sobolev type spaces $G^{p,s}_{\alpha,\beta}$ associated with Jacobi differential operators $\Delta_{\alpha,\beta}$, and examined that the Jacobi potential $\mathcal{H}^s_{\alpha,\beta}$, is a pseudodifferential operator with a precise symbol. Using the aforesaid results they extended the operator $\mathcal{H}^s_{\alpha,\beta}$, to the distributional space [8]. Hassen and Belgacem [35], introduced the Sobolev spaces of exponential type associated with the Weinstein operator, and found many results. Mehrez, [30], Mejjaoli, et al. [31–33], Salem [56], Saoudi [60, 61], and Trimèche, [81], observed useful results regarding the Weinstein transform.

In this present chapter the author made a proper framework for the generalised Sobolev type space with the help of definitions and properties of the Weinstein transform. In this chapter, the generalized Sobolev space of type $G^{p,\,s}_{\omega}$ is introduced, and with the help of the Weinstein transform, and various properties are obtained. The L^p-space of Hankel potential, $W^{m,p}_\omega(\mathbb{R}^{n+1}_+)$ is considered, and many results related to the Hankel potential $\mathcal{H}^k$ are studied by exploiting the theory of Weinstein transform.

4.2 Generalized Sobolev type space $G_{\omega}^{p,\,s}$

In this section, the generalized Sobolev space $G_{\omega}^{p,\,s}$ is defined and its various properties including completeness and inclusion are obtained by exploiting the theory of the Weinstein transform.

Theorem 4.2.1. *The Weinstein transform $\mathcal{F}_w f$ is an automorphism on $S_{\omega}(\mathbb{R}_+^{n+1})$.*

Proof: Let $\phi \in S_{\omega}(\mathbb{R}_+^{n+1})$, then first we show that $\mathcal{F}_w(\phi) \in S_{\omega}(\mathbb{R}_+^{n+1})$. From (3.2.21), we have

$$\begin{aligned}
&p_{\alpha,\lambda}[\mathcal{F}_w\phi(\xi)]\\
&\quad= \sup_{\xi\in\mathbb{R}_+^{n+1}} e^{\lambda\omega(\xi)} \mid (\Delta_{W,\beta}^{n})_{\xi}^{\alpha}(\mathcal{F}_w\phi)(\xi) \mid\\
&\quad= \sup_{\xi\in\mathbb{R}_+^{n+1}} e^{\lambda\omega(\xi)} \mid (\Delta_{W,\beta}^{n})_{\xi}^{\alpha} \int_{\mathbb{R}_+^{n+1}} e^{-i\langle \xi',\, y'\rangle} J_{\beta}(\xi_{n+1}y_{n+1})\phi(y)\, d\mu_{\beta}(y) \mid .
\end{aligned}$$

Using (1.4.5), we get

$$\begin{aligned}
&p_{\alpha,\lambda}[\mathcal{F}_w\phi(\xi)]\\
&\quad= \sup_{\xi\in\mathbb{R}_+^{n+1}} e^{\lambda\omega(\xi)} \mid \int_{\mathbb{R}_+^{n+1}} e^{-i\langle \xi',\, y'\rangle} J_{\beta}(\xi_{n+1}y_{n+1})(-\|y\|^2)^{\alpha})\phi(y)\, d\mu_{\beta}(y) \mid .
\end{aligned}$$

Now, we can write $g(y) = (-\|y\|^2)^{\alpha}\phi(y)$ therefore above expression yeilds

$$\begin{aligned}
p_{\alpha,\lambda}[\mathcal{F}_w\phi(\xi)] &= \sup_{\xi\in\mathbb{R}_+^{n+1}} e^{\lambda\omega(\xi)} \mid \int_{\mathbb{R}_+^{n+1}} e^{-i\langle \xi',\, y'\rangle} J_{\beta}(\xi_{n+1}y_{n+1})g(y)\, d\mu_{\beta}(y) \mid\\
&\leq \sup_{\xi\in\mathbb{R}_+^{n+1}} e^{\lambda|\omega(\xi)|} \mid \int_{\mathbb{R}_+^{n+1}} e^{-i\langle \xi',\, y'\rangle} J_{\beta}(\xi_{n+1}y_{n+1})g(y)\, d\mu_{\beta}(y) \mid .
\end{aligned}$$

From [46] for every $\epsilon > 0$ there exist $C(\epsilon)$ such that

$$|\omega(\xi)| \leq \epsilon \|\xi\|^2 + C(\epsilon).$$

Therefore, we have

$$\begin{aligned}
&p_{\alpha,\lambda}[\mathcal{F}_w \phi(\xi)] \\
&\quad \leq \sup_{\xi \in \mathbb{R}_+^{n+1}} e^{\lambda(\epsilon\|\xi\|^2 + C(\epsilon))} \mid \int_{\mathbb{R}_+^{n+1}} e^{-i\langle \xi', y' \rangle} J_\beta(\xi_{n+1} y_{n+1}) g(y) \, d\mu_\beta(y) \mid \\
&\quad \leq e^{\lambda C(\epsilon)} \sup_{\xi \in \mathbb{R}_+^{n+1}} e^{\lambda\epsilon\|\xi\|^2} \mid \int_{\mathbb{R}_+^{n+1}} e^{-i\langle \xi', y' \rangle} J_\beta(\xi_{n+1} y_{n+1}) g(y) \, d\mu_\beta(y) \mid \\
&\quad \leq e^{\lambda C(\epsilon)} \sup_{\xi \in \mathbb{R}_+^{n+1}} \sum_{m=0}^{\infty} \frac{(\lambda\epsilon)^m}{m!} \|\xi\|^{2m} \mid \int_{\mathbb{R}_+^{n+1}} e^{-i\langle \xi', y' \rangle} J_\beta(\xi_{n+1} y_{n+1}) g(y) \, d\mu_\beta(y) \mid \\
&\quad = e^{\lambda C(\epsilon)} \sum_{m=0}^{\infty} \frac{(\lambda\epsilon)^m}{m!} \sup_{\xi \in \mathbb{R}_+^{n+1}} \mid \int_{\mathbb{R}_+^{n+1}} \|\xi\|^{2m} e^{-i\langle \xi', y' \rangle} J_\beta(\xi_{n+1} y_{n+1}) g(y) \, d\mu_\beta(y) \mid \\
&\quad = e^{\lambda C(\epsilon)} \sum_{m=0}^{\infty} \frac{(\lambda\epsilon)^m}{m!} \sup_{\xi \in \mathbb{R}_+^{n+1}} \mid \int_{\mathbb{R}_+^{n+1}} (\Delta_{W,\beta}^n)_y^m (e^{-i\langle \xi', y' \rangle} J_\beta(\xi_{n+1} y_{n+1})) g(y) \, d\mu_\beta(y) \mid .
\end{aligned}$$

By integrating by parts, we get

$$\begin{aligned}
&p_{\alpha,\lambda}[\mathcal{F}_w \phi(\xi)] \\
&\quad \leq e^{\lambda C(\epsilon)} \sum_{m=0}^{\infty} \frac{(\lambda\epsilon)^m}{m!} \sup_{\xi \in \mathbb{R}_+^{n+1}} \mid (-1)^m \int_{\mathbb{R}_+^{n+1}} e^{-i\langle \xi', y' \rangle} J_\beta(\xi_{n+1} y_{n+1}) (\Delta_{W,\beta}^n)_y^m g(y) \, d\mu_\beta(y) \mid \\
&\quad \leq e^{\lambda C(\epsilon)} \sum_{m=0}^{\infty} \frac{(\lambda\epsilon)^m}{m!} \sup_{\xi \in \mathbb{R}_+^{n+1}} \int_{\mathbb{R}_+^{n+1}} e^{-i\langle \xi', y' \rangle} J_\beta(\xi_{n+1} y_{n+1}) (\Delta_{W,\beta}^n)_y^m g(y) \, d\mu_\beta(y) \mid
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&p_{\alpha,\lambda}[\mathcal{F}_w\phi(\xi)] \\
&\leq e^{\lambda C(\epsilon)}\sum_{m=0}^{\infty}\frac{(\lambda\epsilon)^m}{m!}\left(\sup_{\xi\in\mathbb{R}_+^{n+1}} \mid e^{-i\langle\xi',\,y'\rangle}J_\beta(\xi_{n+1}y_{n+1})\mid\right)\mid\int_{\mathbb{R}_+^{n+1}}(\Delta_{W,\beta}^n)_y^m g(y)\,d\mu_\beta(y)\mid \\
&\leq e^{\lambda C(\epsilon)}\sum_{m=0}^{\infty}\frac{(\lambda\epsilon)^m}{m!}\mid\int_{\mathbb{R}_+^{n+1}} e^{\lambda\omega(y)}(\Delta_{W,\beta}^n)_y^m g(y)e^{-\lambda\omega(y)}\,d\mu_\beta(y)\mid \\
&\leq e^{\lambda C(\epsilon)}\sum_{m=0}^{\infty}\frac{(\lambda\epsilon)^m}{m!}\sup_{y\in\mathbb{R}_+^{n+1}}\mid e^{\lambda\omega(x)}(\Delta_{W,\beta}^n)_y^m g(y)\mid\mid\int_{\mathbb{R}_+^{n+1}} e^{-\lambda\omega(y)}\,d\mu_\beta(y)\mid \\
&= e^{\lambda C(\epsilon)}\sum_{m=0}^{\infty}\frac{(\lambda\epsilon)^m}{m!}\left(\sup_{y\in\mathbb{R}_+^{n+1}} e^{\lambda\omega(y)}\mid(\Delta_{W,\beta}^n)_y^m g(y)\mid\right)\mid\int_{\mathbb{R}_+^{n+1}} e^{-\lambda\omega(y)}\,d\mu_\beta(y)\mid \\
&= e^{\lambda C(\epsilon)}\sum_{m=0}^{\infty}\frac{(\lambda\epsilon)^m}{m!}p_{m,\lambda}(g(y))\mid\int_{\mathbb{R}_+^{n+1}} e^{-\lambda\omega(y)}\,d\mu_\beta(y)\mid \\
&< \infty.
\end{aligned}$$

This shows that $\mathcal{F}_w(\phi)\in S_\omega(\mathbb{R}_+^{n+1})$.

Since by the inversion formula for the Weinstein transform we have $\mathcal{F}_w^{-1}\phi(y) = \mathcal{F}_w\phi(-y)$. Therefore $\mathcal{F}_w$ is one-one. So that $\mathcal{F}_w$ is an automorphism.

Definition 4.2.2. The generalized Sobolev type space $G_\omega^{p,\,s}$ is the set of all ultradistributions $u\in S_\omega'(\mathbb{R}_+^{n+1})$, such that its the Weinstein transform $(F_w u)$ corresponds to a locally integrable function u over $\mathbb{R}_+^{n+1}$ for which

$$\|u\|_{G_\omega^{p,\,s}} = \|e^{s\omega(\xi)}(\mathcal{F}_w u)(\xi)\|_p < \infty. \tag{4.2.1}$$

Theorem 4.2.3. *The space* $G_\omega^{p,\,s}$, $1\leq p<\infty$ *is complete.*

Proof: Let $\{u_n\}$ be a Cauchy sequence in $G_\omega^{p,\,s}$. Then $e^{s\omega(\xi)}(\mathcal{F}_w u_n(\xi))$ is a Cauchy sequence in $L^p(\mathbb{R}_+^{n+1})$. Since $L^p(\mathbb{R}_+^{n+1})$ is complete, therefore there exists a function

$f \in L^p(\mathbb{R}^{n+1}_+)$ such that

$$e^{s\omega(\xi)}(\mathcal{F}_w u_n)(\xi) \to f(\xi) \in L^p(\mathbb{R}^{n+1}_+) \; as \; n \to \infty. \tag{4.2.2}$$

Let

$$g(\xi) = e^{-s\omega(\xi)} f(\xi), \qquad s > 0. \tag{4.2.3}$$

Then $f \in S'_\omega(\mathbb{R}^{n+1}_+)$ and $g(\xi) = e^{-s\omega(\xi)} f(\xi)$ are in $S'_\omega(\mathbb{R}^{n+1}_+)$. Therefore, the inverse Weinstein transform of the above function exists. Using (4.2.3) we have

$$e^{s\omega(\xi)} g(\xi) = e^{s\omega(\xi)} \mathcal{F}_w(\mathcal{F}_w^{-1} g)(\xi) \in L^p(\mathbb{R}^{n+1}_+).$$

In view of (4.2.1), we get

$$\mathcal{F}_w^{-1} g \in G^{p,\,s}_\omega.$$

Let $\mathcal{F}_w^{-1} g = u$. Then

$$e^{s\omega(\xi)}(\mathcal{F}_w u)(\xi) = e^{s\omega(\xi)}(\mathcal{F}_w(\mathcal{F}_w^{-1} g))(\xi) = e^{s\omega(\xi)} g(\xi) = e^{s\omega(\xi)} e^{-s\omega(\xi)} f(\xi) = f(\xi).$$

This implies

$$e^{s\omega(\xi)}(\mathcal{F}_w u)(\xi) = f(\xi). \tag{4.2.4}$$

Using (4.2.2) and (4.2.4) we get $u_n \to u$ in $G^{p,\,s}_\omega$.

This shows that $G^{p,\,s}_\omega$ is complete.

Theorem 4.2.4. *For $t > r$, $G^{p,\,t}_\omega \subseteq G^{p,\,r}_\omega$, the inclusion map is continuous.*

Proof: Let $u \in G^{p,\,t}_{\omega}$ and $t > r$ then we take $t = r + \epsilon$, for some $\epsilon > 0$. Then

$$\begin{aligned}
\|u\|_{G^{p,\,t}_{\omega}} &= \|e^{t\omega(\xi)}(\mathcal{F}_w u)(\xi)\|_p = \left(\int_{\mathbb{R}^{n+1}_+} \mid e^{t\omega(\xi)}(\mathcal{F}_w u)(\xi) \mid^p \, d\mu_\beta(\xi)\right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{R}^{n+1}_+} \mid e^{(r+\epsilon)\omega(\xi)}(\mathcal{F}_w u)(\xi) \mid^p \, d\mu_\beta(\xi)\right)^{\frac{1}{p}} \\
&\geq \left(\int_{\mathbb{R}^{n+1}_+} \mid e^{r\omega(\xi)}(\mathcal{F}_w u)(\xi) \mid^p \, d\mu_\beta(\xi)\right)^{\frac{1}{p}} = \|e^{r\omega(\xi)}(\mathcal{F}_w u)(\xi)\|_p = \|u\|_{G^{p,\,r}_{\omega}}.
\end{aligned}$$

Therefore,

$$\|u\|_{G^{p,\,r}_{\omega}} \leq \|u\|_{G^{p,\,t}_{\omega}}.$$

Hence

$$G^{p,\,t}_{\omega} \subseteq G^{p,\,r}_{\omega}.$$

Now let $\{u_n\} \to 0$ as $n \to \infty$ in $G^{p,\,t}_{\omega}$ then

$$\|u_n\|_{G^{p,\,t}_{\omega}} = \|e^{t\omega(\xi)}(\mathcal{F}_w u_n)(\xi)\|_p \to 0.$$

This implies that

$$\|u_n\|_{G^{p,\,r}_{\omega}} = \|e^{r\omega(\xi)}(F_w u_n)(\xi)\|_p \to 0 \; as \; n \to \infty.$$

This shows that $\{u_n\} \to 0$ in $G^{p,\,r}_{\omega}$ as $n \to \infty$.

This proves the continuity of the inclusion.

Theorem 4.2.5. *The space* $S_\omega(\mathbb{R}^{n+1}_+)$ *is dense in* $G^{p,\,s}_{\omega}$.

Proof: Let $u \in G^{p,\,s}_{\omega}$. Then $e^{s\omega(\xi)}(\mathcal{F}_w u)(\xi) \in L^p(\mathbb{R}^{n+1}_+)$. Since space $D_\omega(\mathbb{R}^{n+1}_+)$ is dense in $L^p(\mathbb{R}^{n+1}_+)$, then there exists a sequence of functions $\{\phi_n\}$ belonging to

$D_\omega(\mathbb{R}^{n+1}_+)$ such that

$$\phi_n(\xi) \to e^{s\omega(\xi)}(F_w u)(\xi) \ in \ L^p(\mathbb{R}^{n+1}_+). \tag{4.2.5}$$

Now, define

$$\psi_n(x) = \mathcal{F}_w^{-1}(e^{-s\omega(\xi)}\phi_n)(x). \tag{4.2.6}$$

Since $\phi_n \in D_\omega(\mathbb{R}^{n+1}_+)$, therefore $(e^{-s\omega(\xi)}\phi_n) \in D_\omega(\mathbb{R}^{n+1}_+)$, as $D_\omega(\mathbb{R}^{n+1}_+)$ is subspace of $S_\omega(\mathbb{R}^{n+1}_+), (e^{-s\omega(\xi)}\phi_n)$ also in $S_\omega(\mathbb{R}^{n+1}_+)$. Hence, its inverse Weinstein transform exists, so that

$$\psi_n(x) = \mathcal{F}_w^{-1}(e^{-s\omega(\xi)}\phi_n)(x) \in H_\omega(\mathbb{R}^{n+1}_+).$$

From (4.2.1) we can get

$$\begin{aligned} \|\psi_n - u\|_{G^{p,\,s}_\omega} &= \|e^{s\omega(\xi)}\big(\mathcal{F}_w(\psi_n - u)(\xi)\big)\|_p \\ &= \|e^{s\omega(\xi)}(\mathcal{F}_w\psi_n)(\xi) - e^{s\omega(\xi)}(\mathcal{F}_w u)(\xi)\|_p. \end{aligned}$$

In view of (4.2.6), we have

$$\|\psi_n - u\|_{G^{p,\,s}_\omega} = \|\phi_n - e^{s\omega(\xi)}(\mathcal{F}_w u)(\xi)\|_p.$$

Using (4.2.5), we find

$$\|\psi_n - u\|_{G^{p,\,s}_\omega} = \|\phi_n - e^{s\omega(\xi)}(\mathcal{F}_w u)(\xi)\|_p \to 0 \ as \ n \to \infty.$$

Hence,

$$\psi_n \to u \ in \ G^{p,\,s}_\omega.$$

This shows that the space $S_\omega(\mathbb{R}^{n+1}_+)$ is dense in $G^{p,\,s}_\omega$.

Lemma 4.2.6. *Let s_1, s and s_2 be three real numbers with $s_1 < s < s_2$ then for every $\epsilon > 0$ there exists $C(\epsilon) > 0$ such that*

$$e^{s\omega(\xi)} \leq \epsilon e^{s_2\omega(\xi)} + C(\epsilon)e^{s_1\omega(\xi)}. \tag{4.2.7}$$

Proof: The proof of above Lemma can be obtained from [45].

Theorem 4.2.7. *Let s_1, s and s_2 be three real numbers with $s_1 < s < s_2$ then for every $\epsilon > 0$ there exist $C(\epsilon) > 0$ such that*

$$\|u\|_{G^{p,\,s}_{\omega}} \leq \epsilon\|u\|_{G^{p,\,s_2}_{\omega}} \leq C(\epsilon)\|u\|_{G^{p,\,s_1}_{\omega}} \qquad \forall u \in \|u\|_{G^{p,\,s_2}_{\omega}}$$

Proof: Let $u \in G^{p,\,s_2}_{\omega}$. Then by Theorem (4.2.4), we have

$$u \in G^{p,\,s}_{\omega} \text{ and } u \in G^{p,\,s_1}_{\omega}.$$

Then

$$\begin{aligned} \|u\|_{u\in G^{p,\,s}_{\omega}} &= \|e^{s\omega(\xi)}(\mathcal{F}_w u)(\xi)\|_p \\ &= \left(\int_{\mathbb{R}^{n+1}_+} \mid e^{s\omega(\xi)}(\mathcal{F}_w u)(\xi) \mid^p \ d\mu_\beta(\xi)\right)^{\frac{1}{p}}. \end{aligned}$$

By using (4.2.7), we get

$$\|u\|_{u\in G^{p,\,s}_{\omega}} \leq \left(\int_{\mathbb{R}^{n+1}_+} \mid \left(\epsilon e^{s_2\omega(\xi)} + C(\epsilon)e^{s_1\omega(\xi)}\right)(\mathcal{F}_w u)(\xi) \mid^p \ d\mu_\beta(\xi)\right)^{\frac{1}{p}}$$

.

So that we find that

$$\begin{aligned}
&\|u\|_{u\in G^{p,\,s}_{\omega}}\\
&\qquad\leq \left(\int_{\mathbb{R}^{n+1}_{+}} \mid \epsilon e^{s_2\omega(\xi)}(\mathcal{F}_w u)(\xi) \mid^p \ d\mu_\beta(\xi)\right)^{\frac{1}{p}}\\
&\qquad\quad + \left(\int_{\mathbb{R}^{n+1}_{+}} \mid C(\epsilon) e^{s_1\omega(\xi)}\mathcal{F}_w u(\xi) \mid^p \ d\mu_\beta(\xi)\right)^{\frac{1}{p}}\\
&\qquad\leq \epsilon\left(\int_{\mathbb{R}^{n+1}_{+}} \mid e^{s_2\omega(\xi)}(\mathcal{F}_w u)(\xi) \mid^p \ d\mu_\beta(\xi)\right)^{\frac{1}{p}}\\
&\qquad\quad + C(\epsilon)\left(\int_{\mathbb{R}^{n+1}_{+}} \mid e^{s_1\omega(\xi)}\mathcal{F}_w u(\xi) \mid^p \ d\mu_\beta(\xi)\right)^{\frac{1}{p}}\\
&\qquad= \epsilon\|e^{s_2\omega(\xi)}(\mathcal{F}_w u)(\xi)\|_p + C(\epsilon)\|e^{s_1\omega(\xi)}(\mathcal{F}_w u)(\xi)\|_p\\
&\qquad= \epsilon\|u\|_{u\in G^{p,\,s_2}_{\omega}} + C(\epsilon)\|u\|_{u\in G^{p,\,s_1}_{\omega}}.
\end{aligned}$$

This shows that

$$\|u\|_{G^{p,\,s}_{\omega}} \leq \epsilon\|u\|_{G^{p,\,s_2}_{\omega}} \leq C(\epsilon)\|u\|_{G^{p,\,s_1}_{\omega}}.$$

This proves the theorem.

Theorem 4.2.8. *Let* $s<0$*, then* $L^2(\mathbb{R}^{n+1}_{+}) \subset G^{2,\,s}_{\omega}$.

Proof: Let $\phi \in L^2(\mathbb{R}^{n+1}_{+})$. Then from (1.4.3), we find

$$\int_{\mathbb{R}^{n+1}_{+}} |(\mathcal{F}_w\phi)(\xi)|^2\, d\mu_\beta(\xi) = \int_{\mathbb{R}^{n+1}_{+}} |\phi(x)|^2\, d\mu_\beta(x).$$

Therefore, above implies

$$\left(\int_{\mathbb{R}^{n+1}_{+}} |(\mathcal{F}_w\phi)(\xi)|^2\, d\mu_\beta(\xi)\right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}^{n+1}_{+}} |\phi(x)|^2\, d\mu_\beta(x)\right)^{\frac{1}{2}}.$$

Therefore, we have

$$\|\mathcal{F}_w\phi\|_{L^2(\mathbb{R}^{n+1}_{+})} = \|\phi\|_{L^2(\mathbb{R}^{n+1}_{+})}. \tag{4.2.8}$$

Hence for $s < 0$, we have

$$\|\phi\|_{G^{2,s}_\omega} = \|e^{s\omega(\xi)}(\mathcal{F}_w\phi)(\xi)\|_{L^2(\mathbb{R}^{n+1}_+)} = \left(\int_{\mathbb{R}^{n+1}_+} \mid e^{s\omega(\xi)}(\mathcal{F}_w\phi)(\xi) \mid^2 \, d\mu_\beta(\xi)\right)^{\frac{1}{2}}$$
$$\leq \left(\int_{\mathbb{R}^{n+1}_+} \mid (\mathcal{F}_w\phi)(\xi) \mid^2 \, d\mu_\beta(\xi)\right)^{\frac{1}{2}} = \|(\mathcal{F}_w\phi)(\xi)\|_{L^2(\mathbb{R}^{n+1}_+)}.$$

By using (4.2.8), we get

$$\|\phi\|_{G^{2,s}_\omega} \leq \mid (\mathcal{F}_w\phi)(\xi)\|_{L^2(\mathbb{R}^{n+1}_+)}.$$

Therefore

$$\|\phi\|_{G^{2,s}_\omega} < \infty.$$

Hence, above implies

$$\phi \in G^{2,s}_\omega.$$

This shows that $L^2(\mathbb{R}^{n+1}_+) \subset G^{2,s}_\omega$.

4.3 The generalized Hankel potential associated with the Weinstein transform

In this section, we discussed various properties of the Hankel potential associated with the Weinstein transform.

Let $a(x) \neq 0$ be a multiplier in $S_\omega(\mathbb{R}^{n+1}_+)$ such that

$$\mathcal{F}_w^{-1}(a^k)(x) \in L^1(\mathbb{R}^{n+1}_+) \;\; \forall \;\; k = 0, 1, 2, 3 \; ..... \tag{4.3.1}$$

Then, for $f \in S'_\omega(\mathbb{R}^{n+1}_+)$, the products fa^k and $fa^{-k} \in S'_\omega(\mathbb{R}^{n+1}_+)$ are defined by the following relations.

$$\langle f\ a^k,\ \phi\rangle = \langle f,\ a^k\ \phi\rangle \quad \forall\ \ \phi \in S_\omega(\mathbb{R}^{n+1}_+) \tag{4.3.2}$$

and

$$\langle f\ a^{-k},\ a^k\ \phi\rangle = \langle f,\ \phi\rangle \quad \forall\ \ \phi \in S_\omega(\mathbb{R}^{n+1}_+). \tag{4.3.3}$$

Now, the generalized Hankel potential associated with the Weinstein transform is

$$\mathcal{H}^k u = \mathcal{F}_w^{-1}(a^{-k}(\mathcal{F}_w u)) \qquad \forall\ \ u \in\ S'_\omega(\mathbb{R}^{n+1}_+). \tag{4.3.4}$$

Since the generalized Weinstein transformation $\mathcal{F}'_w$ corresponds to a continuous linear mapping between $S'_\omega(\mathbb{R}^{n+1}_+)$ onto $S'_\omega(\mathbb{R}^{n+1}_+)$ and inverse of the Weinstein transform holds for $(\mathcal{F}'_w)^{-1}$ as well . Therefore we conclude that the generalized Hankel potential associated with the Weinstein transform (4.3.5) is a continuous linear mapping of $S'_\omega(\mathbb{R}^{n+1}_+)$ onto $S'_\omega(\mathbb{R}^{n+1}_+)$. From (1.4.1) and (1.4.6) we have

$$(\mathcal{H}^k u)(x) = \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1}) a^{-k}(\xi)(\mathcal{F}_w u)(\xi)\, d\mu_\beta(\xi). \tag{4.3.5}$$

From [84, p.10], the generalized Hankel potential associated with the Weinstein transform is a pseudo differential operator with symbol $a^{-k}(\xi)$

Theorem 4.3.1. *Let $u \in S'_\omega(\mathbb{R}^{n+1}_+)$. Then for any $m, l \in \mathbb{Z}$ we have*

(i) $\quad \mathcal{H}^l \mathcal{H}^m u = \mathcal{H}^{l+m} u.$

(ii) $\quad \mathcal{H}^0 u = u.$

Proof:(i): Let $u \in S'_\omega(\mathbb{R}^{n+1}_+)$ then from (4.3.4), we have

$$(\mathcal{H}^l\mathcal{H}^m u)(x) = \mathcal{F}_w^{-1}\Big(a^{-l}(\mathcal{F}_w(\mathcal{H}^m u)\Big)(x).$$

Using (1.4.1), we get

$$\begin{aligned}(\mathcal{H}^l\mathcal{H}^m u)(x) &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1})(a^{-l}\mathcal{F}_w(\mathcal{H}^m u))(\xi)\, d\mu_\beta(\xi)\\ &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1})a^{-l}(\xi)\mathcal{F}_w(\mathcal{H}^m u)(\xi)\, d\mu_\beta(\xi).\end{aligned}$$

In view of (4.3.4), we have

$$(\mathcal{H}^m u)(\xi) = (\mathcal{F}_w^{-1}(a^{-m}(\mathcal{F}_w u)))(\xi).$$

Then, by the Weinstein transform we find

$$\mathcal{F}_w(\mathcal{H}^m u)(\xi) = a^{-m}(\xi)((\mathcal{F}_w u))(\xi). \tag{4.3.6}$$

Using (4.3.6), we find

$$\begin{aligned}(\mathcal{H}^l\mathcal{H}^m u)(x) &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1})a^{-l}(\xi)a^{-m}(\xi)(\mathcal{F}_w u)(\xi)\, d\mu_\beta(\xi)\\ &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1})a^{-(l+m)}(\xi)(\mathcal{F}_w u)(\xi)\, d\mu_\beta(\xi)\\ &= \mathcal{F}_w^{-1}\Big(a^{-(l+m)}(\xi)\mathcal{F}_w u(\xi)\Big)(x)\\ &= (\mathcal{H}^{l+m}u)(x).\end{aligned}$$

Therefore,

$$\mathcal{H}^l\mathcal{H}^m u = \mathcal{H}^{l+m} u.$$

(ii): Now, we have

$$\begin{aligned}(\mathcal{H}^0 u)(x) =& \mathcal{F}_w^{-1}(a^0(\mathcal{F}_w u))(x)\\ =& \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1})(a^{-0}(\xi))(\mathcal{F}_w(u))(\xi)\, d\mu_\beta(\xi)\\ =& \mathcal{F}_w^{-1}(\mathcal{F}_w u)(x)\\ =& u(x).\end{aligned}$$

This shows that

$$\mathcal{H}^0 u = u.$$

The Space $W_\omega^{m,p}(\mathbb{R}_+^{n+1})$

In this section we introduce the L^p-space of all Hankel potential $\mathcal{H}^k$ which is represented by the $W_\omega^{m,p}(\mathbb{R}_+^{n+1})$ and exploiting the theory of Weinstein transform, many properties of this space will be discussed.

Definition 4.3.2. Let $m \in \mathbb{Z}$ and $1 < p < \infty$, the space $W_\omega^{m,p}(\mathbb{R}_+^{n+1})$ is the set of all those ultradistributions $u \in S_\omega'(\mathbb{R}_+^{n+1})$ for which

$$\mathcal{H}^{-m} u \in L^p(\mathbb{R}_+^{n+1}). \tag{4.3.7}$$

The norm of this space is defind by

$$||u||_{m,p,\omega} = ||\mathcal{H}^{-m}u||_p = \left(\int_{\mathbb{R}_+^{n+1}} |(\mathcal{H}^{-m}u)(\xi)|^p \, d\mu_\beta(\xi)\right)^{\frac{1}{p}}. \tag{4.3.8}$$

In the next theorem we are going to prove that the space $W_{\omega}^{m,p}(\mathbb{R}_{+}^{n+1})$ is a Banach space.

Theorem 4.3.3. *The space* $W_{\omega}^{m,p}(\mathbb{R}_{+}^{n+1})$ *is a Banach space with respect to the norm* $|| \, ||_{m,p,\omega}$.

Proof: Let $\{u_k\}$ be a Cauchy sequence in $W_{\omega}^{m,p}(\mathbb{R}_{+}^{n+1})$. Then $\{\mathcal{H}^{-m}u_k\}$ is a Cauchy sequence in $L^p(\mathbb{R}_{+}^{n+1})$. Since $L^p(\mathbb{R}_{+}^{n+1})$ is complete, therefore there exists a function $u \in L^p(\mathbb{R}_{+}^{n+1})$ such that

$$\mathcal{H}^{-m}u_k \to u \in L^p(\mathbb{R}_{+}^{n+1}) \text{ as } k \to \infty. \tag{4.3.9}$$

Take $v = \mathcal{H}^m u$. Then from (4.3.5), we have

$$\begin{aligned} v(x) =& \int_{\mathbb{R}_{+}^{n+1}} e^{i\langle x',\xi'\rangle} J_{\beta}(x_{n+1}\xi_{n+1}) a^{-m}(\xi)(\mathcal{F}_w u)(\xi)\, d\mu_{\beta}(\xi) \\ =& \mathcal{F}_w^{-1}(a^{-m}(\mathcal{F}_w u))(x). \end{aligned}$$

Therefore, by the property of the Weinstein transform (1.4.6), we get

$$u(x) = \mathcal{F}_w^{-1}(a^m(\mathcal{F}_w v))(x) = (\mathcal{H}^{-m}v)(x) \in L^p(\mathbb{R}_{+}^{n+1}). \tag{4.3.10}$$

Hence, we find

$$v \in W_p^{m,p}.$$

From (4.3.9) and (4.3.10), it follows that

$$u_k \to v \in W_p^{m,p} \quad as \quad k \to \infty.$$

This proves that the space $W_{\omega}^{m,p}(\mathbb{R}_{+}^{n+1})$ is a Banach space.

Theorem 4.3.4. *The generalized Hankel potential* $\mathcal{H}^t$ *is an isometry of* $W_\omega^{m,p}(\mathbb{R}_+^{n+1})$ *onto* $W_\omega^{m+t,p}(\mathbb{R}_+^{n+1})$*; and we have*

$$||\mathcal{H}^t\phi||_{m+t,p,\omega} = ||\phi||_{m,p,\omega}, \quad \forall\phi \in W_\omega^{m,p}(\mathbb{R}_+^{n+1}). \tag{4.3.11}$$

Proof: Let $\phi \in W_\omega^{m,p}$, then from (4.3.8) we have

$$||\mathcal{H}^t\phi||_{m+t,p,\omega} = ||\mathcal{H}^{-(m+t)}(\mathcal{H}^t\phi)||_p. \tag{4.3.12}$$

Now, we have

$$(\mathcal{H}^{-(m+t)}(\mathcal{H}^t\phi))(x) = \mathcal{F}_w^{-1}(a^{m+t}(x)\mathcal{F}_w(\mathcal{H}^t\phi))(x)$$

By inversion formula of the Weinstein transform (1.4.4), above expression yields

$$(\mathcal{H}^{-(m+t)}(\mathcal{H}^t\phi))(x) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1})a^{m+t}(\xi)\mathcal{F}_w(\mathcal{H}^t\phi)(\xi)\, d\mu_\beta(\xi).$$

Taking (4.3.6), we get

$$\begin{aligned}
&(\mathcal{H}^{-(m+t)}(\mathcal{H}^t\phi))(x)\\
&\quad= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1})a^{m+t}(\xi)a^{-t}(\xi)(\mathcal{F}_w\phi)(\xi)\, d\mu_\beta(\xi)\\
&\quad= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1})a^{m}(\xi)(\mathcal{F}_w\phi)(\xi)\, d\mu_\beta(\xi)\\
&\quad= (\mathcal{H}^{-m}\phi)(x).
\end{aligned}$$

Therefore, from above expression we find that

$$(\mathcal{H}^{-(m+t)}(\mathcal{H}^t\phi)) = (\mathcal{H}^{-m}\phi).$$

In veiw of (4.3.12), we have

$$\begin{aligned} ||\mathcal{H}^{t}\phi||_{m+t,p,\omega} &= ||(\mathcal{H}^{-m}\phi)||_{p} \\ &= ||\phi||_{m,p,\omega}. \end{aligned}$$

Let $\phi \in W_{\omega}^{m+t,p}(\mathbb{R}_{+}^{n+1})$. Then from (4.3.7), we obtain

$$\mathcal{H}^{-(m+t)}\phi \in L^{p}(\mathbb{R}_{+}^{n+1}).$$

Again, we get

$$\begin{aligned} &(\mathcal{H}^{-(m+t)}\phi)(x) \\ &\quad= \mathcal{F}_{w}^{-1}(a^{m+t}(\xi)(\mathcal{F}_{w}\phi)(\xi))(x) \\ &\quad= \int_{\mathbb{R}_{+}^{n+1}} e^{i\langle x',\xi'\rangle} J_{\beta}(x_{n+1}\xi_{n+1}) a^{m+t}(\xi)(\mathcal{F}_{w}\phi)(\xi)\, d\mu_{\beta}(\xi) \\ &\quad= \int_{\mathbb{R}_{+}^{n+1}} e^{i\langle x',\xi'\rangle} J_{\beta}(x_{n+1}\xi_{n+1}) a^{m}(\xi) a^{t}(\xi)(\mathcal{F}_{w}\phi)(\xi)\, d\mu_{\beta}(\xi). \end{aligned}$$

Then applying (4.3.6), we find

$$\begin{aligned} (\mathcal{H}^{-(m+t)}\phi)(x) &= \int_{\mathbb{R}_{+}^{n+1}} e^{i\langle x',\xi'\rangle} J_{\beta}(x_{n+1}\xi_{n+1}) a^{m}(\xi)\mathcal{F}_{w}(\mathcal{H}^{-t}\phi)(\xi)\, d\mu_{\beta}(\xi) \\ &= (\mathcal{H}^{-m}(\mathcal{H}^{-t}\phi))(x). \end{aligned}$$

This show that

$$\mathcal{H}^{-(m+t)}\phi = \mathcal{H}^{-m}(\mathcal{H}^{-t})\phi \in L^{p}(\mathbb{R}_{+}^{n+1}).$$

Hence,

$$(\mathcal{H}^{-t}\phi) \in W_{\omega}^{m,p}.$$

And, we have to obtain

$$\begin{aligned}
&\mathcal{H}^t(\mathcal{H}^{-t}\phi)(x)\\
&\quad= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1}) a^{-t}(\xi)\mathcal{F}_w(\mathcal{H}^{-t}\phi)(\xi)\, d\mu_\beta(\xi)\\
&\quad= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1}) a^{-t}(\xi) b^{-t}(\xi)\mathcal{F}_w\phi)(\xi)\, d\mu_\beta(\xi)\\
&\quad= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1})(\mathcal{F}_w\phi)(\xi)\, d\mu_\beta(\xi)\\
&\quad= \mathcal{F}^{-1}((\mathcal{F}_w\phi)(\xi))(x) = \phi(x).
\end{aligned}$$

Therefore, for each $\phi \in W^{m,p}_\omega$, there exists an $\mathcal{H}^{-t}\phi \in W^{m+t,p}_\omega$ such that

$$\mathcal{H}^t\mathcal{H}^{-t}\phi = \phi.$$

Hence, $\mathcal{H}^t$ is onto.

Theorem 4.3.5. *Let* $1 < p < \infty$ *and* $m = 0, 1, 2....$ *then,*

$$||\mathcal{H}^m\phi|| \leq C||\phi||_p \quad \forall\ \phi \in S_\omega(\mathbb{R}^{n+1}_+). \tag{4.3.13}$$

Proof: From (1.4.13), we have

$$\mathcal{F}_w(f\#_\beta\phi)(\xi) = (\mathcal{F}_w f)(\xi)(\mathcal{F}_w\phi)(\xi). \tag{4.3.14}$$

Take

$$a^{-m}(\xi) = \mathcal{F}_w(f(t))(\xi). \tag{4.3.15}$$

Then,

$$\begin{aligned}
f(t) &= \mathcal{F}_w^{-1}(a^{-m}(\xi))(t) = \mathcal{F}_w(a^{-m}(\xi))(-t)\\
&= \int_{\mathbb{R}_+^{n+1}} e^{i\langle t',\xi'\rangle} J_\beta(t_{n+1}\xi_{n+1}) a^{-m}(\xi)\, d\mu_\beta(\xi)\\
&= \mathcal{F}_w^{-1}(a^{-m})(t).
\end{aligned}$$

Utilising (4.3.1), it follows that

$$f(t) = \mathcal{F}_w^{-1}(a^{-m})(t) \in L^1(\mathbb{R}_+^{n+1}).$$

In view of (4.3.14), we have

$$\begin{aligned}
\mathcal{F}_w(f\#_\beta\phi)(\xi) &= \mathcal{F}_w(\mathcal{F}_w^{-1}(a^{-m}(t))(\xi)(\mathcal{F}_w\phi)(\xi)\\
&= a^{-m}(\xi)(\mathcal{F}_w\phi)(\xi).
\end{aligned}$$

Therefore,

$$\begin{aligned}
(f\#_\beta\phi)(t) &= \mathcal{F}_w^{-1}(a^{-m}(\xi)(\mathcal{F}_w\phi)(\xi))(t)\\
&= \mathcal{F}_w^{-1}(a^{-m}\mathcal{F}_w\phi)(t)\\
&= \mathcal{H}^m\phi.
\end{aligned}$$

Then

$$||\mathcal{H}^m\phi||_p = ||f\#_\beta\phi||_p.$$

Using (1.4.14), we obtain

$$||\mathcal{H}^m\phi||_p \leq ||f||_1||\phi||_p \leq C||\phi||_p.$$

This proves the theorem.

Theorem 4.3.6. *Let* $1 < p < \infty$ *and* $m \leq l$. *Then*

$$W_{\omega}^{l,p}(\mathbb{R}_{+}^{n+1}) \subseteq W_{\omega}^{m,p}(\mathbb{R}_{+}^{n+1}),$$

and

$$||\phi||_{m,p,\omega} \leq C||\phi||_{l,p,\omega}.$$

Proof: Let $\phi \in W_{\omega}^{l,p}(\mathbb{R}_{+}^{n+1})$. Then from (4.3.7), we have

$$\mathcal{H}^{-l}\phi \in L^{p}(\mathbb{R}_{+}^{n+1}).$$

Then

$$\begin{aligned}(\mathcal{H}^{-m}\phi)(x) &= \mathcal{F}_{w}^{-1}(a^{m}\mathcal{F}_{w}\phi)(x)\\ &= \int_{\mathbb{R}_{+}^{n+1}} e^{i\langle x',\xi'\rangle} J_{\beta}(x_{n+1}\xi_{n+1}) a^{m}(\xi)(\mathcal{F}_{w}\phi)(\xi)\,d\mu_{\beta}(\xi)\\ &= \int_{\mathbb{R}_{+}^{n+1}} e^{i\langle x',\xi'\rangle} J_{\beta}(x_{n+1}\xi_{n+1}) a^{-(l-m)}(\xi) a^{l}(\xi)(\mathcal{F}_{w}\phi)(\xi)\,d\mu_{\beta}(\xi).\end{aligned}$$

From (4.3.6), we obtain

$$(\mathcal{H}^{-m}\phi)(x) = \int_{\mathbb{R}_{+}^{n+1}} e^{i\langle x',\xi'\rangle} J_{\beta}(x_{n+1}\xi_{n+1}) a^{-(l-m)}(\xi)\mathcal{F}_{w}(\mathcal{H}^{l}\phi)(\xi)\,d\mu_{\beta}(\xi).$$

From (4.3.4), we have

$$(\mathcal{H}^{-m}\phi)(x) \qquad = \mathcal{H}^{l-m}(\mathcal{H}^{-l}\phi)(x).$$

Therefore, in view of (4.3.7), we have

$$||\phi||_{m,p,\omega} =||\mathcal{H}^{-m}\phi||_p = ||\mathcal{H}^{l-m}(\mathcal{H}^{-l}\phi)||_p.$$

Using (4.3.13), we get

$$||\phi||_{m,p,\omega} \leq C||\mathcal{H}^{-l}\phi||_p = ||\phi||_{l,p,\omega}.$$

Hence,

$$||\phi||_{m,p,\omega} \leq ||\phi||_{l,p,\omega}.$$

This proves the theorem.

Theorem 4.3.7. *Let $1 < p < \infty$ and for any symbol $a^m, m \in \mathbb{Z}$, define the pseudo differential operator*

$$(A(D)u)(x) = \mathcal{F}_w^{-1}(a^m \mathcal{F}_w u)(x).$$

Then

$$A(D) : W_\omega^{m,p}(\mathbb{R}_+^{n+1}) \longrightarrow W_\omega^{0,p}(\mathbb{R}_+^{n+1})$$

is a bounded linear operator.

Proof: Consider the following operators;

$$\mathcal{H}^{-l} : W_\omega^{l,p}(\mathbb{R}_+^{n+1}) \longrightarrow W_\omega^{0,p}(\mathbb{R}_+^{n+1}),$$

$$A(D)\mathcal{H}^{m} : W_\omega^{0,p}(\mathbb{R}_+^{n+1}) \longrightarrow W_\omega^{0,p}(\mathbb{R}_+^{n+1})$$

$$\mathcal{H}^{l-m} : W_\omega^{0,p}(\mathbb{R}_+^{n+1}) \longrightarrow W_\omega^{l-m,p}(\mathbb{R}_+^{n+1}).$$

We have to show that operator $\mathcal{H}^{-l}$ is bounded. Now we take $\phi \in W_{\omega}^{l,p}$, then

$$||\phi||_{l,p,\omega} = ||\mathcal{H}^{-l}\phi||_p < \infty, \tag{4.3.16}$$

and

$$||\mathcal{H}^{-l}\phi||_{0,p,\omega} = ||\mathcal{H}^0(\mathcal{H}^{-l}\phi)||_p. \tag{4.3.17}$$

Now,

$$(\mathcal{H}^0(\mathcal{H}^{-l}\phi))(x) = \mathcal{F}_w^{-1}(a^0\mathcal{F}_w(\mathcal{H}^{-l}\phi))(x).$$

By the inverse Weinstein transform (1.4.4), we get

$$\begin{aligned}(\mathcal{H}^0(\mathcal{H}^{-l}\phi))(x) &= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\ \xi'\rangle} J_\beta(x_{n+1}\xi_{n+1}) a^0(\xi)\mathcal{F}_w(\mathcal{H}^{-l}\phi)(\xi)\, d\mu_\beta(\xi)\\ &= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\ \xi'\rangle} J_\beta(x_{n+1}\xi_{n+1})\mathcal{F}_w(\mathcal{H}^{-l}\phi)(\xi)\, d\mu_\beta(\xi)\\ &= \mathcal{F}_w^{-1}(\mathcal{F}_w(\mathcal{H}^{-l}\phi))(x) = (\mathcal{H}^{-l}\phi)(x).\end{aligned}$$

Therefore,

$$\mathcal{H}^0(\mathcal{H}^{-l}\phi) = (\mathcal{H}^{-l}\phi). \tag{4.3.18}$$

In veiw of (4.3.17), we get

$$||\mathcal{H}^{-l}\phi||_{0,p,\omega} = ||(\mathcal{H}^{-l}\phi)||_p < \infty.$$

We have to show that the second operator is bounded. Let $\phi \in W_{\omega}^{0,p}$. Then

$$||\phi||_{0,p,\omega} = ||\mathcal{H}^0\phi||_p = ||\phi||_p < \infty, \tag{4.3.19}$$

and

$$||A(D)\mathcal{H}^m\phi||_{0,p,\omega} = ||A(D)\mathcal{H}^m\phi||_p. \tag{4.3.20}$$

Now, we have to calculate

$$\begin{aligned}(A(D)\mathcal{H}^m\phi)(x) &= \mathcal{F}_w^{-1}(a^m(\xi)\mathcal{F}_w(\mathcal{H}^m\phi))(x)\\ &= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1})a^m(\xi)\mathcal{F}_w(\mathcal{H}^m\phi)(\xi)\,d\mu_\beta(\xi).\end{aligned}$$

Therefore, using (4.3.6), we can obtain

$$\begin{aligned}(A(D)\mathcal{H}^m\phi)(x) &= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1})a^m(\xi)a^{-m}(\xi)(\mathcal{F}_w\phi)(\xi)\,d\mu_\beta(\xi)\\ &= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1})(\mathcal{F}_w\phi)(\xi)\,d\mu_\beta(\xi)\\ &= \mathcal{F}_w^{-1}(\mathcal{F}_w\phi)(x) = \phi(x).\end{aligned}$$

Using (4.3.19) and (4.3.20) we get

$$||A(D)\mathcal{H}^m\phi||_{0,p,\omega} = ||\phi||_p < \infty.$$

Third operator $\mathcal{H}^{l-m}$, is bounded by the following way:

$$||\mathcal{H}^{l-m}\phi||_{l-m,p,\omega} = ||\mathcal{H}^{-(l-m)}\mathcal{H}^{l-m}\phi||_p = ||\phi||_p < \infty.$$

Hence, the product $\mathcal{H}^{l-m}A(D)\mathcal{H}^{m-l}$ is a bounded linear operator from $W_\omega^{l,p}(\mathbb{R}_+^{n+1})$ into $W_\omega^{l-m,p}(\mathbb{R}_+^{n+1})$.

Theorem 4.3.8. *For $m \in \mathbb{Z}$ and symbol a^m, the pseudo differential operator*

$$A(D) = \mathcal{F}_w^{-1}a^m\mathcal{F}_w : W_\omega^{l,p}(\mathbb{R}_+^{n+1}) \longrightarrow W_\omega^{l-m,p}(\mathbb{R}_+^{n+1})$$

is a bounded linear operator.

Proof: Let $\phi \in W_{\omega}^{l,p}(\mathbb{R}_+^{n+1})$. Then

$$||\phi||_{l,p,\omega} = ||\mathcal{H}^{-l}\phi||_p < \infty,$$

and

$$||A(D)\phi||_{l-m,p,\omega} = ||\mathcal{H}^{m-l}(A(D)\phi)||_p < \infty. \tag{4.3.21}$$

Therefore, we have

$$\begin{aligned}
\mathcal{H}^{m-l}(A(D)\phi)(x) &= \mathcal{F}_w^{-1}(a^{-(m-l)}\mathcal{F}_w A(D)\phi)(x)\\
&= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x', \xi' \rangle} J_\beta(x_{n+1}\xi_{n+1}) a^{-(m-l)}(\xi)\mathcal{F}_w(A(D)\phi)(\xi)\, d\mu_\beta(\xi)\\
&= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x', \xi' \rangle} J_\beta(x_{n+1}\xi_{n+1}) a^{-(m-l)}(\xi)\mathcal{F}_w(\mathcal{F}_w^{-1} a^m \mathcal{F}_w\phi)(\xi)\, d\mu_\beta(\xi)\\
&= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x', \xi' \rangle} J_\beta(x_{n+1}\xi_{n+1}) a^{-(m-l)}(\xi) a^m(\xi)(\mathcal{F}_w\phi)(\xi)\, d\mu_\beta(\xi)\\
&= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x', \xi' \rangle} J_\beta(x_{n+1}\xi_{n+1}) a^{l}(\xi)(\mathcal{F}_w\phi)(\xi)\, d\mu_\beta(\xi)
\end{aligned}$$

By (1.4.6), we have

$$\begin{aligned}
\mathcal{H}^{m-l}(A(D)\phi)(x) &= \mathcal{F}_w^{-1}(a^l\mathcal{F}_w\phi)(x)\\
&= \mathcal{H}^{-l}\phi(x).
\end{aligned}$$

Therefore, we have

$$\mathcal{H}^{m-l}(A(D)\phi)(x) = \mathcal{H}^{-l}\phi(x). \tag{4.3.22}$$

Using (4.3.21) and (4.3.22) the above expression yeilds

$$||A(D)\phi||_{l-m,p,\omega} = ||\mathcal{H}^{-l}\phi)||_p < \infty.$$

Above shows that the mapping

$$A(D) : W_{\omega}^{l,p}(\mathbb{R}_{+}^{n+1}) \longrightarrow W_{\omega}^{l-m,p}(\mathbb{R}_{+}^{n+1})$$

is a bounded linear operator.

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