Chapter 5

Higher Order Tempered Fractional Sturm-Liouville Problem: Transformation in Integral Form and Numerical Approximation

This chapter deals with the higher order tempered fractional Sturm-Liouville problem. Firstly, the introduction of the problem is given in Section 5.1. In Section 5.2, we have stated the problem and converted it into integral form. The main numerical scheme is presented in Section 5.3. In the proposed numerical scheme, the approximations of left and right tempered fractional integrals are presented. Section 5.4 consists of numerical results. Finally, this chapter is concluded with summary and discussion.

5.1 Introduction

In addition to the traditional fractional calculus, the variable order and tempered fractional operators were introduced recently. Furthermore, the definition of fractional integration with weak singular and exponential kernel was introduced in [76]. The applications of tempered fractional calculus (TFC) were used to study the phenomena of random walks, brownian motion etc. [77, 78].

In comparison with other types of fractional differential equations (FDEs), the TFDE drew less attention and numerical schemes for solving them are still under development. Often it is difficult to derive analytical solutions for TFDEs due to the nonlocal property of the tempered fractional operator. The most commonly used algorithms for solving TFDEs are the discrete implicit scheme [79], Petrov–Galerkin spectral [21], Chebyshev pseudo-spectral [80] and finite difference scheme [81] and the predictor-corrector methods using the trapezoidal method [82].

In recent studies, it is observed that the TFDs model many applications in different fields like heterogeneous systems [20], stochastic process [77] etc. For a detailed study on TFDs, we refer the readers to [78]. Now, we define a higher order tempered fractional Sturm-Liouville problem.

5.2 Statement of the Problem

In this chapter, we consider a tempered fractional Sturm-Liouville problem (TFSLP) of the following form:

$$ {}^{c}_{x}D^{\alpha,\tau}_{b}\ {}_{a}D^{\alpha,\tau}_{x} y(x) + q(x)y(x) = \lambda r(x)y(x), \tag{5.1} $$

where $N-1 < \alpha < N,\ \ y(x) \in AC^N[a,b]$ is an unknown function satisfying boundary conditions

$$y^k(x)|_{x=a} = 0, \quad {}_aD_x^{\alpha+k,\tau} y(x)|_{x=b} = C_k \text{ for } \ k = 0, 1, 2, ..., N-1. \tag{5.2}$$

5.2.1 Transformation of TFSLP into the Integral Form

Theorem 5.1. *The TFSLP given by Eqs. (5.1)-(5.2) is equivalent to the integral equation of the form*

$$y(x) - \sum_{k=0}^{N-1} \frac{(-1)^k}{k!}\ {}_aD_x^{\alpha+k,\tau} y(x)|_{x=b}\ e^{-\tau(x+b)} \sum_{j=0}^{k} (-1)^j b^{k-j}\Gamma(j+1) x^{\alpha+j} E_{1,\alpha+j+1}^{j+1}(2\tau x)$$
$$= \ {}_aI_x^{\alpha,\tau}\ {}_xI_b^{\alpha,\tau}(-q(x) + \lambda r(x))y(x). \tag{5.3}$$

Proof. Since,

$${}_x^cD_b^{\alpha,\tau}\ {}_aD_x^{\alpha,\tau} y(x) = (-q(x) + \lambda r(x))y(x). \tag{5.4}$$

Now, applying ${}_xI_b^{\alpha,\tau}$ on both sides of Eq. (5.4), we have

$${}_xI_b^{\alpha,\tau}\ {}_x^cD_b^{\alpha,\tau}\ {}_aD_x^{\alpha,\tau} y(x) = \ {}_xI_b^{\alpha,\tau}(-q(x) + \lambda r(x))y(x),$$
$${}_aD_x^{\alpha,\tau} y(x) - \sum_{k=0}^{N-1} e^{\tau(x-b)} \frac{(-1)^k}{k!}(b-x)^k\ {}_aD_x^{\alpha+k,\tau} y(x)|_{x=b} = \ {}_xI_b^{\alpha,\tau}(-q(x) + \lambda r(x))y(x). \tag{5.5}$$

Again, operating ${}_aI_x^{\alpha,\tau}$ both sides of Eq. (5.5) and using composition rule ${}_aI_x^{\alpha,\tau}\ {}_a^cD_x^{\alpha,\tau}$ with $y^k(0) = 0, k = 0, 1, 2, ..., N-1$ then we obtain following form of differential equation

$$y(x) - \sum_{k=0}^{N-1} \frac{(-1)^k}{k!} \; {}_aD_x^{\alpha+k,\tau} y(x)|_{x=b} \; {}_aI_x^{\alpha,\tau}(e^{\tau(x-b)}(b-x)^k) = \; {}_aI_x^{\alpha,\tau} \; {}_xI_b^{\alpha,\tau}(-q(x) + \lambda r(x))y(x),$$

$$y(x) - \sum_{k=0}^{N-1} \frac{(-1)^k}{k!} \; {}_aD_x^{\alpha+k,\tau} y(x)|_{x=b} \; e^{(-\tau(x+b))} \sum_{j=0}^{k} \sum_{i=0}^{\infty} \frac{(-1)^j (2\tau)^i}{\Gamma(i+1)} b^{k-j} \binom{k}{j} \frac{\Gamma(i+j+1)}{\Gamma(i+j+\alpha+1)} x^{i+j+\alpha}$$
$$= \; {}_aI_x^{\alpha,\tau} \; {}_xI_b^{\alpha,\tau}(-q(x) + \lambda r(x))y(x),$$
$$y(x) - \sum_{k=0}^{N-1} \frac{(-1)^k}{k!} \; {}_aD_x^{\alpha+k,\tau} y(x)|_{x=b} \; e^{-\tau(x+b)} \sum_{j=0}^{k} (-1)^j b^{k-j} \binom{k}{j} \Gamma(j+1) x^{\alpha+j} E_{1,\alpha+j+1}^{j+1}(2\tau x)$$
$$= \; {}_aI_x^{\alpha,\tau} \; {}_xI_b^{\alpha,\tau}(-q(x) + \lambda r(x))y(x). \tag{5.6}$$

□

Now, we obtain the form of Eq. (5.6) for different range of α as follows:

1. Case for $\alpha \in (0, 1]$

For $\alpha \in (0, 1]$, Eq. (5.6) is simplified into the following form

$$y(x) + \; {}_aI_x^{\alpha,\tau} \; {}_xI_b^{\alpha,\tau}(q(x) - \lambda r(x))y(x) = C_0 e^{-\tau(x+b)} x^\alpha E_{1,\alpha+1}(2\tau x), \tag{5.7}$$

where C_0 is defined as in Eq. (5.2).

2. Case for $\alpha \in (1, 2]$

For $\alpha \in (1, 2]$, Eq. (5.6) is simplified into the following form

$$y(x) + \; {}_aI_x^{\alpha,\tau} \; {}_xI_b^{\alpha,\tau}(q(x) - \lambda r(x))y(x) = C_0 e^{-\tau(x+b)} x^\alpha E_{1,\alpha+1}(2\tau x) - C_1 e^{-\tau(x+b)} b x^\alpha$$
$$E_{1,\alpha+1}(2\tau x) + C_1 e^{-\tau(x+b)} x^{\alpha+1} E_{1,\alpha+2}^2(2\tau x). \tag{5.8}$$

where C_0 and C_1 are defined as in Eq. (5.2).

5.3 Numerical Scheme

Here, we present an approximate scheme to solve the integral forms of considered TFSLP. First we discretize the given interval $[a, b]$ into N number of subintervals, each of length $h = (b-a)/N$, using $N+1$ points, $a = x_0 < x_1 < x_2 < ... < x_N = b$. Then we approximate tempered integral in each subinterval using the following scheme.

At node x_0, we have ${}_aI_x^{\alpha,\tau}y(x)|_{x=x_0} = 0$. The discrete form of the left integral operator at nodes x_i for $i = 1, 2, .., N$ is approximated by the formula

$$\begin{aligned}
{}_aI_x^{\alpha,\tau}y(x)|_{x=x_k} &= \frac{e^{-\tau x_k}}{\Gamma(\alpha)}\int_0^{x_k}\frac{e^{\tau s}y(s)}{(x_k-s)^{1-\alpha}}ds \\
&= \frac{e^{-\tau x_k}}{\Gamma(\alpha)}\sum_{j=0}^{k-1}\int_{x_j}^{x_{j+1}}\frac{e^{\tau s}y(s)}{(x_k-s)^{1-\alpha}}ds \\
&\approx \frac{e^{-\tau x_k}}{\Gamma(\alpha)}\sum_{j=0}^{k-1}\int_{x_j}^{x_{j+1}}\frac{[(s-x_j)e^{\tau x_{j+1}}y_{j+1}-(s-x_{j+1})e^{\tau x_j}y_j]/h}{(x_k-s)^{1-\alpha}}ds \\
&= \sum_{j=0}^{k} l_{k,j}y_j,
\end{aligned} \tag{5.9}$$

where,

$$l_{k,j} = \frac{e^{-\tau x_k}}{\Gamma(\alpha+2)}\begin{cases} 0 & \text{for } k=0, \\ e^{\tau x_0}[(k-1)^{\alpha+1}-k^{\alpha+1}+k^{\alpha}(\alpha+1)] & \text{for } k>0,\ j=0, \\ e^{\tau x_j}[(k-j+1)^{\alpha+1}-2(k-j)^{\alpha+1} & \\ +(k-j-1)^{\alpha+1}] & \text{for } k>0,\ j=1,2,...,k-1, \\ e^{\tau x_k} & \text{for } k>0,\ j=k. \end{cases} \tag{5.10}$$

Similarly, we get the discrete form of the right tempered fractional integral operator as follows:

$$ {}_xI_b^{\alpha,\tau} y(x)|_{x=x_k} \approx \sum_{j=k}^{N} u_{k,j} y_j, \tag{5.11}$$

where,

$$u_{k,j} = \frac{e^{\tau x_k}}{\Gamma(\alpha+2)} \begin{cases} 0 & \text{for } k = N, \\ e^{-\tau x_m}[(m-k-1)^{\alpha+1} - (m-k)^{\alpha+1} & \\ +(m-k)^{\alpha}(\alpha+1)] & \text{for } k < m,\ j = N, \\ e^{-\tau x_j}[(j-k+1)^{\alpha+1} - 2(j-k)^{\alpha+1} & \\ +(j-k-1)^{\alpha+1}] & \text{for } k < N,\ k < j < N, \\ e^{-\tau x_k} & \text{for } k < N,\ j = k. \end{cases} \tag{5.12}$$

$$ {}_aI_x^{\alpha,\tau}\ {}_xI_b^{\alpha,\tau} y(x)|_{x=x_k} \approx \sum_{j=0}^{k} l_{k,j} \sum_{i=j}^{N} u_{j,i} y_i. \tag{5.13}$$

1. Case for $\alpha \in (0,1]$

The discrete form of Eq. (5.7) can be written in the form of the system of $N+1$ linear equations. For every node x_k, we write the following equation,

$$y_k + \sum_{j=0}^{k} l_{k,j} \sum_{i=j}^{N} (-\lambda w_i + q_i) u_{j,i} y_k = C_0 e^{-\tau(x_k+b)} x_k^{\alpha} E^1_{1,\alpha+1}(2\tau x_k). \tag{5.14}$$

2. Case for $\alpha \in (1, 2]$

Similarly, for this case, the discrete forms of Eq. (5.8) is given by

$$y_k + \sum_{j=0}^{k} l_{k,j} \sum_{i=j}^{N} (-\lambda w_i + q_i) u_{j,i} y_k = C_0 e^{-\tau(x_k+b)} x_k^{\alpha} E_{1,\alpha+1}(2\tau x_k)$$
$$- C_1 e^{-\tau(x_k+b)} b x_k^{\alpha} E_{1,\alpha+1}(2\tau x_k) + C_1 e^{-\tau(x_k+b)} x_k^{\alpha+1} E^2_{1,\alpha+2}(2\tau x_k). \quad (5.15)$$

5.4 Numerical Results and Discussion

In this section, we present the numerical results. Table 5.1 presents the numerical values of the function y at different node values and rate of convergence.

TABLE 5.1: Numerical values of y at nodes x_i and rate of convergence R_λ for fixed $\lambda = 10$, $\alpha = 0.3$ and $\tau = 0, 1$

		$x = 0.25$		$x = 0.5$		$x = 0.75$	
τ	$1/N$	y	R_λ	y	R_λ	y	R_λ
0	1/64	-6.0824		0.95044		5.2719	
	1/128	-1.2686	1.6149	-1.8454	1.6178	-0.6998	1.6190
	1/256	0.76061	1.6168	1.3842	1.6193	0.53279	1.6203
	1/512	0.46441	1.6160	0.9121	1.6181	0.44904	1.6189
	1/1024	0.41498		0.83481		0.4365	
1	1/64	-7.5526		1.2599		6.8061	
	1/128	-0.8752	1.5896	-1.0390	1.5903	-0.4461	1.6032
	1/256	2.1590	1.5738	3.8473	1.5994	1.2726	1.6083
	1/512	0.8759	1.5756	1.7559	1.5832	0.8424	1.5989
	1/1024	0.74193		1.5399		0.8008	

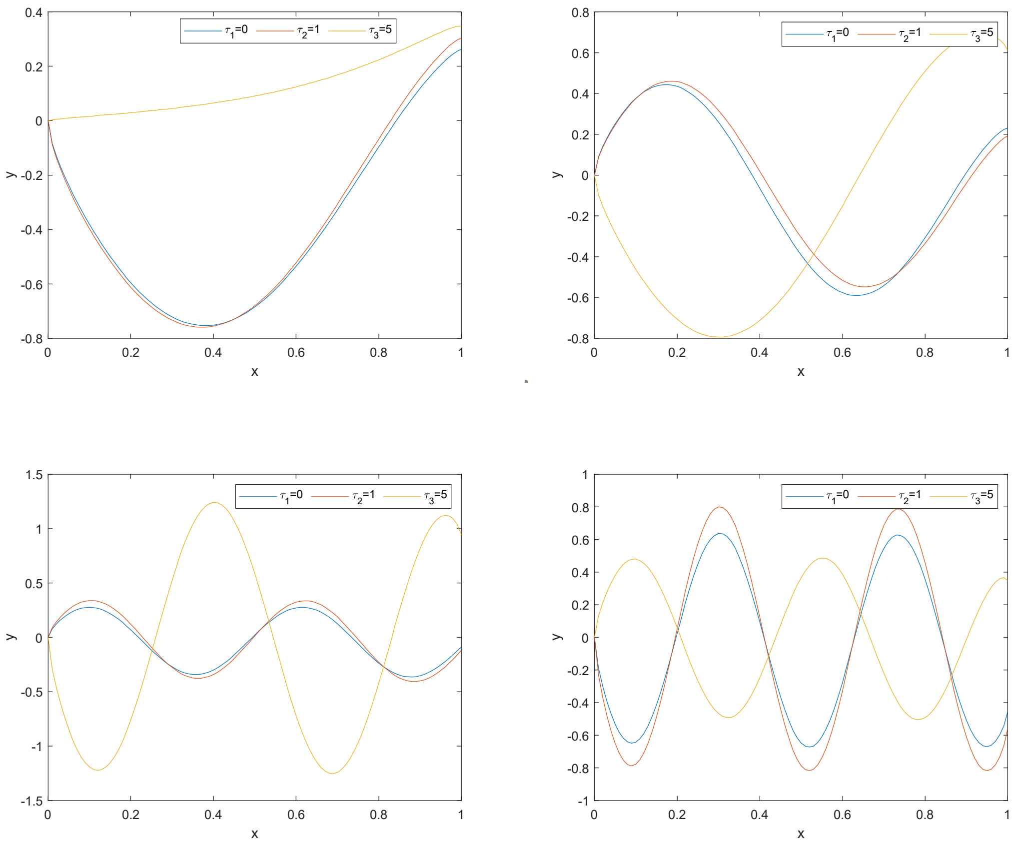

FIGURE 5.1: Numerical solution of Eq. (5.14) for $\alpha = 0.6$ and different values of λ and τ.

Figure 5.1 shows the numerical solution of Eq. (5.14) for $\alpha = 0.6$ and different values of the parameter $\lambda = \{5, 10, 15, 20\}$ and $\tau = \{0, 1, 5\}$.

5.5 Error Estimation of Approximation

Here, we discuss the error estimates of the numerical scheme presented in Section 5.3 as follows:

Theorem 5.2. *For any $0 < \alpha < 1$ and $y(x) \in C^2[0,1]$, the error bound is given by*

$$\|y(x) - y_{1,j}(x)\|_{L^2[0,1]} \le C_4 h^{2-\alpha} \max_{0 \le x \le x_j} |y''(x)|, \tag{5.16}$$

where C_4 is a constant and $y_{1,j}(x)$ is linear interpoltaing function of $y(x)$ on $[x_{j-1}, x_j]$ such that

$$|y(x) - y_{1,j}(x)| = \frac{y''(\xi_k)}{2!}(x - x_j)(x - x_{j-1}), \qquad \xi_k \in [x_{j-1}, x_j], \tag{5.17}$$

Proof. Here, first we estimate the difference $|{}_aD_x^{\alpha,\tau} y(x) - {}_aD_x^{\alpha,\tau} y_{1,j}(x)|$, such that

$$\begin{aligned} |{}_aD_x^{\alpha,\tau} y(x) - {}_aD_x^{\alpha,\tau} y_{1,j}(x)| &= |{}_aD_x^{\alpha,\tau}(y(x) - y_{1,j}(x))| \\ &= \left| \frac{e^{-\tau x}}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{e^{\tau s}(y(s) - y_{1,j}(s))}{(x-s)^\alpha} ds \right|. \end{aligned} \tag{5.18}$$

Let $\zeta(x) = \int_0^x \frac{e^{\tau s}(y(s) - y_{1,j}(s))}{(x-s)^\alpha} ds$ and now by approximating the first derivative, we get

$$\frac{d\zeta}{dx} = \frac{\zeta(x_j) - \zeta(x_{j-1})}{h}, \tag{5.19}$$

so that

$$\begin{aligned}
\zeta(x_j) &= \left|\frac{1}{\Gamma(1-\alpha)}\int_0^{x_j}\frac{e^{\tau s}(y(s)-y_{1,j}(s))}{(x_j-s)^\alpha}ds\right| \\
&= \left|\frac{1}{\Gamma(1-\alpha)}\sum_{k=1}^{j}\int_{t_{k-1}}^{x_k}\frac{e^{\tau s}(y(s)-y_{1,j}(s))}{(x_j-s)^\alpha}ds\right| \\
&\leq \left|\frac{1}{\Gamma(1-\alpha)}\sum_{k=1}^{j}\frac{h^2y''(\xi_k)}{8}\int_{x_{k-1}}^{x_k}\frac{e^{\tau s}}{(x_j-s)^\alpha}ds\right| \\
&\leq \left|\frac{1}{\Gamma(1-\alpha)}\frac{h^2y''(\xi_k)}{8}\sum_{k=1}^{j}\int_{x_{k-1}}^{x_k}\frac{e^{\tau s}}{(x_j-s)^\alpha}ds\right|. \qquad (5.20)
\end{aligned}$$

Using Eq. (11) of ref. [83]

$$\begin{aligned}
\zeta(x_j) &\leq \frac{h^2}{8\Gamma(1-\alpha)}\max_{0\leq x\leq x_j}|y''(x)|\frac{h^{1-\alpha}}{\Gamma(\alpha)}\|e^{\tau x}\|_{L^2} \\
&\leq \frac{Lh^{3-\alpha}}{8\Gamma(1-\alpha)\Gamma(\alpha)}\max_{0\leq x\leq x_j}|y''(x)| \\
&\leq C_1\max_{0\leq x\leq x_j}|y''(x)|h^{3-\alpha} \qquad (5.21)
\end{aligned}$$

where $\max\limits_{0\leq x\leq 1}|e^{\tau x}| < e^\tau < L$.

Similary, we can find $\zeta(x_{j-1})$

$$\zeta(x_{j-1}) \leq C_2\max_{0\leq x\leq x_j}|y''(x)|h^{3-\alpha}. \qquad (5.22)$$

Now, we have

$$\begin{aligned}
|{}_aD_x^{\alpha+k,\tau}y(x) - {}_aD_x^{\alpha+k,\tau}y_{1,j}(x)| &= \frac{e^{-\tau x}}{\Gamma(1-\alpha)}\frac{d\zeta}{dx} \\
&\leq C_3h^{2-\alpha}\max_{0\leq x\leq x_j}|y''(\xi_k)|. \qquad (5.23)
\end{aligned}$$

Now, using fractional Poincare-Friedrichs inequality [28]

$$
\begin{aligned}
|y(x) - y_{1,j}(x)| &\leq C\|{}_aD_x^{\alpha,\tau}y(x) - {}_aD_x^{\alpha,\tau}y_{1,j}(x)\|_{L^2[0,1]} \\
&= C\bigg(\int_0^1 |{}_aD_x^{\alpha,\tau}y(x) - {}_aD_x^{\alpha,\tau}y_{1,j}(x)|^2 dx\bigg)^{1/2} \\
&\leq C\ C_3 h^{2-\alpha} \max_{0\leq x\leq x_j} |y''(x)| \bigg(\int_0^1 1 dx\bigg)^2 \\
&\leq C_4 h^{2-\alpha} \max_{0\leq x\leq x_j} |y''(x)|. \qquad (5.24)
\end{aligned}
$$

□

5.6 Conclusions

In this chapter, a TFSLP with mixed boundary conditions given by Eqs. (5.1)-(5.2) was considered. Firstly, TFSLP was transformed into an integral form using the composition rule of integral and derivatives. After that, we have converted this integral equation into a system of linear equation by discretization of tempered integral using linear approximation. Further, the equation was solved for different order of derivatives $\alpha \in (0, 1]$ and $\alpha \in (1, 2]$ and different values of tempered factor τ. The obtained results are shown through Table 5.1 and graph of eigenfunctions are shown in Figure 5.1.

****