Study on Numerical Schemes for Solving Coupled Fractional Order Diffusion Equations

Thesis submitted in partial fulfillment

for the Award of Degree

DOCTOR OF PHILOSOPHY

by

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Mohd Kashif

CERTIFICATE

It is certified that the work contained in this thesis titled "***Study on Numerical Schemes for Solving Coupled Fractional Order Diffusion Equations***" by ***Mohd Kashif*** has been carried out under our joint supervision and co-supervision, and this work has not been submitted elsewhere for a degree.

It is further certified that the student has fulfilled all the requirements of Comprehensive Examination, Candidacy and SOTA for the award of Ph.D. degree.

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DECLARATION BY THE CANDIDATE

I, ***Mohd Kashif***, certify that the work embodied in this thesis is my own bonafide work and carried out by me under the supervision of ***Prof. Tanmoy Som*** and the co-supervision of ***Prof. Subir Das*** from ***January, 2019*** to ***December, 2023*** at the ***Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi***. The matter embodied in this thesis has not been submitted for the award of any other degree/diploma. I declare that I have faithfully acknowledged and given credits to the research workers wherever their works have been cited in my work in this thesis. I further declare that I have not willfully copied any other's work, paragraphs, text, data, results, *etc.*, reported in journals, books, magazines, reports dissertations, theses, *etc.*, or available at websites and have not included them in this thesis and have not cited as my own work.

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It is certified that the above statement made by the student is correct to the best of my/our knowledge.

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PREFACE

There are six chapters in this thesis, in which the numerical study of the coupled fractional order diffusion equation has been done. Chapter 1 contains the introductory section of the thesis, which provides an overview of the history of fractional calculus with evolution. The definition of different types of fractional order derivatives, like Caputo and Riemann-Liouville derivatives with constant and variable order and their properties are given, which will be used throughout the thesis. In the last, the background, derivation of the fractional diffusion equation and list of numerical methods dealing with it are incorporated in this chapter.

In chapter 2, a non-standard finite difference collocation method is developed by using the Fibonacci polynomial to solve the coupled fractional order Burgers' equation. To show the efficiency of the method, compared the obtained numerical results with the existing results through error analysis. Based on the tabular presentation of the results, it is shown that the proposed method is performs much better as compared to the existing methods. After validation, the method is used for solving a nonlinear fractional order coupled Burgers' equation and simulate the results for different fractional order spatial derivative for different values of the parameters.

In chapter 3, a nonlinear coupled system of variable order reaction-diffusion equations with given initial and boundary conditions has been studied. The Bernstein operational matrix with collocation method has been used for solving this coupled system. A few examples are presented to demonstrate the accuracy and stability of the scheme by comparing L_2 and L_∞ norm errors between the obtained numerical and exact solutions. In the end, the graphical exhibitions of the effects of variable order derivatives on the solutions of the considered nonlinear coupled reaction-diffusion equation for different particular cases have been given.

Chapter 4 introduces the fractional variable order Gray-Scott model by using the notion of variable order fractional derivative in the Caputo sense. An efficient numerical method based on the Vieta–Lucas polynomial and the spectral collocation method for solving this model has been designed. The convergence analysis of the approximation has been presented and shown that a high order of convergence can be achieved despite a smaller number of approximations. A few numerical results are presented in order to verify the reliability and accuracy of the demonstrated scheme. The results of absolute errors for the considered Gray-Scott model with its exact solution show that the technique is very suitable for finding the solutions to the said kind of complex physical problem.

Chapter 5 presents an approximate numerical method for solving time fractional advection-diffusion-reaction predator prey equations of variable order. Results for Ulam-Hyers stability are shown, as well as the existence and uniqueness of solutions. It is suggested to use a numerical approximation based on the shifted second kind of airfoil polynomials to solve the equations under consideration. A fractional derivative operational matrix with variable order is derived for shifted airfoil polynomials, which will be used to compute the unknown function. With the help of a few examples, the accuracy and effectiveness of the presented method are verified.