

**STUDY OF STABILITY AND SYNCHRONIZATION OF  
SOME PROBLEMS RELATED TO NEURAL  
NETWORKS**



*The thesis submitted in partial fulfilment  
for the Award of Degree  
DOCTOR OF PHILOSOPHY*

*by*

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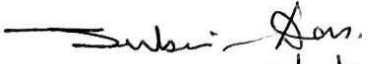
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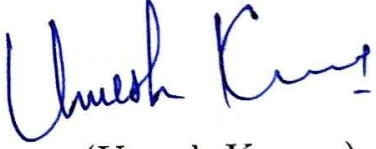
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*DEDICATED*

*To  
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---

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