Chapter 3

Lagrange's Operational Approach for the Approximate Solution of Two-dimensional Hyperbolic Telegraph Equation Subject to Dirichlet Boundary Conditions

3.1 Introduction

Most of the physical problems can be described in the form of mathematical models and these models consist of partial differential equations (PDEs). PDEs are observed in many fields of applied sciences and engineering. Among these partial differential equations, hyperbolic PDEs play an important role in several areas of applied sciences. The propagation of signal (digital and analog) through media, the propagation of electromagnetic waves in the earth–ionosphere waveguide [101], mechanical wave [102], ecological and cosmological phenomena are modeled using hyperbolic PDEs [103]. Recently, many methodologies have been investigated to find the numerical solution of the telegraph equation due to their universal applications in the area of applied mathematics. In [104], the phenomena of propagation of the electric signal in a cable of transmission is described by one-dimensional telegraph equation which can be derived by using the basic principles of electricity. But Goldstein [105] was the first who derived the one-dimensional telegraph equation with probabilistic argument. He proved that a particle which moves forward

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and backward direction with speed c satisfies the following hyperbolic one dimensional telegraph equation:

$$\frac{\partial^2 p}{\partial t^2} + 2\lambda \frac{\partial p}{\partial t} = c^2 \frac{\partial^2 p}{\partial x^2}. \tag{3.1}$$

In [106], an effort has been taken to extend the results for two-dimensional case of random planar motion, but this is different from the two-dimensional telegraph equation. Most of the authors extend the one-dimensional telegraph equation simply replacing $\frac{\partial^2 p}{\partial x^2}$ by Laplacian operator $\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right)$ in equation (3.1). But, in some articles [107, 108] authors consider that such replacement is not justified. In [109], a planar random motion governed by two-dimensional telegraph equation has been presented. For the mathematical formulation of the telegraph equation consider a particle P performing random motion with speed c on a planar lattice with spacing $c\Delta t$ [109]. At time t=0, the particle choose any one of the following directions of velocity:

$(x^+, y^+), \quad (x^+, y^-), \quad (x^-, y^-), \quad (x^-, y^-).$

If the particle choose the direction (x^+, y^+) then the particle will move horizontally forward or vertically upward. Also, all other directions of the velocity are likely equal. Moreover, the change in the velocity follows the homogeneous Poisson process with the rate λ (for more details refer to[109]).

Let us introduce some symbols for the mathematical formulation of two-dimensional telegraph equation which describes the planar random motion. Let σ_{++} denote the probability distribution of the particle position at time t and with velocity direction (x^+, y^+). Similarly, we can define $\sigma_{+-}, \sigma_{-+}, \sigma_{--}$. Hence, $p = \sigma_{++} + \sigma_{+-} + \sigma_{-+} + \sigma_{--}$ denotes the probability distribution of the particle and if we define $w = \sigma_{++} + \sigma_{+-} - \sigma_{-+} - \sigma_{--}$ and $u = \sigma_{++} + \sigma_{-+} - \sigma_{+-} - \sigma_{--}$, then p satisfies the following differential equation

$$\frac{\partial w}{\partial t} = -c\frac{\partial p}{\partial x} - 2\lambda w, \tag{3.2}$$

$$\frac{\partial u}{\partial t} = -c\frac{\partial p}{\partial y} - 2\lambda u, \tag{3.3}$$

$$\frac{\partial p}{\partial t} = \frac{-c}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial y}\right), \tag{3.4}$$

where $w, u, p \in C^{2,2}(\Omega \times (0,T])$ and $\Omega = [0,1] \times [0,1]$ (for proof see [109]). Now, differentiating equation (3.4) with respect to time t, we obtain

$$\frac{\partial^2 p}{\partial t^2} = \frac{-c}{2}\left(\frac{\partial^2 w}{\partial x \partial t} + \frac{\partial^2 u}{\partial t \partial y}\right).$$

Using equations (3.2) and (3.3) in the above equation, one can obtain

$$\frac{\partial^2 p}{\partial t^2} = \frac{-c}{2}\left\{-c\frac{\partial^2 p}{\partial x^2} - 2\lambda\frac{\partial w}{\partial x}\right\} + \frac{-c}{2}\left\{-c\frac{\partial^2 p}{\partial y^2} - 2\lambda\frac{\partial u}{\partial y}\right\},$$

or,

$$= \frac{c^2}{2}\left\{\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right\} + \lambda c\left\{\frac{\partial w}{\partial x} + \frac{\partial u}{\partial y}\right\},$$

$$\frac{\partial^2 p}{\partial t^2} = \frac{c^2}{2}\left\{\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right\} - 2\lambda\frac{\partial p}{\partial t}.$$

After rearranging the above equation, we obtain the following equation

$$\frac{\partial^2 p}{\partial t^2} + 2\lambda\frac{\partial p}{\partial t} = \frac{c^2}{2}\left\{\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right\}. \tag{3.5}$$

Equation (3.5) is known as telegraph equation and $p(x,y,t)$ is the solution of this equation.

In this chapter the more general form of equation (3.5) is considered as

$$\begin{aligned}\frac{\partial^2 p(\nu,\varsigma,\tau)}{\partial \tau^2} + 2\alpha\frac{\partial p(\nu,\varsigma,\tau)}{\partial \tau} + \beta^2 p(\nu,\varsigma,\tau) = \frac{\partial^2 p(\nu,\varsigma,\tau)}{\partial \nu^2} &+ \frac{\partial^2 p(\nu,\varsigma,\tau)}{\partial \varsigma^2} \\ &+ k(\nu,\varsigma,\tau),\end{aligned} \tag{3.6}$$

$(\nu,\varsigma,\tau) \in \Omega \times (0,T]$, where $\Omega = [0,1] \times [0,1]$ and $\tau \in (0,T]$.

The initial conditions and the Dirichlet boundary conditions are

$$\begin{cases} p(\nu,\varsigma,0) = f_1(\nu,\varsigma), \\ p_\tau(\nu,\varsigma,0) = f_2(\nu,\varsigma), \end{cases} \tag{3.7}$$

and

$$\begin{cases} p(0,\varsigma,\tau) = g_1(\varsigma,\tau),\ p(1,\varsigma,\tau) = g_2(\varsigma,\tau), \\ p(\nu,0,\tau) = h_1(\nu,\tau),\ p(\nu,1,\tau) = h_2(\nu,\tau). \qquad (\nu,\varsigma,\tau) \in \Omega \times (0,T] \end{cases} \tag{3.8}$$

respectively, where α and β are constants.

Various numerical methods have been developed to solve hyperbolic partial differential equations. In articles [21, 22], R.K. Mohanty has developed three level unconditionally stable schemes based on finite difference. A semi discretization method which is unconditionally stable has been proposed by F. Gao and C. Chi in [110]. In [111], the authors have used compact finite difference approximation for space derivative. Dehghan and Shokri [112] have used radial basis function. However, Dehghan and Yousefi [113] used He's variational iteration approach to solve one-dimensional telegraph equations numerically. Moreover, Saadatmandi and Dehghan [114] have approximated the solution in terms of shifted Chebyshev polynomials to get the numerical solution. In [115], M. Lakestani and B. N. Saray developed an operational matrix approach based on interpolating scaling functions to solve the telegraph equation. In [116], the authors proposed Taylor matrix method for the numerical solution of telegraph equation. In [117], R. Mittal and R. Bhatia have solved the telegraph equation using the B-spline method. Also, in [118], the authors have implemented the Chebyshev wavelet method to solve one-dimensional telegraph equation numerically.

In the progress of the solution of telegraph equation, various meshless techniques have also been developed. Dehghan and Salehi [119] have proposed a meshfree technique which is the combination of boundary knot method and analog equation method to solve two-dimensional telegraph equation. Dehghan and Ghesmati [120] have introduced a combination of meshless local weak and strong (MLWS) forms and Rostamy et al. [121] studied pseudospectral radial basis meshless technique to solve the two-dimensional hyperbolic telegraph equation. Dehghan and Shokri In [122], presented a meshless method for two-dimensional linear hyperbolic equation with variable coefficients. Bulbul and Sezer [116] used Taylor matrix method. Dehgan and Mohebbi [29] have proposed a higher-order implicit collocation method. An unconditionally stable implicit scheme was presented by Mohanty et al.[123]. Ding and Zhang [124] have developed a fourth-order compact finite difference scheme. Mittal and Bhatia proposed [125] differential quadrature method based on B-spline with space discretization. Jiwari et al. [126] proposed a numerical technique based on the polynomial differential quadrature method (PDQM). Recently, an operational matrix approach based on Bernoulli polynomials is proposed by Singh et al. [2].

Recently, the methods based on operational matrices have proved to be very effective. The main advantage of using operational matrices is the sparsity of the operational matrices [125]. The operational matrices also reduce existing singularities in integral and differential equations and convert them into algebraic equations. It simplifies the proposed problem as well as speed up the computation ([36, 56, 127]). In this chapter, two numerical schemes based on operational matrices have been constructed. In the scheme-I, the roots of Legendre polynomial are used as node points for Lagrange polynomials and in the scheme- II, random node points are taken in the domain $[0, 1]$ and use the Gram-Schmidt orthogonalization process to obtain the orthonormal polynomials. Firstly, equations (3.6)-(3.8) are converted into the equivalent form of PIDEs by using boundary and initial conditions. After the transformation of two-dimensional telegraph equation into PIDEs, the operational matrices of differentiation and integration based on ISF and OLBF are used to convert PIDEs into the Sylvester equation [128]. The robust Krylov subspace iterative method (i.e., generalized BICGSTAB, [129]) is used to solve Sylvester equation numerically.

3.2 Preliminaries: Construction of basis functions and their properties

In this chapter, those schemes are applied which have already been discussed in chapter 2.

3.3 Operational matrices

Let, $\Psi(\nu) = [\Psi_0(\nu), \Psi_1(\nu), ..., \Psi_N(\nu)]^T$ be the basis functions. Then,

$$\frac{d}{d\nu}\begin{bmatrix}\Psi_0(\nu)\\ \Psi_1(\nu)\\ \vdots\\ \Psi_N(\nu)\end{bmatrix} \approx \begin{bmatrix} d_{00} & d_{01} & d_{02} & \cdots & d_{0N}\\ d_{10} & d_{11} & d_{12} & \cdots & d_{1N}\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ d_{N0} & d_{N1} & d_{N2} & \cdots & d_{NN}\end{bmatrix}\begin{bmatrix}\Psi_0(\nu)\\ \Psi_1(\nu)\\ \vdots\\ \Psi_N(\nu)\end{bmatrix} = M_D\Psi^T(\nu),$$

$$\int_0^{\nu}\begin{bmatrix}\Psi_0(\nu')\\ \Psi_1(\nu')\\ \vdots\\ \Psi_N(\nu')\end{bmatrix}d\nu' \approx \begin{bmatrix} i_{00} & i_{01} & i_{02} & \cdots & i_{0N}\\ i_{10} & i_{11} & i_{12} & \cdots & i_{1N}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ i_{N0} & i_{N1} & i_{N2} & \cdots & i_{NN}\end{bmatrix}\begin{bmatrix}\Psi_0(\nu)\\ \Psi_1(\nu)\\ \vdots\\ \Psi_N(\nu)\end{bmatrix} = M_I\Psi^T(\nu),$$

where $M_D = [d_{jk}]$ and $M_I = [i_{jk}]$ ($\forall j, k = 0, \cdots, N$) are the operational matrices of differentiation and integration, respectively. Since two schemes have already been established, one can obtain the operational matrices with respect to the first and second schemes by using equations (2.12) and (2.20), respectively. The approximation given in section 2.2 of chapter 2 for both schemes can be extended for the higher dimension. For this purpose, the concept of Kronecker product [129] is used. Suppose that $\Psi(\nu, \varsigma) = [\Psi_0(\nu, \varsigma)\ \Psi_1(\nu, \varsigma)\ \cdots\ \Psi_N(\nu, \varsigma)]^T_{(N+1)^2\times 1}$, where $\Psi_i(\nu, \varsigma) = [\Psi_{i,0}\ (\nu, \varsigma)\ \Psi_{i,1}(\nu, \varsigma)\ \cdots\ \Psi_{i,N}(\nu, \varsigma)]^T \forall\ i = 0, 1, ..., N$ and $\Psi_{r,s}(\nu, \varsigma) = \Psi_r(\nu)\Psi_s(\varsigma)\ \ \forall\ \ r, s = 0, 1, ..., N$. Also, $\Psi(\nu, \varsigma) = \Psi(\nu) \otimes \Psi(\varsigma)$.

Two variables functions, namely $g_1(\varsigma, \tau), g_2(\varsigma, \tau), h_1(\nu, \tau)$ and $h_2(\nu, \tau) \in L^2(\Omega)$ can be approximated as

$$\begin{cases} g_1(\varsigma, \tau) \approx \sum_{m=0}^{N}\sum_{n=0}^{N} g_{1,mn}\Psi_m(\varsigma)\Psi_n(\tau) = \Psi^T(\varsigma)G_1\Psi(\tau),\\ g_2(\varsigma, \tau) \approx \sum_{m=0}^{N}\sum_{n=0}^{N} g_{2,mn}\Psi_m(\varsigma)\Psi_n(\tau) = \Psi^T(\varsigma)G_2\Psi(\tau),\\ h_1(\nu, \tau) \approx \sum_{m=0}^{N}\sum_{n=0}^{N} h_{1,mn}\Psi_m(\nu)\Psi_n(\tau) = \Psi^T(\nu)H_1\Psi(\tau),\\ h_2(\nu, \tau) \approx \sum_{m=0}^{N}\sum_{n=0}^{N} h_{2,mn}\Psi_m(\nu)\Psi_n(\tau) = \Psi^T(\nu)H_2\Psi(\tau),\end{cases} \tag{3.9}$$

where

$$G_1 = \begin{bmatrix} g_{1,00} & g_{1,01} & \cdots & g_{1,0N} \\ g_{1,10} & g_{1,11} & \cdots & g_{1,1N} \\ \vdots & \vdots & \ddots & \vdots \\ g_{1,N0} & g_{1,N1} & \cdots & g_{1,NN} \end{bmatrix}, \ G_2 = \begin{bmatrix} g_{2,00} & g_{2,01} & \cdots & g_{2,0N} \\ g_{2,10} & g_{2,11} & \cdots & g_{2,1N} \\ \vdots & \vdots & \ddots & \vdots \\ g_{2,N0} & g_{2,N1} & \cdots & g_{2,NN} \end{bmatrix},$$

$$H_1 = \begin{bmatrix} h_{1,00} & h_{1,01} & \cdots & h_{1,0N} \\ h_{1,10} & h_{1,11} & \cdots & h_{1,1N} \\ \vdots & \vdots & \ddots & \vdots \\ h_{1,N0} & h_{1,N1} & \cdots & h_{1,NN} \end{bmatrix}, \ H_2 = \begin{bmatrix} h_{2,00} & h_{2,01} & \cdots & h_{2,0N} \\ h_{2,10} & h_{2,11} & \cdots & h_{2,1N} \\ \vdots & \vdots & \ddots & \vdots \\ h_{2,N0} & h_{2,N1} & \cdots & h_{2,NN} \end{bmatrix}.$$

The elements of the above matrices can be calculated $\forall i, j = 0, 1, 2, \cdots, N$ by using the formulae

$g_{1,ij} = \int_0^1 \int_0^1 g_1(\varsigma, \tau)\Psi_{i,j}(\varsigma, \tau) d\varsigma d\tau, \quad g_{2,ij} = \int_0^1 \int_0^1 g_2(\varsigma, \tau)\Psi_{i,j}(\varsigma, \tau) d\varsigma d\tau,$

$h_{1,ij} = \int_0^1 \int_0^1 h_1(\nu, \tau)\Psi_{i,j}(\nu, \tau) d\nu d\tau, \quad h_{2,ij} = \int_0^1 \int_0^1 h_2(\nu, \tau)\Psi_{i,j}(\nu, \tau) d\nu d\tau.$

Similarly, the function in three variable $f(\nu, \varsigma, \tau) \in L^2(\Omega \times (0, T])$ is approximated as

$f(\nu, \varsigma, \tau) \approx \sum_{i=0}^{N} \sum_{j=0}^{N} \sum_{k=0}^{N} f_{ijk} \Psi_i(\nu)\Psi_j(\varsigma)\Psi_k(\tau) = \Psi(\nu, \varsigma)^T F \Psi(\tau),$

where $\Psi(\nu, \varsigma)^T = \Psi(\nu) \otimes \Psi(\varsigma)$ and F is $(N+1)^2 \times (N+1)$ matrix given as

$$F = \begin{bmatrix} f_{000} & f_{001} & \cdots & f_{00N} \\ \vdots & \vdots & \ddots & \vdots \\ f_{0N0} & f_{0N1} & \cdots & f_{0NN} \\ f_{100} & f_{101} & \cdots & f_{10N} \\ \vdots & \vdots & \ddots & \vdots \\ f_{1N0} & f_{1N1} & \cdots & f_{0NN} \\ \vdots & \vdots & \vdots & \vdots \\ f_{N00} & f_{N01} & \cdots & f_{N0N} \\ \vdots & \vdots & \ddots & \vdots \\ f_{NN0} & f_{NN1} & \cdots & f_{NNN} \end{bmatrix},$$

and the coefficients of the matrix F can be calculated by using the formula:

$$f_{ijk} = \int_0^1 \int_0^1 \int_0^1 f(\nu, \varsigma, \tau)\Psi_i(\nu)\Psi_j(\varsigma)\Psi_k(\tau)d\nu d\varsigma d\tau.$$

Lemma 3.3.1. Let, $\omega = \int_0^1 \Psi^T(\nu)d\nu$ and $\omega_{M_I} = \omega M_I^T$ are the $1 \times (N+1)$ vectors, then the following relations hold.

(i) $\Psi^T(\varsigma) = \Psi^T(\nu, \varsigma)\gamma$,

(ii) $\nu\Psi^T(\varsigma) = \Psi^T(\nu, \varsigma)\overline{\gamma}$,

(iii) $\nu(\omega_{M_I} \otimes \Psi^T(\varsigma)) = \Psi^T(\nu, \varsigma)\lambda$,

(iv) $\varsigma(\Psi^T(\nu) \otimes \omega_{M_I}) = \Psi^T(\nu, \varsigma)\overline{\lambda}$,

(v) $\Psi^T(\nu) = \Psi^T(\nu, \varsigma)\zeta$,

(vi) $\varsigma\Psi^T(\nu) = \Psi^T(\nu, \varsigma)\overline{\zeta}$.

Proof. Let, $\Psi(\nu) = [\Psi_0(\nu), \Psi_1(\nu), ..., \Psi_N(\nu)]^T$ be the basis of $L^2[0, 1]$. Then $\nu\Psi^T(\varsigma)$ can be written in the following form

$$\nu\Psi^T(\varsigma) = \nu \otimes \Psi^T(\varsigma). \tag{3.10}$$

Let $\nu = f(\nu) \in L^2[0, 1]$. Then $f(\nu)$ can be approximated in terms of basis functions as

$$f(\nu) = \nu \approx \sum_{i=0}^{N} a_i\Psi_i(\nu) = A^T\Psi(\nu),$$

where, $A = [a_0, a_1, ..., a_N]^T$ and the coefficients of A are calculated by the formula

$$a_i = \int_0^1 \nu\Psi_i(\nu)d\nu.$$

Hence from equation 3.10, we obtain

$$\nu\Psi^T(\varsigma) = \Psi^T(\nu)A \otimes \Psi^T(\varsigma) = \left(\Psi^T(\nu) \otimes \Psi^T(\varsigma)\right)(A \otimes I_{N+1}) = \Psi^T(\nu, \varsigma)\overline{\gamma}$$

, where

$$\overline{\gamma} = A \otimes I_{N+1}.$$

Similarly, we can write (iii) of the Lemma as

$$\nu\left(\omega_{M_I} \otimes \Psi^T(\varsigma)\right) = (\nu\omega_{M_I}) \otimes \Psi^T(\varsigma).$$

Since, $\nu\omega_{M_I}$ is $1 \times (N+1)$ vector and each element of this vector is a function of ν, so we can make similar argument as given in the proof of (ii) of the Lemma.

Hence,

$$\begin{aligned}(\nu\omega_{M_I}) \otimes \Psi^T(\varsigma) &= \left(\Psi^T(\nu)B^T\right) \otimes \Psi^T(\varsigma)\\ &= \left(\Psi^T(\nu) \otimes \Psi^T(\varsigma)\right)\left(B^T \otimes I_{N+1}\right)\\ &= \Psi^T(\nu,\varsigma)\lambda,\end{aligned}$$

where $\lambda = \left(B^T \otimes I_{N+1}\right)$ and $B = \begin{bmatrix} b_{00} & b_{01} & \cdots & b_{0N} \\ b_{10} & b_{11} & \cdots & b_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ b_{N0} & b_{N1} & \cdots & b_{NN} \end{bmatrix} = [b_{ij}]_{(N+1)\times(N+1)}.$

The coefficients b_{ij} are calculated by using the formula

$$b_{ij} = \int_0^1 \left[(\nu\omega_{M_I})_i \Psi_j(\nu)\right] d\nu. \quad \forall i,j = 0,\cdots,N.$$

Note: (a) All other matrices $\overline{\lambda}, \zeta$ and $\overline{\zeta}$ are calculated in the similar manner.

(b) If we take $\nu = 1$, then the proof of (i) is same as (ii). Similarly, the proof of $(iv), (v)$ and (vi) are same as $(i), (ii)$ and (iii), respectively.

Lemma 3.3.2. If $M_\varsigma = I_{N+1} \otimes M_D, P_\varsigma = I_{N+1} \otimes M_I$ and $P_\nu = M_I \otimes I_{N+1}$, where I_{N+1} denotes the identity matrix of dimension $(N+1)\times(N+1)$, then we have

(i) $\frac{\partial \Psi(\nu,\varsigma)}{\partial \varsigma} = \Psi_\varsigma(\nu,\varsigma) = M_\varsigma \Psi(\nu,\varsigma)$,

(ii) $\frac{\partial^2 \Psi(\nu,\varsigma)}{\partial \varsigma^2} = \Psi_{\varsigma\varsigma}(\nu,\varsigma) = (M_\varsigma)^2 \Psi(\nu,\varsigma)$,

(iii) $\int_0^{\varsigma} \Psi(\nu,\varsigma')d\varsigma' \approx P_{\varsigma}\Psi(\nu,\varsigma)$,

(iv) $\int_0^{\varsigma}\int_0^{\varsigma'} \Psi(\nu,\varsigma'')d\varsigma''d\varsigma' \approx (P_{\varsigma})^2\Psi(\nu,\varsigma)$,

(v) $\int_0^{\nu} \Psi(\nu',\varsigma)d\nu' \approx P_{\nu}\Psi(\nu,\varsigma)$,

(vi) $\int_0^{\nu}\int_0^{\nu'} \Psi(\nu'',\varsigma)d\nu''d\nu' \approx (P_{\nu})^2\Psi(\nu,\varsigma)$.

3.4 Numerical method of solution

To solve equations (3.6)-(3.8) numerically, we first use all the initial and boundary conditions to convert the equations (3.6)-(3.8) into PIDEs. For this purpose let us rewrite the equation (3.6) as:

$$\frac{\partial^2 p(\nu,\varsigma,\tau)}{\partial\nu^2} = \frac{\partial^2 p(\nu,\varsigma,\tau)}{\partial\tau^2} + 2\alpha\frac{\partial p(\nu,\varsigma,\tau)}{\partial\tau} + \beta^2 p(\nu,\varsigma,\tau) - \frac{\partial^2 p(\nu,\varsigma,\tau)}{\partial\varsigma^2} - k(\nu,\varsigma,\tau).$$

Integrating the above equation in the interval $[0,\nu]$,

$$\begin{aligned} p_{\nu}(\nu,\varsigma,\tau) = p_{\nu}(0,\varsigma,\tau) + \int_0^{\nu} (p_{\tau\tau}(\nu',\varsigma,\tau) + 2\alpha p_{\tau}(\nu',\varsigma,\tau) + \beta^2 p(\nu',\varsigma,\tau) \\ -p_{\varsigma\varsigma}(\nu',\varsigma,\tau) - k(\nu',\varsigma,\tau))d\nu'. \end{aligned} \tag{3.11}$$

Again integrating equation (3.11) in the interval $[0,\nu]$, we get

$$\begin{aligned} p(\nu,\varsigma,\tau) = \underbrace{p(0,\varsigma,\tau)}_{=g_1(\varsigma,\tau)} + \nu p_{\nu}(0,\varsigma,\tau) + \int_0^{\nu}\int_0^{\nu'} (p_{\tau\tau}(\nu'',\varsigma,\tau) + 2\alpha p_{\tau}(\nu'',\varsigma,\tau) \\ +\beta^2 p(\nu'',\varsigma,\tau) - p_{\varsigma\varsigma}(\nu'',\varsigma,\tau) - k(\nu'',\varsigma,\tau))d\nu''d\nu'. \end{aligned} \tag{3.12}$$

Putting $\nu = 1$ in equation (3.12), we obtain

$$\begin{aligned} p_{\nu}(0,\varsigma,\tau) = \underbrace{p(1,\varsigma,\tau)}_{=g_2(\varsigma,\tau)} - g_1(\varsigma,\tau) - \int_0^{1}\int_0^{\nu} (p_{\tau\tau}(\nu',\varsigma,\tau) + 2\alpha p_{\tau}(\nu',\varsigma,\tau) \\ +\beta^2 p(\nu',\varsigma,\tau) - p_{\varsigma\varsigma}(\nu',\varsigma,\tau) - k(\nu',\varsigma,\tau))d\nu'd\nu. \end{aligned} \tag{3.13}$$

Substituting the value of $p_\nu(0,\varsigma,\tau)$ from (3.13) to equation (3.12), one can obtain

$$\begin{aligned} p(\nu,\varsigma,\tau) &= \nu g_2(\varsigma,\tau) + (1-\nu) g_1(\varsigma,\tau) - \nu \int_0^1 \int_0^\nu (p_{\tau\tau}(\nu',\varsigma,\tau) + 2\alpha p_\tau(\nu',\varsigma,\tau) \\ &+ \beta^2 p(\nu',\varsigma,\tau) - p_{\varsigma\varsigma}(\nu',\varsigma,\tau) - k(\nu',\varsigma,\tau)) d\nu' d\nu + \int_0^\nu \int_0^{\nu'} (p_{\tau\tau}(\nu'',\varsigma,\tau) \\ &+ 2\alpha p_\tau(\nu'',\varsigma,\tau) + \beta^2 p(\nu'',\varsigma,\tau) - p_{\varsigma\varsigma}(\nu'',\varsigma,\tau) - k(\nu'',\varsigma,\tau)) d\nu'' d\nu'. \end{aligned} \tag{3.14}$$

To use boundary conditions in ς, we integrate $p_{\varsigma\varsigma}(\nu,\varsigma,\tau)$ from 0 to ς

$$\int_0^\varsigma p_{\varsigma\varsigma}(\nu,\varsigma',\tau) d\varsigma = p_\varsigma(\nu,\varsigma,\tau) - p_\varsigma(\nu,0,\tau).$$

Again integrating above equation from 0 to ς, we get

$$\int_0^\varsigma \int_0^{\varsigma'} p_{\varsigma\varsigma}(\nu,\varsigma'',\tau) d\varsigma'' d\varsigma' = p(\nu,\varsigma,\tau) - p(\nu,0,\tau) - \varsigma p_\varsigma(\nu,0,\tau). \tag{3.15}$$

Using $p(\nu,0,\tau) = h_1(\nu,\tau)$, the equation (3.15) can be rearranged as

$$p(\nu,\varsigma,\tau) = h_1(\nu,\tau) + \varsigma p_\varsigma(\nu,0,\tau) + \int_0^\varsigma \int_0^{\varsigma'} p_{\varsigma\varsigma}(\nu,\varsigma'',\tau) d\varsigma'' d\varsigma'. \tag{3.16}$$

To use the condition $p(\nu,1,\tau) = h_2(\nu,\tau)$, putting $\varsigma = 1$ in equation (3.16), we get

$$p_\varsigma(\nu,0,\tau) = h_2(\nu,\tau) - h_1(\nu,\tau) - \int_0^1 \int_0^\varsigma p_{\varsigma\varsigma}(\nu,\varsigma',\tau) d\varsigma' d\varsigma'. \tag{3.17}$$

Substituting the value of $p_\varsigma(\nu,0,\tau)$ obtained from (3.17) into equation (3.16), we get

$$\begin{aligned} p(\nu,\varsigma,\tau) = \varsigma h_2(\nu,\tau) + (1-\varsigma) h_1(\nu,\tau) - \varsigma \int_0^1 \int_0^\varsigma p_{\varsigma\varsigma}(\nu,\varsigma',\tau) d\varsigma' d\varsigma \\ + \int_0^\varsigma \int_0^{\varsigma'} p_{\varsigma\varsigma}(\nu,\varsigma'',\tau) d\varsigma'' d\varsigma'. \end{aligned} \tag{3.18}$$

Now, the goal is to use the initial conditions (3.7). For this purpose, the same process is used which is applied to use the boundary conditions in ς i.e, integrate $p_{\tau\tau}(\nu,\varsigma,\tau)$ two times from 0 to τ and obtain

$$p(\nu,\varsigma,\tau) = f_1(\nu,\varsigma) + \tau f_2(\nu,\varsigma) + \int_0^\tau \int_0^{\tau'} p_{\tau\tau}(\nu,\varsigma,\tau'') d\tau'' d\tau'.$$

The above equation can be rewritten as

$$p(\nu,\varsigma,\tau) = f(\nu,\varsigma,\tau) + \int_0^{\tau}\int_0^{\tau'} p_{\tau\tau}(\nu,\varsigma,\tau'')d\tau''d\tau', \tag{3.19}$$

where $f(\nu,\varsigma,\tau) = f_1(\nu,\varsigma) + \tau f_2(\nu,\varsigma)$.

Since, we can obtain the original equation (3.6) with boundary conditions $p(0,\varsigma,\tau) = g_1(\varsigma,\tau)$ and $p(1,\varsigma,\tau) = g_2(\varsigma,\tau)$ by differentiating two times equations (3.14) and replacing ν by 0 and 1 in equation (3.14), respectively. Boundary conditions $p(\nu,0,\tau) = h_1(\nu,\tau)$, $p(\nu,1,\tau) = h_2(\nu,\tau)$ can be obtained by replacing ς by 0 and 1 in equation (3.18). Also, one can obtain initial conditions $p(\nu,\varsigma,0) = f_1(\nu,\varsigma)$ and $p_\tau(\nu,\varsigma,0) = f_2(\nu,\varsigma)$ by replacing τ by 0 and 1 in equation (3.19), respectively. Hence, the equations (3.14), (3.18) and (3.19) are the equivalent formulation of the proposed problem (3.6)-(3.8). To solve these PIDEs, all the known and unknown functions are approximated in terms of basis functions. The known function of three variables $k(\nu,\varsigma,\tau)$ can be approximated in terms of basis functions as follows

$$k(\nu,\varsigma,\tau) \approx \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{j=0}^{N} k_{ijk}\Psi_i(\nu)\Psi_j(\varsigma)\Psi_k(\tau) = \Psi^T(\nu,\varsigma)K\Psi(\tau), \tag{3.20}$$

where $\Psi^T(\nu,\varsigma) = \Psi^T(\nu)\otimes\Psi^T(\varsigma)$ and K is a matrix calculated in similar manner as F. Similarly, we have

$$f(\nu,\varsigma,\tau) \approx \Psi^T(\nu,\varsigma)F\Psi(\tau), \tag{3.21}$$

and

$$p(\nu,\varsigma,\tau) \approx \sum_{m=0}^{N}\sum_{n=0}^{N}\sum_{p=0}^{N} p_{mnp}\Psi_m(\nu)\Psi_n(\varsigma)\Psi_p(\tau) = \Psi^T(\nu,\varsigma)P_{apx}\Psi(\tau), \tag{3.22}$$

where

$$P_{apx} = \begin{bmatrix} p_{000} & p_{001} & \cdots & p_{00N} \\ \vdots & \vdots & \ddots & \vdots \\ p_{0N0} & p_{0N1} & \cdots & p_{0NN} \\ p_{100} & p_{101} & \cdots & p_{10N} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1N0} & p_{1N1} & \cdots & p_{0NN} \\ \vdots & \vdots & \vdots & \vdots \\ p_{N00} & p_{N01} & \cdots & p_{N0N} \\ \vdots & \vdots & \ddots & \vdots \\ p_{NN0} & p_{NN1} & \cdots & p_{NNN} \end{bmatrix}_{(N+1)^2\times(N+1)}.$$

It is to be noted that we have to calculate the unknown matrix P_{apx}.

Now, substituting the approximated value of $k(\nu,\varsigma,\tau)$, $p(\nu,\varsigma,\tau)$, $g_1(\varsigma,\tau)$ and $g_2(\varsigma,\tau)$ which are given by equations (3.20), (3.22) and (3.9) in the right hand side of equation (3.14). After using the operational matrices of differentiation and integration and Lemma 3.3.2, the equation (3.14) reduces to the following form

$$\begin{aligned} p(\nu,\varsigma,\tau) &\approx \nu(\Psi^T(\varsigma)G_2\Psi(\tau)) + (1-\nu)(\Psi^T(\varsigma)G_1\Psi(\tau)) \\ -\nu\int_0^1\int_0^\nu &\Psi^T(\nu',\varsigma)(P_{apx}M_D^2 + 2\alpha P_{apx}M_D + \beta^2 P_{apx} - (M_\varsigma^T)^2 P_{apx} - K)\Psi(\tau)d\nu' d\nu \\ +\int_0^\nu\int_0^{\nu'} &\Psi^T(\nu'',\varsigma)(P_{apx}M_D^2 + 2\alpha P_{apx}M_D + \beta^2 P_{apx} - (M_\varsigma^T)^2 P_{apx} - K)\Psi(\tau)d\nu'' d\nu', \end{aligned}$$

or,

$$\begin{aligned} p(\nu,\varsigma,\tau) &\approx \nu\Psi^T(\varsigma)(G_2 - G_1)\Psi(\tau) + \Psi^T(\varsigma)G_1\Psi(\tau) \\ -\nu(\omega_{M_I}\otimes\Psi^T(\varsigma))&(P_{apx}M_D^2 + 2\alpha P_{apx}M_D + \beta^2 P_{apx} - (M_\varsigma^T)^2 P_{apx} - K)\Psi(\tau) \\ +\Psi^T(\nu,\varsigma)(P_\nu^T)^2&(P_{apx}M_D^2 + 2\alpha P_{apx}M_D + \beta^2 P_{apx} - (M_\varsigma^T)^2 P_{apx} - K)\Psi(\tau). \end{aligned} \tag{3.23}$$

Using (i), (ii) and (iii) of Lemma 3.3.1, we can rewrite equation (3.23) as

$$\begin{aligned} p(\nu,\varsigma,\tau) &= \Psi^T(\nu,\varsigma)\overline{\gamma}(G_2-G_1)\Psi(\tau)+\Psi^T(\nu,\varsigma)\gamma G_1\Psi(\tau)\\ &-\Psi^T(\nu,\varsigma)\lambda(P_{apx}M_D^2+2\alpha P_{apx}M_D+\beta^2P_{apx}-(M_\varsigma^T)^2P_{apx}-K)\Psi(\tau)\\ &+\Psi^T(\nu,\varsigma)(P_\nu^T)^2(P_{apx}M_D^2+2\alpha P_{apx}M_D+\beta^2P_{apx}-(M_\varsigma^T)^2P_{apx}-K)\Psi(\tau). \end{aligned} \tag{3.24}$$

The above equation can be rewritten as

$$p(\nu,\varsigma,\tau)\approx\Psi^T(\nu,\varsigma)Z\Psi(\tau), \tag{3.25}$$

in which

$$Z=\overline{\gamma}(G_2-G_1)+\gamma G_1+((P_\nu^T)^2-\lambda)(P_{apx}M_D^2+2\alpha P_{apx}M_D+\beta^2P_{apx}-(M_\varsigma^T)^2P_{apx}-K).$$

Now, substituting the approximated value of $p(\nu,\varsigma,\tau)$ from equation (3.25) and $h_1(\nu,\tau),\ h_2(\nu,\tau)$ from equation(3.9) in equation (3.18) and using the operational matrices and Lemma 3.3.2, we get

$$\begin{aligned} p(\nu,\varsigma,\tau) &\approx \varsigma(\Psi^T(\nu)H_2\Psi(\tau))+(1-\varsigma)\Psi^T(\nu)H_1\Psi(\tau)\\ &-\varsigma\int_0^1\int_0^\varsigma\Psi^T(\nu,\varsigma)(M_\varsigma^T)^2Z\Psi(\tau)d\varsigma' d\varsigma\\ &+\int_0^\varsigma\int_0^{\varsigma'}\Psi^T(\nu,\varsigma)(M_\varsigma^T)^2Z\Psi(\tau)d\varsigma'' d\varsigma'. \end{aligned} \tag{3.26}$$

We can rearrange the above equation as

$$\begin{aligned} p(\nu,\varsigma,\tau) &\approx \varsigma\Psi^T(\nu)(H_2-H_1)\Psi(\tau)+\Psi^T(\nu)H_1\Psi(\tau)\\ &-\varsigma(\Psi^T\otimes\omega_{M_I})(M_\varsigma^T)^2Z\Psi(\tau)+\Psi^T(\nu,\varsigma)(P_\nu^T)^2(M_\varsigma^T)^2Z\Psi(\tau). \end{aligned} \tag{3.27}$$

By using (iv),(v) and (vi) of Lemma (3.3.1), the above equation can be written as

$$p(\nu,\varsigma,\tau)\approx\Psi^T(\nu,\varsigma)Y\Psi(\tau), \tag{3.28}$$

$$\text{where}\quad Y=\overline{\zeta}(H_2-H_1)+\zeta H_1+((P_\varsigma^T)^2-\overline{\lambda})(M_\varsigma^T)^2Z.$$

In the next step, equation (3.28) can be taken as an approximation for $p(\nu,\varsigma,\tau)$. Substituting the approximated values of $f(\nu,\varsigma,\tau)$, $p(\nu,\varsigma,\tau)$ together with operational matrices of differentiation and integration in the right hand side of equation (3.19), we get

$$p(\nu,\varsigma,\tau) \approx \Psi^T(\nu,\varsigma)F\Psi(\tau) + \Psi^T(\nu,\varsigma)YM_D^2M_I^2\Psi(\tau).$$

On combining above equation with equation (3.22), we get

$$\Psi^T(\nu,\varsigma)P_{apx}\Psi(\tau) = \Psi^T(\nu,\varsigma)F\Psi(\tau) + \Psi^T(\nu,\varsigma)\widetilde{Y}M_D^2M_I^2\Psi(\tau), \tag{3.29}$$

where

$$\widetilde{Y} = \overline{\zeta}(H_2 - H_1) + \zeta H_1 + ((P_\varsigma^T)^2 - \overline{\lambda})(M_\varsigma^T)^2\widetilde{Z},$$

$$\widetilde{Z} = \overline{\gamma}(G_2 - G_1) + \gamma G_1 + ((P_\nu^T)^2 - \lambda)(\Psi M_D^2 + 2\alpha P_{apx}M_D + \beta^2 P_{apx} - (M_\varsigma^T)^2 P_{apx} - K).$$

Equation (3.29) can be reduced into the following matrix equation

$$P_{apx} + A_1P_{apx}B_1 + A_2P_{apx}B_2 = C, \tag{3.30}$$

which is in the form of Sylvester equation,

where

$$\begin{aligned} A_1 &= ((P_\varsigma^T)^2 - \overline{\lambda})(M_\varsigma^T)^2((P_\nu^T)^2 - \lambda), \\ B_1 &= -(M_D^2 + 2\alpha M_D + \beta^2)M_D^2M_I^2, \\ A_2 &= ((P_\varsigma^T)^2 - \overline{\lambda})(M_\varsigma^T)^2((P_\nu^T)^2 - \lambda)(M_\varsigma^T)^2, \\ B_2 &= M_D^2M_I^2 \qquad , \end{aligned}$$

and

$$\begin{aligned} C = F + \zeta H_1M_D^2M_I^2 + \overline{\zeta}(H_2 - H_1)M_D^2M_I^2 + ((P_\varsigma^T)^2 - \overline{\lambda})(M_\varsigma^T)^2 \\ (\overline{\gamma}(G_2 - G_1) + \gamma G_1 - ((P_\nu^T)^2 - \lambda)K)M_D^2M_I^2. \end{aligned}$$

Finally, we get the system of equations in the form of equation (3.30) which is known as the Sylvester equation. Many authors have proposed numerical schemes to solve Sylvester equation(see [130–132], [128]). In this chapter Sylvester equation (3.30) is solved for P_{apx} by robust Krylov subspace iterative method (i.e., generalized BICGSTAB, [129]) and using equation (3.22), the approximate solution of equations (3.6)-(3.8) is obtained in terms of basis functions by using scheme-I and scheme-II.

Remark 3.4.1. We can obtain the approximate solutions of equation (3.6) using scheme-I and scheme-II by using Θ and Φ, respectively in place of Ψ in equation (3.22).

3.5 Numerical experiments

In this section, the procedure of solving the Sylvester equation (3.30) is illustrated which determines the solution of telegraph equations (3.6)-(3.8). For this aim, three examples are included to test the applicability and effectiveness of the proposed schemes. The results of numerical experiments for different number of basis functions are depicted in the form of Tables 3.1-3.6 and the Figures 3.1-3.6. Also, the comparison of pointwise errors with the Bernoulli matrix method is given in Table 3.7. The comparison of L_∞-error with the differential quadrature method is provided in Table 3.8.
Numerical simulations have been done with the help of Matlab. The error at point $(\nu_i, \varsigma_j, \tau_k)$ and L_∞-error are given by the following formulae:

$$e_{i,j,k} = (p_{exact})_{i,j,k} - (p_{apx})_{i,j,k},$$

$$L_\infty - error = \|p - p_{apx}\|_\infty = \max_{i,j,k} |(p_{exact})_{i,j,k} - (p_{apx})_{i,j,k}|,$$

where p denotes the solution of the telegraph equation.

Example 3.5.1. In this example, consider TDTE (3.6) in the domain $(\nu, \varsigma) \in \Omega$ and $k(\nu, \varsigma, \tau) = (-2\alpha + \beta^2 - 1)e^{-\tau} \sinh \nu \sinh \varsigma$. The initial and the Dirichlet boundary conditions are given by

$$\begin{cases} p(\nu, \varsigma, 0) = \sinh(\nu) \sinh(\varsigma), \\ p_\tau(\nu, \varsigma, 0) = -\sinh(\nu) \sinh(\varsigma), \end{cases} \tag{3.31}$$

$$\begin{cases} p(0,\varsigma,\tau) = 0, \ p(1,\varsigma,\tau) = e^{-\tau}\sinh(1)\sinh(\varsigma), \\ p(\nu,0,\tau) = 0, \ p(\nu,1,\tau) = e^{-\tau}\sinh(\nu)\sinh(1). \end{cases} \tag{3.32}$$

The exact solution is given by

$$p(\nu,\varsigma,\tau) = e^{-\tau}\sinh(\nu)\sinh(\varsigma).$$

In this example, we have taken $\alpha = 10, \beta = 5$ and solved for $N = 3,4,5,6,7$ and 8. Numerical results of absolute error at different points of the domain Ω are given in Table 3.1 and Table 3.2 by using scheme-I and scheme-II, respectively. The graphs of error for $N = 6$ are shown in Figure 3.1 and Figure 3.2 for scheme-I and scheme-II, respectively.

Table 3.1: Absolute error $e_N(\nu,\varsigma,1)$ for values of $N = 3,4,5,6,7$ and 8 of example 3.5.1 via scheme-I.

Grid Points	$N=3$	$N=4$	$N=5$	$N=6$	$N=7$	$N=8$
(0.0,0.0)	1.0753e-16	1.2423e-15	7.5909e-15	1.3207e-15	7.2501e-14	2.3954e-13
(0.1,0.1)	2.6215e-04	1.9482e-04	1.3003e-04	7.0226e-05	3.9404e-05	2.7964e-05
(0.2,0.2)	1.0136e-03	7.5973e-04	4.8444e-04	2.6253e-04	1.4604e-04	7.7700e-05
(0.3,0.3)	2.0764e-03	1.5776e-03	9.6718e-04	5.2271e-04	2.8799e-04	1.1436e-04
(0.4,0.4)	3.0924e-03	2.4140e-03	1.4381e-03	7.7175e-04	4.2090e-04	1.2422e-04
(0.5,0.5)	3.5815e-03	2.9897e-03	1.7430e-03	9.2614e-04	4.9986e-04	1.0958e-04
(0.6,0.6)	3.0434e-03	3.05953e-03	1.7573e-03	9.2355e-04	4.9324e-04	8.1058e-05
(0.7,0.7)	1.0997e-03	2.4941e-03	1.4360e-03	7.4795e-04	3.9544e.04	5.0147e-05
(0.8,0.8)	2.3249e-03	1.3629e-03	8.5765e-04	4.4799e-04	2.3490e-04	2.4467e-05
(0.9,0.9)	6.7866e-03	1.3996e-05	2.4542e-04	1.4126e-04	7.4203e-06	7.0349e-06
(1.0,1.0)	1.1074e-02	8.5581e-04	5.600 e-05	2.7846e-06	1.2180e-07	4.3713e-09

Table 3.2: Absolute error $e_N(\nu,\varsigma,1)$ for values of $N = 3,4,5,6,7$ and 8 of example 3.5.1 via scheme-II.

Grid Points	$N=3$	$N=4$	$N=5$	$N=6$	$N=7$	$N=8$
(0.0,0.0)	0.0000	1.0000e-15	8.0000e-15	1.3210e-15	7.2501e-14	7.7102e-13
(0.1,0.1)	2.6215e-04	1.9482e-04	1.3003e-04	7.0226e-05	3.9404e-05	7.2636e-06
(0.2,0.2)	1.0136e-03	7.5973e-04	4.8444e-04	2.6253e-04	1.4604e-04	2.4871e-05
(0.3,0.3)	2.0764e-03	1.5776e-03	9.6718e-04	5.2271e-04	2.8799e-04	4.7405e-05
(0.4,0.4)	3.0924e-03	2.4140e-03	1.4381e-03	7.7175e-04	4.2090e-04	7.0071e-05
(0.5,0.5)	3.5815e-03	2.9897e-03	1.7430e-03	9.2614e-04	4.9986e-04	8.7937e-05
(0.6,0.6)	3.0434e-03	3.0595e-03	1.7573e-03	9.2355e-04	4.9324-04	9.5398e-05
(0.7,0.7)	1.0997e-03	2.4941e-03	1.4360e-03	7.4795e-04	3.9544e-04	8.6890e-05
(0.8,0.8)	2.3249e-03	1.3629e-03	8.5765e-04	4.4799e-04	2.3490e-04	6.0127e-05
(0.9,0.9)	6.7866e-03	1.3996e-05	2.4542e-04	1.4126e-04	7.4203e-05	2.2528e-05
(1.0,1.0)	1.1074e-02	8.5581e-04	5.6452e-05	2.7846e-06	1.2180e-07	4.6617e-09

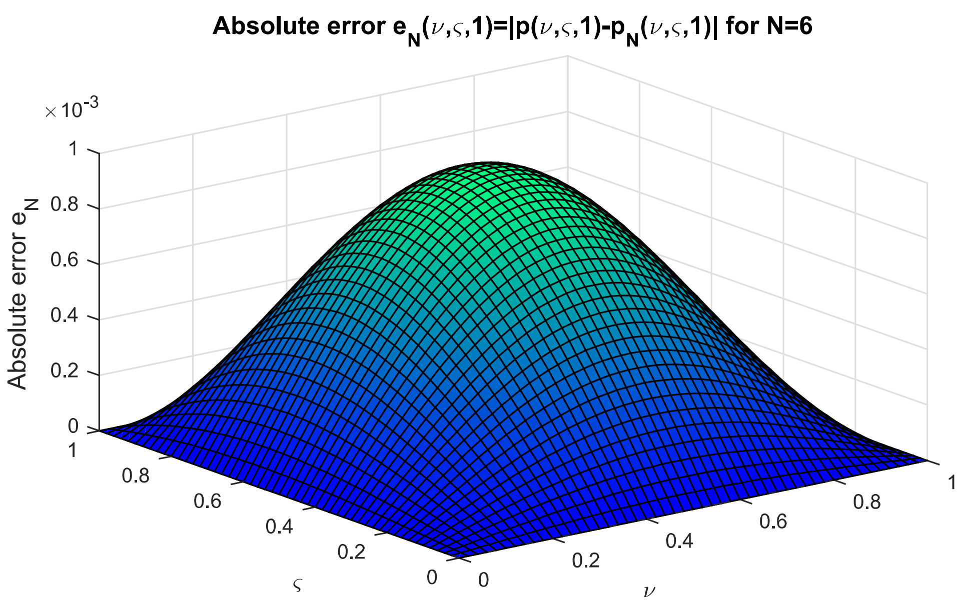

Figure 3.1: Absolute error of example 3.5.1 obtained for N=6 using scheme-I.

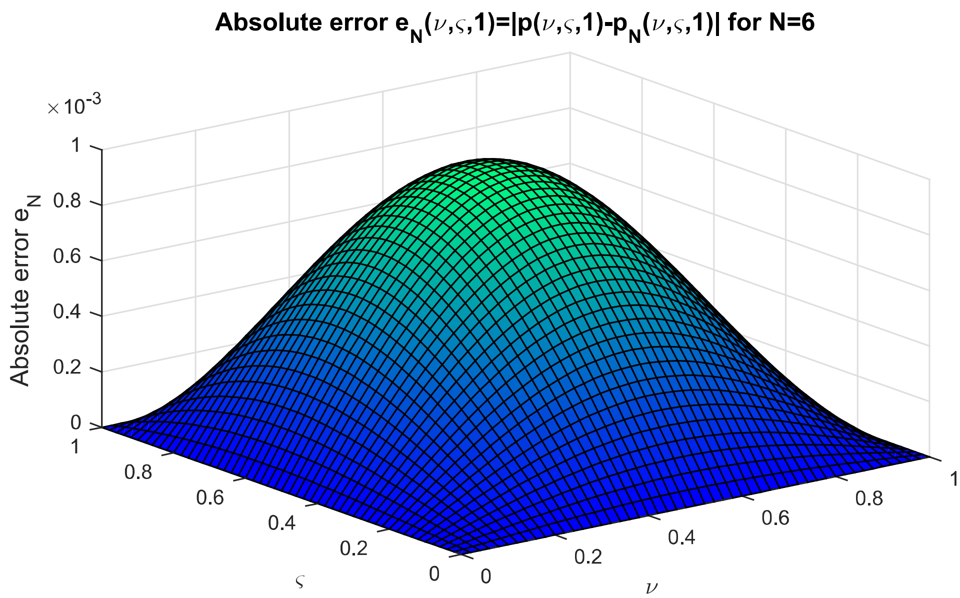

Figure 3.2: Absolute error of example 3.5.1 obtained for N=6 using scheme-II.

Example 3.5.2. In this example, consider the TDTE (3.6) in the domain $(\nu, \varsigma) \in \Omega$, with $\alpha = \beta = 1$ and $k(\nu, \varsigma, \tau) = 2\cos(\tau)\sin(\nu)\sin(\varsigma) - 2\sin(\tau)\sin(\nu)\sin(\varsigma)$. The initial and boundary conditions are given by

$$\begin{cases} p(\nu, \varsigma, 0) = \sin(\nu)\sin(\varsigma), \\ p_\tau(\nu, \varsigma, 0) = 0, \end{cases} \tag{3.33}$$

and

$$\begin{cases} p(0, \varsigma, \tau) = 0, \;\; p(1, \varsigma, \tau) = \cos(\tau)\sin(1)\sin(\varsigma), \\ p(\nu, 0, \tau) = 0, \;\; p(\nu, 1, \tau) = \cos(\tau)\sin(\nu)\sin(1). \end{cases} \tag{3.34}$$

The exact solution for this example is

$$p(\nu, \varsigma, \tau) = \cos\tau \sin(\nu)\sin(\varsigma).$$

Again, taking $N = 3, 4, 5$ and 6, the numerical solutions at some points of domain for Example 3.5.2 are given in Table 3.3 and Table 3.4 by scheme-I and scheme-II, respectively. Also, the graphs for absolute errors for $N = 6$ are depicted in Figure 3.3 and Figure 3.4

by scheme-I and scheme-II, respectively.

Table 3.3: Absolute error $e_N(\nu, \varsigma, 1)$ for values of $N = 3, 4, 5, 6, 7$ and 8 of example 3.5.2 via scheme-I.

Grid Points	$N = 3$	$N = 4$	$N = 5$	$N = 6$	$N = 7$	$N = 8$
(0.0,0.0)	6.3877e-08	8.6472e-10	4.4574e-13	2.3498e-14	2.0009e-14	2.6755e-14
(0.1,0.1)	1.8800e-04	7.1051e-05	1.3918e-05	6.6214e-06	2.4532e-06	1.7522e-05
(0.2,0.2)	7.0241e-04	2.1391e-04	4.7968e-05	1.7545e-05	8.5974e-06	5.7300e-05
(0.3,0.3)	1.5050e-03	3.3910e-04	8.9930e-05	2.3625e-05	1.6242e-05	1.0088e-04
(0.4,0.4)	2.4953e-03	3.8179e-04	1.2800e-04	2.1472e-05	2.3010e-05	1.3317e-04
(0.5,0.5)	3.5399e-03	3.1947e-04	1.5228e-04	1.2526e-05	2.6780e-05	1.4454e-04
(0.6,0.6)	4.5119e-03	1.7398e-04	1.5602e-04	1.2576e-06	2.6171e-05	1.3191e-04
(0.7,0.7)	5.3358e-03	1.1347e-06	1.3700e-04	7.1051e-06	2.1003e-05	9.9022e-05
(0.8,0.8)	6.0376e-03	1.4635e-04	9.9702e-05	8.9926e-06	1.2640e-05	5.5777e-05
(0.9,0.9)	6.7947e-03	2.3284e-04	5.8319e-05	4.8208e-06	4.1345e-06	1.6935e-05
(1.0,1.0)	7.9843 e-04	3.0769e-04	4.0858e-05	1.0288e-06	8.8486e-08	1.6434e-09

Table 3.4: Absolute error $e_N(\nu, \varsigma, 1)$ for values of $N = 3, 4, 5, 6, 7$ and 8 of example 3.5.2 via scheme-II.

Grid Points	$N = 3$	$N = 4$	$N = 5$	$N = 6$	$N = 7$	$N = 8$
(0.0,0.0)	6.3876e-08	8.6472e-10	4.45741e-13	2.3498e-14	1.7119e-13	3.730e-13
(0.1,0.1)	1.8800e-04	7.1051e-05	1.3918e-05	6.6214e-06	2.4849e-06	1.8668e-04
(0.2,0.2)	7.0241e-04	2.1391e-04	4.7968e-05	1.7545e-05	8.7049e-06	6.1473e-04
(0.3,0.3)	1.5050e-03	3.3910e-04	8.9930e-05	2.3625e-05	1.6438e-05	1.0900e-03
(0.4,0.4)	2.4953e-03	3.8179e-04	1.2800e-04	2.1472e-05	2.3279e-05	1.4493e-03
(0.5,0.5)	3.5399e-03	3.1947e-04	1.5228e-04	1.2526e-05	2.7082e-05	1.5840e-03
(0.6,0.6)	4.5119e-03	1.7398e-04	1.5602e-04	1.2576e-06	2.6457e-05	1.4551e-03
(0.7,0.7)	5.3358e-03	1.1347e-06	1.3700e-04	7.1051e-06	2.1225e-05	1.0987e-03
(0.8,0.8)	6.0376e-03	1.4635e-04	9.9702e-05	8.9926e-06	1.2769e-05	6.2199e-04
(0.9,0.9)	6.7947e-03	2.3284e-04	5.8319e-05	4.8208e-06	4.1752e-06	1.8960e-04
(1.0,1.0)	7.9843e-03	3.0769e-04	4.0858e-05	1.0288e-06	8.8484e-08	1.6439e-09

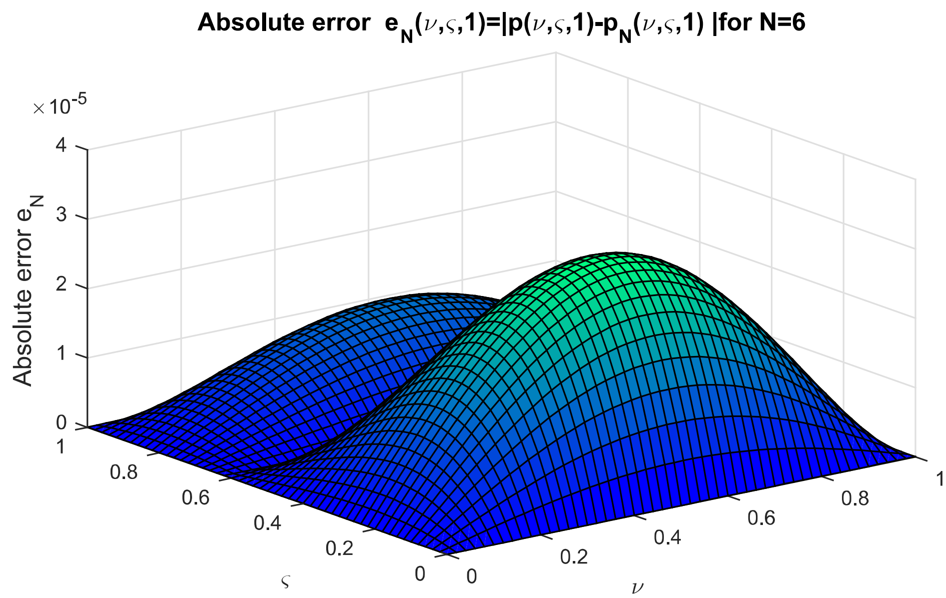

Figure 3.3: Absolute error of example 3.5.2 obtained for $N = 6$ by scheme-I.

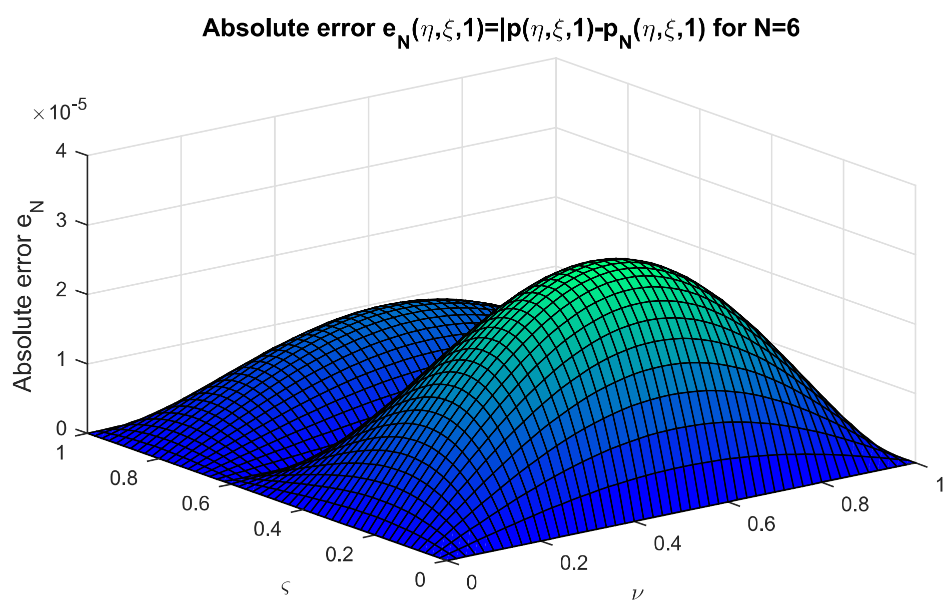

Figure 3.4: Absolute error of example 3.5.2 obtained for $N = 6$ by scheme-II.

Example 3.5.3. In Example 3.5.3, we consider the TDTE (3.6) in the domain $(\nu, \varsigma) \in \Omega$ with $\alpha = 10, \beta = 5$ and $k(\nu, \varsigma, \tau) = 22\cos(\tau)\sinh(\nu)\sinh(\varsigma) - 20\sin(\tau)\sinh(\nu)\sinh(\varsigma)$.

The initial and Dirichlet boundary conditions are given below

$$\begin{cases} p(\nu,\varsigma,0) = \sinh(\nu)\sinh(\varsigma), \\ p_\tau(\nu,\varsigma,0) = 0, \end{cases} \tag{3.35}$$

and

$$\begin{cases} p(0,\varsigma,\tau) = 0,\ p(1,\varsigma,\tau) = \cos(\tau)\sinh(1)\sinh(\varsigma), \\ p(\nu,0,\tau) = 0,\ p(\nu,1,\tau) = \cos(\tau)\sinh(\nu)\sinh(1). \end{cases} \tag{3.36}$$

The exact solution is given by

$$p(\nu,\varsigma,\tau) = \cos\tau\sinh(\nu)\sinh(\varsigma).$$

The above example is solved for $N = 3,4,5,6,7$ and 8 and the absolute error for $N = 3,4,5,6,7$ and 8 at some points of domain are given in Table 3.5 and Table 3.6 by using scheme-I and scheme-II, respectively. The error graphs for $N = 6$ are depicted through Figure 3.5 and Figure 3.6 by scheme-I and scheme-II, respectively.

Table 3.5: Absolute error $e_N(\nu, \varsigma, 1)$ for values of $N = 3, 4, 5, 6, 7$ and 8 of example 3.5.3 via scheme-I.

Grid Points	$N=3$	$N=4$	$N=5$	$N=6$	$N=7$	$N=8$
(0.0,0.0)	7.6614e-08	1.3633e-09	5.1400 e-13	7.1777 e-14	2.738e-15	2.7066e-13
(0.1,0.1)	2.6487e-04	1.9894e-04	1.8252e-04	6.4114e-05	5.7161e-05	4.9982e-05
(0.2,0.2)	1.0897e-03	7.3939e-04	6.8181e-04	2.3827e-04	2.1183e-04	1.7232e-04
(0.3,0.3)	2.2118e-03	1.4814e-03	1.3633e-03	4.7225e-04	4.1761e-04	3.208e-04
(0.4,0.4)	3.1930e-03	2.1990e-03	2.0287e-03	6.9461e-04	6.1009e-04	4.4852e-04
(0.5,0.5)	3.4858e-03	2.6452e-03	2.4595e-03	8.3089e-04	7.2415e-04	5.1560e-04
(0.6,0.6)	2.5415e-03	2.6273e-03	2.4794e-03	8.2623e-04	7.1405e-04	4.9783e-04
(0.7,0.7)	4.2872e-05	2.0796e-03	2.0251e-03	6.6753e-04	5.7193e-04	3.9447e-04
(0.8,0.8)	4.3415e-03	1.1184e-03	1.2082e-03	3.9910e-04	3.3932e-04	2.3383e-04
(0.9,0.9)	9.8864e-03	6.2624e-05	3.4478e-04	1.2588e-04	1.0702e-04	7.4434e-05
(1.0,1.0)	1.5428e-02	5.9913e-04	7.6212e-05	1.7609e-06	1.7209e-07	3.2008e-09

Table 3.6: Absolute error $e_N(\nu, \varsigma, 1)$ for values of $N = 3, 4, 5, 6, 7$ and 8 of example 3.5.3 via scheme-II.

Grid Points	$N=3$	$N=4$	$N=5$	$N=6$	$N=7$	$N=8$
(0.0,0.0)	7.6614e-08	1.3633e-09	5.080e-13	5.4830e-15	5.3737e-13	6.770e-13
(0.1,0.1)	2.6487e-04	1.98947e-04	1.8252e-04	6.4110e-05	5.7164e-05	2.4258e-04
(0.2,0.2)	1.0897e03	7.3939e-04	6.8181e-04	2.3826e-04	2.1185e-04	7.9292e-04
(0.3,0.3)	2.2118e-03	1.4814e-04	1.3633e-03	4.7223e-04	4.1768e-04	1.4063e-03
(0.4,0.4)	3.1930e-03	2.1990e-03	2.0287e-03	6.9458e-04	6.1023e-04	1.8843e-03
(0.5,0.5)	3.4858e-03	2.6452e-03	2.4595e-03	8.3086e-04	7.2435e-04	2.0903e-03
(0.6,0.6)	2.5415e-03	2.6273e-03	2.4794e-03	8.2621e-04	7.1431e-04	1.9618e-03
(0.7,0.7)	4.2872e-05	2.0796e-03	2.0251e-03	6.6751e-04	5.7218e-04	1.5225e-03
(0.8,0.8)	4.3415e-03	1.1184e-03	1.2082e-03	3.9909e-04	3.3951e-04	8.9062e-03
(0.9,0.9)	9.8864e-03	6.2624e-05	3.4478e-04	1.2588e-04	1.0709e-04	2.8179e-04
(1.0,1.0)	1.5428e-02	5.9913e-04	7.6212e-05	1.7609e-06	1.7209e-07	1.3969e-09

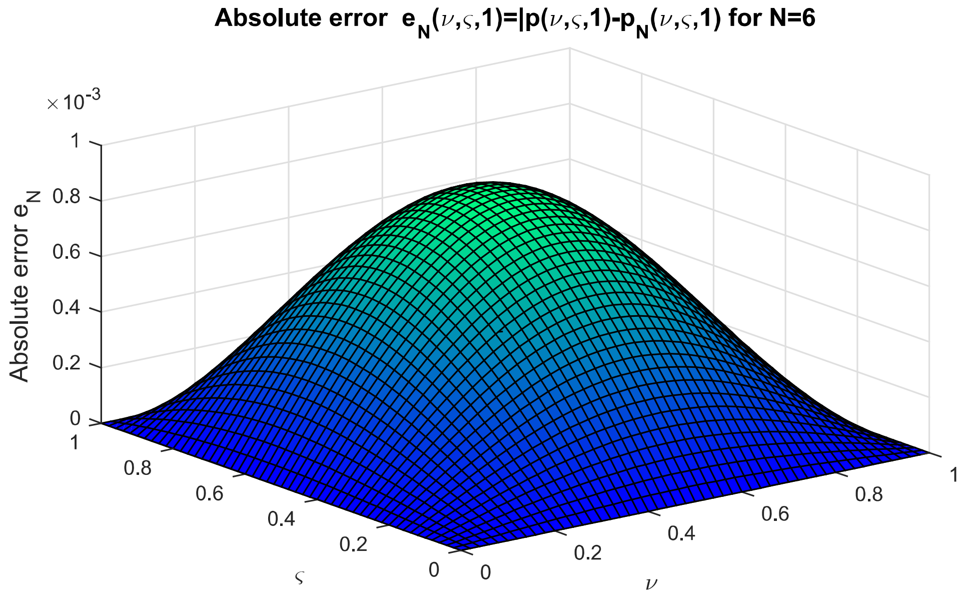

Figure 3.5: Absolute error of example 3.5.3 obtained for N=6 by scheme-I.

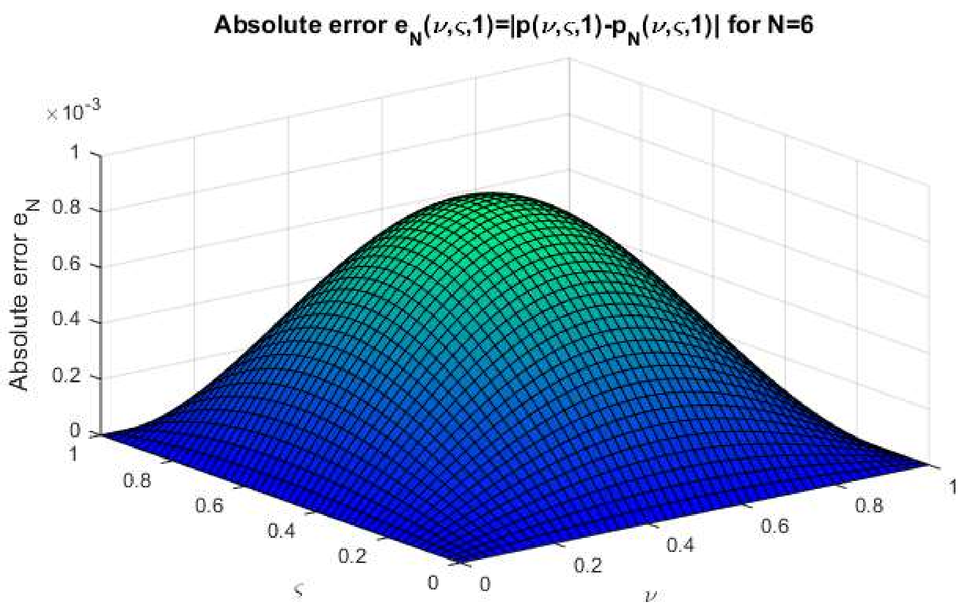

Figure 3.6: Absolute error of example 3.5.3 obtained for N=6 by scheme-II.

Table 3.7: Comparison of absolute errors with Bernoulli matrix method (BMM) [2] for $N = 6$.

x	Example 3.5.1			Example 3.5.2			Example 3.5.3		
	BMM [2]	Scheme-I	Scheme-II	BMM [2]	Scheme-I	Scheme-II	BMM [2]	Scheme-I	Scheme-II
(.1,.1)	7.7e-05	7.0e-05	7.0e-05	3.4e-06	6.6e-06	6.6e-06	6.7e-05	6.4e-05	6.4e-05
(.2,.2)	2.6e-04	2.6e-04	2.6e-04	7.9e-06	1.7e-05	1.7e-05	2.3e-04	2.3e-04	2.3e-04
(.3,.3)	5.0e-04	5.2e-04	5.2e-04	8.1e-06	2.3e-05	2.3e-05	4.3e-04	4.7e-04	4.7e-04
(.4,.4)	7.1e-04	7.7e-04	7.7e-04	3.0e-06	2.1e-05	2.1e-05	6.1e-04	6.9e-04	6.9e-04
(.5,.5)	8.3e-04	9.2e-04	9.2e-04	4.9e-06	1.2e-05	1.2e-05	7.0e-04	8.3e-04	8.3e-04
(.6,.6)	6.6e-04	9.2e-04	9.2e-04	1.2e-05	1.2e-06	1.2e-06	6.9e-04	8.2e-04	8.2e-04
(.7,.7)	4.0e-04	7.4e-04	7.4e-04	1.3e-05	7.1e-06	7.1e-06	5.5e-04	6.6e-04	6.6e-04
(.8,.8)	1.2e-04	4.4e-04	4.4e-04	1.1e-05	8.9e-06	8.9e-06	3.3e-04	3.9e-04	3.9e-04
(.9,.9)	2.8e-06	1.4e-04	1.4e-04	9.5e-05	4.8e-06	4.8e-06	9.6e-05	1.2e-04	1.2e-04

Table 3.8: Comparison of L_∞-error by scheme-I and scheme-II for $N = 7$ with B-spline differential quadrature method [125] .

Example No.	L_∞ **error by scheme-I**	L_∞ **error by scheme-II**	L_∞ **error in [125].**
3.5.1	5.1441e-04	5.1480e-04	3.3082e-03
3.5.2	2.7342e-05	2.7642e-05	2.8706e-03
3.5.3	7.4538e-04	7.4565e-04	5.6395e-04

3.6 Conclusion

The telegraph equations are of special interest as they are used to understand various physical and complex phenomena. This chapter deals with two numerical schemes namely, scheme-I and scheme-II, which are matrix techniques based on ISF and OLBF, respectively, to find the numerical solution of two-dimensional hyperbolic telegraph equation. Operational matrices based on ISF and OLBF are constructed for integration and differentiation to convert the telegraph equation to algebraic PIDEs which are solved by the BICGSTAB method. To show the accuracy and effectiveness the proposed schemes, three examples are considered whose results are compared with the results of the existing methods ([2],[125]). It can be easily seen from the obtained numerical results based on the error figures (Figures 3.1, 3.3, 3.5 for scheme-I and Figures 3.2, 3.4, 3.6 for scheme-II) and tables (Tables 3.1, 3.3, 3.5 for scheme-I and Tables 3.2, 3.4, 3.6 for scheme-II) that the proposed schemes are very accurate and efficient for small number of basis functions. Also, the L_∞-error of the proposed schemes are compared with the existing scheme [125]. The advantage of the scheme-II over the scheme-I is that one can construct the orthonormal polynomial without finding the roots of the Legendre polynomial. In this chapter, theoretically error bounds for the solution approximation are not provided, which is one of the aims and a topic for the future study.

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