Chapter 2

Fractional Order Operational Matrix Methods for Fractional Singular Integro-Differential Equation

2.1 Introduction

In this chapter, we obtain the numerical method for the fractional order singular volterra integro-differential equations which are typical for the theory of Brownian motion based on the fractional Langevin equation and Basset force. Fractional order singular integro-differential equation appears in the theory of classical Brownian motion based on the fractional Langevin equation which is introduced by Kubo (1966). One of the physical applications of the fractional order integro differential equation based on Basset force (Mainardi and Pironi (1996)) is directly connected with the Caupto Fractional derivative of half order and is given as:

$$F_v = -6\pi a \rho_f v \left\{ V(t) + \frac{a}{\sqrt{\pi v}} \int_{-\infty}^{t} \frac{dV(\tau)/d\tau}{\sqrt{t-\tau}} d\tau \right\}.$$

Singular integro-differential equation has many applications in several fields like, magnetic field induction in dielectric media (Tarasov (2009)), anomalous diffusion (Sokolov and Klafter (2005)), micro and nanotechnology (Grzybowski *et al.* (2005)), velocity fluctuation of a hard-core Brownian particle (Widom (1971)), transfer of dust and fog particles in atmosphere (Aartijk and Clercx (2010)). Analytical evaluation of singular integro-differential equation of fractional order is difficult so the numerical methods are important. There exist many numerical algorithms to find the solution of singular integro-differential equation of fractional order as well as for the integer order. All the existing numerical methods like Legendre approximation method (Khater *et al.* (2007)), variable coefficient quadratic approximation (Shamardan (1989)), Mckee scheme (Mckee (1979)), exponentially fitted scheme (El-Gendi *et al.* (1988)) and spline collocation method with Gauss and Radau points (Brunne and Tang (1989)) are based on the multi-step process. These methods required a sufficiently dense discretization mesh shell and many iterations for solving large systems of algebraic equations, i.e. these methods need high calculation cost.

The construction of operational matrices and their applications to solve integral and differential equations are given by the authors (Saadatmandi and Dehghan (2010), Saadatmandi and Dehghan (2011), Eslahchi and Dehghan (2011), Lakestani *et al.* (2012), Abdi-mazraeh *et al.* (2014), Saadatmandi and Dehghan (2011)).Recently, Singh and Postnikov (2013) have also given the method based on operational and almost operational matrices for singular integro-differential equation of integer order only. The aim of this chapter is to find the numerical methods for singular integro-differential equation of fractional order based on fractional operational matrices of integration and fractional operational matrices of differentiation. The present alogorithm is followed by an error analysis in which we have given an upper bound for the error involved.

2.2 Some Basic Preliminaries of Legendre Scaling Function

There are several definitions of fractional order derivatives and integrals. These are not necessarily equivalent. In this chapter, the fractional order differentiations and integrations are used in Caputo and Riemann-Liouville sense. This means that the operational matrices of fractional order integration and differentiation are derived by using the Riemann-Liouville fractional order integral operator and the Caputo fractional derivative operator respectively.

Definition 2.2.1. The Riemann-Liouville fractional order integral operator is given as (Miller and Ross (1993)):

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt \qquad \alpha > 0, x > 0,$$

$$I^0 f(x) = f(x).$$

For the fractional Riemann-Liouville integration

$$I^{\alpha} x^k = \frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)} x^{k+\alpha}.$$

Definition 2.2.2. The Caputo fractional derivative of order β is defined as

$$D^{\beta} f(x) = I^{m-\beta} D^m f(x) = \frac{1}{\Gamma(m-\beta)} \int_0^x (x-t)^{m-\beta-1} \frac{d^m}{dt^m} f(t)dt, \quad m-1 < \beta < m, x > 0.$$

For the Caputo derivative, we have (Diethelm *et al.* (2005)):

$$D^{\beta} A = 0 \quad (A \text{ is a constant}),$$

$$D^{\beta} x^k = \begin{cases} \dfrac{\Gamma(k+1)}{\Gamma(k+1-\beta)} x^{k-\beta}, & \text{for } k \in N_0 \text{ and } k \geq \lceil \beta \rceil \text{ or } k \in N \text{ and } k \in \lfloor \beta \rfloor; \\ 0 & , \text{ for } k \in N_0 \text{ and } k < \lceil \beta \rceil, \end{cases}$$

where $\lceil \beta \rceil$ and $\lfloor \beta \rfloor$ are the ceiling and floor functions respectively, while $\mathrm{N} = \{1,2,3,......\}$ and $\mathrm{N}_0 = \{0,1,2,......\}$.

The Caputo fractional differentiation is a linear operator similar to the integer order differentiation.

The Legendre scaling functions $\{\phi_i(t)\}$ are defined by

$$\phi_i(t) = \begin{cases} \sqrt{(2i+1)} P_i(2t-1), & \text{for } 0 \leq t < 1. \\ 0, & \text{otherwise}, \end{cases}$$

where $P_i(t)$ is Legendre polynomial of order i which are orthogonal on the interval $[-1,1]$ with respect to the weight function $w(t)=1$. Legendre scaling functions are constructed by normalizing the shifted Legendre polynomials. So the collection $\{\phi_i(t)\}$ forms an orthonormal basis for $L^2[0,1]$. A function $f \in L^2[0,1]$, with bounded second derivative $\left|f^{''}(t)\right| \le M$, may be expanded as infinite sum of Legendre scaling function and the series converges uniformly to the function $f(t)$ i.e.

$$f(t)=\lim_{n\to\infty}\sum\nolimits_{i=0}^{n} c_i\phi_i(t), \tag{2.2.1}$$

where, $c_i=\langle f(t),\phi_i(t)\rangle$, and $\langle .,.\rangle$ is standard inner product on $L^2[0,1]$.

If the series is truncated at $n=m$, then we have

$$f \cong \sum\nolimits_{i=0}^{m} c_i\phi_i = C^T\Phi(t), \tag{2.2.2}$$

where, C and $\Phi(t)$ are $(m+1)\times 1$ matrices given by

$$C=[c_0,c_1,....,c_m]^T \text{ and } \Phi(t)=[\phi_0(t),\phi_1(t),....,\phi_m(t)]^T. \tag{2.2.3}$$

The Legendre scaling function of degree i is given by

$$\phi_i(t)=(2i+1)^{\frac{1}{2}}\sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!}{(i-k)!}\frac{t^k}{(k!)^2}, \tag{2.2.4}$$

where,

$$D^{\beta}\phi_i(t)=0, \qquad i=0,1,........,\lceil\beta\rceil-1, \quad \beta>0. \tag{2.2.5}$$

2.3 Operational Matrices for Fractional Integration and Differentation

Theorem 2.3.1. Let $\Phi(t)$ be a Legendre scaling vector defined in Eqs. (2.2.3) and (2.2.4), then

$$\int_0^x \frac{\phi_i(t)}{(x-t)^b}dt = P^{(b)}\Phi(x), \qquad 0<b<1, \tag{2.3.1}$$

where, $P^{(b)}$ is the $(N+1)\times(N+1)$ operational matrix of integration for singular Volterra type integral of order b and is given by

$$P^{(b)} = \begin{pmatrix} \Pi(0,0) & \Pi(0,1) & \Pi(0,2) & . & . & . & . & \Pi(0,N) \\ \Pi(1,0) & \Pi(1,1) & \Pi(1,2) & . & . & . & . & \Pi(1,N) \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ \Pi(i,0) & \Pi(i,1) & \Pi(i,2) & . & . & . & . & \Pi(i,N) \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ \Pi(N,0) & \Pi(N,1) & \Pi(N,2) & . & . & . & . & \Pi(N,N) \end{pmatrix}; \tag{2.3.2}$$

where,

$$\Pi(i,j) = (2i+1)^{1/2}(2j+1)^{1/2}\sum_{k=0}^{i}\sum_{l=0}^{j}(-1)^{i+j+k+l}\frac{(i+k)!(j+l)!\Gamma(1-b)}{(i-k)!(j-l)!(k)!(l!)^2(l+k+2-b)\Gamma(2-b+k)};$$

$i=0,1,2,\ldots\ldots\ldots,N$ and $j=0,1,2,\ldots\ldots\ldots,N.$

Proof.

By using the closed form of Legendre scaling function of degree i, we get

$$\int_0^x \frac{\phi_i(t)}{(x-t)^b}dt = (2i+1)^{1/2}\sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!}{(i-k)!}\frac{1}{(k!)^2}\int_0^x \frac{t^k}{(x-t)^b}dt$$

$$= (2i+1)^{1/2}\sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!}{(i-k)!(k)!}\frac{\Gamma(1-b)}{\Gamma(2-b+k)}x^{1-b+k}. \tag{2.3.3}$$

Using Legendre scaling function approximation for x^{1-b+k}, we have

$$x^{1-b+k} = \sum_{j=0}^{N} c_j \phi_j(x), \tag{2.3.4}$$

where ,
$$c_j = (2j+1)^{1/2} \sum_{l=0}^{j} (-1)^{l+j} \frac{(j+l)!}{(j-l)!} \frac{1}{(l!)^2} \frac{1}{(l+k+2-b)}. \tag{2.3.5}$$

By using Eqs. (2.3.4) and (2.3.5), the Eq. (2.3.3) may be written as

$$\int_0^x \frac{\phi_i(t)}{(x-t)^b} dt = \sum_{j=0}^{N} \Pi(i,j) \phi_j(x), \tag{2.3.6}$$

where,

$$\Pi(i,j) = (2i+1)^{1/2}(2j+1)^{1/2} \sum_{k=0}^{i} \sum_{l=0}^{j} (-1)^{i+j+k+l} \frac{(i+k)!(j+l)!\Gamma(1-b)}{(i-k)!(j-l)!(k)!(l!)^2 (l+k+2-b)\Gamma(2-b+k)}. \tag{2.3.7}$$

Hence, we have

$$\int_0^x \frac{\phi_i(t)}{(x-t)^b} dt = [\Pi(i,0), \Pi(i,1), \Pi(i,2), \ldots\ldots., \Pi(i,N)] \Phi(x). \tag{2.3.8}$$

Theorem 2.3.2. Let $\Phi(x)$ be Legendre scaling vector. Then for $\alpha > 0$,

$$I^{\alpha} \phi_i(x) = I^{(\alpha)} \Phi(x), \tag{2.3.9}$$

where $I^{(\alpha)}$ is $(N+1) \times (N+1)$ operational matrix of fractional integral of order α given by

$$I^{(\alpha)} = \begin{pmatrix} \omega(0,0) & \omega(0,1) & \omega(0,2) & . & . & . & \omega(0,N) \\ \omega(1,0) & \omega(1,1) & \omega(1,2) & . & . & . & \omega(1,N) \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \omega(i,0) & \omega(i,1) & \omega(i,2) & . & . & . & \omega(i,N) \\ . & . & . & . & . & . & . \\ \omega(N,0) & \omega(N,1) & \omega(N,2) & . & . & . & \omega(N,N) \end{pmatrix}; \tag{2.3.10}$$

where,

$$\omega(i,j)=(2i+1)^{1/2}(2j+1)^{1/2}\sum_{k=0}^{i}\sum_{l=0}^{j}(-1)^{i+j+k+l}\frac{(i+k)!(j+l)!}{(i-k)!(j-l)!(k)!(l!)^2(\alpha+k+l+1)\Gamma(\alpha+k+1)},$$

$i=0,1,2,........,N$ and $j=0,1,2,........,N$.

Proof.

Using the Legendre scaling function of degree i, we get

$$I^{\alpha}\phi_i(x)=(2i+1)^{1/2}\sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!}{(i-k)!}\frac{1}{(k!)^2}I^{\alpha}x^k$$

$$=(2i+1)^{1/2}\sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!}{(i-k)!(k)!\Gamma(\alpha+k+1)}x^{\alpha+k}. \tag{2.3.11}$$

Replacing $x^{\alpha+k}$ by its Legendre scaling function approximation given by

$$x^{\alpha+k}=\sum_{j=0}^{N}c_j\phi_j(x), \tag{2.3.12}$$

with $$c_j=(2j+1)^{1/2}\sum_{l=0}^{j}(-1)^{l+j}\frac{(j+l)!}{(j-l)!}\frac{1}{(l!)^2}\frac{1}{(\alpha+k+l+1)}, \tag{2.3.13}$$

the Eq. (2.3.11) may be written as

$$I^{\alpha}\phi_i(x)=\sum_{j=0}^{N}\omega(i,j)\phi_j(x), \tag{2.3.14}$$

where

$$\omega(i,j)=(2i+1)^{1/2}(2j+1)^{1/2}\sum_{k=0}^{i}\sum_{l=0}^{j}(-1)^{i+j+k+l}\frac{(i+k)!(j+l)!}{(i-k)!(j-l)!(k)!(l!)^2(\alpha+k+l+1)\Gamma(\alpha+k+1)}. \tag{2.3.15}$$

Thus, we have

$$I^{\alpha}\phi_i(x)=[\omega(i,0),\omega(i,1),\omega(i,2),.......,\omega(i,N)]\Phi(x). \tag{2.3.16}$$

Theorem 2.3.3. If $\Phi(x)$ be the Legendre scaling vector, then for $\beta > 0$,

$$D^{\beta}\phi_i(x) = D^{(\beta)}\Phi(x), \tag{2.3.17}$$

where $D^{(\beta)}$ is $(N+1)\times(N+1)$ operational matrix of Caputo fractional derivative of order β given by

$$D^{(\beta)} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ \eta(\lceil\beta\rceil,0) & \eta(\lceil\beta\rceil,1) & \eta(\lceil\beta\rceil,2) & \cdots & \eta(\lceil\beta\rceil,N) \\ \vdots & \vdots & \vdots & & \vdots \\ \eta(i,0) & \eta(i,1) & \eta(i,2) & \cdots & \eta(i,N) \\ \vdots & \vdots & \vdots & & \vdots \\ \eta(N,0) & \eta(N,1) & \eta(N,2) & \cdots & \eta(N,N) \end{pmatrix}; \tag{2.3.18}$$

with

$$\eta(i,j) = (2i+1)^{1/2}(2j+1)^{1/2}\sum_{k=\lceil\beta\rceil}^{i}\sum_{l=0}^{j}(-1)^{i+j+k+l}\frac{(i+k)!(j+l)!}{(i-k)!(j-l)!(k)!(l!)^2(k+l+1-\beta)\Gamma(k+1-\beta)},$$

$i = \lceil\beta\rceil,\ldots,N$ and $j = 0,1,2,\ldots,N$.

Proof.

Using the Legendre scaling function of degree i, we get

$$D^{\beta}\phi_i(x) = (2i+1)^{1/2}\sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!}{(i-k)!}\frac{1}{(k!)^2}D^{\beta}x^k$$

$$= (2i+1)^{1/2}\sum_{k=\lceil\beta\rceil}^{i}(-1)^{i+k}\frac{(i+k)!}{(i-k)!(k)!\Gamma(k+1-\beta)}x^{k-\beta}, i = \lceil\beta\rceil,\ldots,N. \tag{2.3.19}$$

Similary,

$$x^{k-\beta}=\sum_{j=0}^{N}b_j\phi_j(x), \tag{2.3.20}$$

where, $$b_j=(2j+1)^{1/2}\sum_{l=0}^{j}(-1)^{l+j}\frac{(j+l)!}{(j-l)!}\frac{1}{(l!)^2}\frac{1}{(k+l+1-\beta)}. \tag{2.3.21}$$

So, using Eqs. (2.3.20) and (2.3.22), Eq. (2.3.19) becomes

$$D^{\beta}\phi_i(x)=\sum_{j=0}^{N}\eta(i,j)\phi_j(x), \tag{2.3.22}$$

where,

$$\eta(i,j)=(2i+1)^{1/2}(2j+1)^{1/2}\sum_{k=\lceil\beta\rceil}^{i}\sum_{l=0}^{j}(-1)^{i+j+k+l}\frac{(i+k)!(j+l)!}{(i-k)!(j-l)!(k)!(l!)^2(k+l+1-\beta)\Gamma(k+1-\beta)}. \tag{2.3.23}$$

So, we have

$$D^{\beta}\phi_i(x)=[\eta(i,0),\eta(i,1),\eta(i,2),.......,\eta(i,N)]\Phi(x). \tag{2.3.24}$$

2.4 Error Analysis

Theorem 2.4.1. If $D^{\alpha}f(x)$ and its second derivative are bounded and $D^{\alpha}f_N(x)$ is the truncated part of $D^{\alpha}f(x)$ at nth term when expanded using Legendre scaling functions then the upper bound of error is given by

$$\left\|D^{\alpha}f(x)-D^{\alpha}f_N(x)\right\|_E^2\le\frac{M^2}{256}F_3(-\frac{3}{2}+N), \tag{2.4.1}$$

where, $\|f(x)\|_E=\left(\int_0^1|f(x)|^2\,dx\right)^{\frac{1}{2}}$ and $\left|D^{\alpha+2}f(x)\right|\le M$, $M>0$.

$F_n(z)$ is Polygamma function as defined by (Abramowitz and Stegun (2005)):

$$F_n(z)=(-1)^{n+1}\lfloor n\sum_{k=0}^{\infty}\frac{1}{(z+k)^{n+1}}$$

Proof.

$$\text{Let } D^{\alpha} f(x) = \sum_{i=0}^{\infty} c_i \phi_i(x). \tag{2.4.2}$$

Truncating the above series upto level $N-1$, we have

$$D^{\alpha} f_N(x) = \sum_{i=0}^{N-1} c_i \phi_i(x).$$

So,

$$\left\| D^{\alpha} f(x) - D^{\alpha} f_N(x) \right\|_E^2 = \int_0^1 \left(D^{\alpha} f(x) - D^{\alpha} f_N(x) \right)^2 dx$$

$$= \int_0^1 \left(\sum_{i=N}^{\infty} c_i \phi_i(x) \right)^2 dx$$

$$= \sum_{i=N}^{\infty} c_i^2 \text{ (using orthonormality of } \{\phi_i\}), \tag{2.4.3}$$

Now, c_i in Eq. (2.4.3) may be calculated as

$$c_i = \left\langle D^{\alpha} f(x), \phi_i(x) \right\rangle = \int_0^1 D^{\alpha} f(x) \phi_i(x) dx = \int_0^1 D^{\alpha} f(x) \sqrt{(2i+1)} P_i(2x-1) dx.$$

By changing the variable x as $2x-1=t$, we have

$$c_i = \int_{-1}^1 D^{\alpha} f\left(\frac{t+1}{2}\right) \sqrt{(2i+1)} \frac{P_i(t)}{2} dt$$

$$= \sqrt{\left(\frac{2i+1}{2^2}\right)} \int_{-1}^1 D^{\alpha} f\left(\frac{t+1}{2}\right) P_i(t) dt$$

$$= \sqrt{\left(\frac{1}{2^2(2i+1)}\right)} \int_{-1}^1 D^{\alpha} f\left(\frac{t+1}{2}\right) d(P_{i+1}(t) - P_{i-1}(t)) dt$$

$$= -\frac{1}{4} \sqrt{\left(\frac{1}{(2i+1)}\right)} \int_{-1}^1 D^{\alpha+1} f\left(\frac{t+1}{2}\right) (P_{i+1}(t) - P_{i-1}(t)) dt$$

$$=-\frac{1}{4}\sqrt{\left(\frac{1}{(2i+1)}\right)}\int_{-1}^{1} D^{\alpha+1} f\left(\frac{t+1}{2}\right) d\left(\frac{P_{i+2}(t)-P_i(t)}{2i+3}-\frac{P_i(t)-P_{i-2}(t)}{2i-1}\right) dt$$

$$=\frac{1}{8}\sqrt{\left(\frac{1}{(2i+1)}\right)}\int_{-1}^{1} D^{\alpha+2} f\left(\frac{t+1}{2}\right)\left(\frac{P_{i+2}(t)-P_i(t)}{2i+3}-\frac{P_i(t)-P_{i-2}(t)}{2i-1}\right) dt.$$

Thus

$$\left|c_i\right|^2=\frac{1}{64}\left(\frac{1}{2i+1}\right)\left|\int_{-1}^{1} D^{\alpha+2} f\left(\frac{t+1}{2}\right)\left(\frac{(2i-1)P_{i+2}(t)-(4i+2)P_i(t)+(2i+3)P_{i-2}(t)}{(2i+3)(2i-1)}\right) dt\right|^2.$$

Using Cauchy-Schwartz inequality in an inner product space, we have

$$\left|c_i\right|^2 \le \frac{1}{64}\left(\frac{1}{2i+1}\right)\int_{-1}^{1}\left|D^{\alpha+2} f\left(\frac{t+1}{2}\right)\right|^2 dt \int_{-1}^{1}\left|\frac{(2i-1)P_{i+2}(t)-(4i+2)P_i(t)+(2i+3)P_{i-2}(t)}{(2i+3)(2i-1)}\right|^2 dt$$

$$\le \frac{M^2}{32}\left(\frac{1}{2i+1}\right)\int_{-1}^{1}\left|\frac{(2i-1)P_{i+2}(t)-(4i+2)P_i(t)+(2i+3)P_{i-2}(t)}{(2i+3)(2i-1)}\right|^2 dt$$

$$< \frac{M^2}{32}\left(\frac{1}{2i+1}\right)\int_{-1}^{1}\frac{(2i-1)^2 P_{i+2}^2(t)+(4i+2)^2 P_i^2(t)+(2i+3)^2 P_{i-2}^2(t)}{(2i+3)^2(2i-1)^2} dt$$

$$=\frac{M^2}{32(2i+1)(2i-1)^2(2i+3)^2}\left[\frac{2(2i-1)^2}{2i+5}+\frac{8(2i+1)^2}{2i+1}+\frac{2(2i+3)^2}{2i-3}\right]$$

$$<\frac{M^2}{32(2i+1)(2i-1)^2(2i+3)^2}\frac{12(2i+3)^2}{(2i-3)}$$

$$<\frac{3M^2}{8(2i-3)^4}.$$

So,

$$\sum_{i=N}^{\infty} c_i^2 < \sum_{i=N}^{\infty}\frac{3M^2}{8(2i-3)^4}$$

$$= \frac{3M^2}{8} \sum_{i=N}^{\infty} \frac{1}{(2i-3)^4}$$

$$= \frac{3M^2}{8} \left(\frac{1}{96} \right) F_3(-\frac{3}{2}+N)$$

$$= \frac{M^2}{256} F_3(-\frac{3}{2}+N) .$$

Hence, Eq. (2.4.1) gives

$$\left\| D^{\alpha} f(x) - D^{\alpha} f_N(x) \right\|_E^2 \leq \frac{M^2}{256} F_3(-\frac{3}{2}+N).$$

2.5 Methods of Solution

Type 1:

A special form of Abel's integral equation is given by Singh *et al.* (2012)

$$I(y) = 2\int_y^1 \frac{\varepsilon(r) r}{(r^2 - y^2)^b} dr, \qquad 0 \leq y \leq 1, 0 < b < 1. \tag{2.5.1}$$

By change of variables, the Abel integral Eq. (2.5.1) reduces to

$$I(\sqrt{y}) = \int_y^1 \frac{\varepsilon(\sqrt{r})}{(r-y)^b} dr, \qquad 0 \leq y \leq 1, 0 < b < 1.$$

Which may be written as

$$I_1(y) = \int_y^1 \frac{\eta(r)}{(r-y)^b} dr , \quad \text{where } I_1(y) = I(\sqrt{y}) \text{ and } \eta(r) = \varepsilon(\sqrt{r}).$$

In above form of Abel integral we also introduce a fractional derivative of order $0 < \beta < 1$ inside the integral then we have

$$I_2(y) = \int_y^1 \frac{D^{\beta}\eta(r)}{(r-y)^b} dr , \text{ with initial condition } \eta(0) = a, \ a \text{ is constant.} \tag{2.5.2}$$

Approximating various terms involved in the above integral by their Legendre's scaling function approximations as

$$D^{\beta}\eta(r)=C^{T}\phi(r)\ ,\ \eta(0)=a=A^{T}\phi(r)\ \text{ and } I_{2}(y)=F^{T}\phi(y), \tag{2.5.3}$$

and integrating $D^{\beta}\eta(r)=C^{T}\phi(r)$, we get

$$\eta(r)=C^{T}I^{(\beta)}\phi(r)+A^{T}\phi(r). \tag{2.5.4}$$

Using the Eqs. (2.5.2) and (2.5.3), we have

$$F^{T}\phi(y)=\int_{y}^{1}\frac{C^{T}\phi(r)}{(r-y)^{b}}dr. \tag{2.5.5}$$

Thus
$$\int_{y}^{1}\frac{\phi(r)}{(r-y)^{b}}dr=Q\phi(y), \tag{2.5.6}$$

where $Q=P^{T}$ is $(N+1)\times(N+1)$ matrix and P is an operational matrix of integration for singular Volterra type integral equation.

From Eq. (2.5.5) and (2.5.6), we have

$$C^{T}=F^{T}Q^{-1}. \tag{2.5.7}$$

From Eqs. (2.5.4) and (2.5.7), the approximate solution $\eta(r)\ (=\varepsilon(\sqrt{r}))$ for the Abel integral Eq. (2.5.4) is given by

$$\eta(r)=(F^{T}Q^{-1}I^{(\beta)}+A^{T})\phi(r).$$

Type 2:

This sub-section deals with solution of generalized Abel integral equation (Chakrabarti (2008)) of the kind

$$a(y)\int_{\alpha}^{y}\frac{r^{\gamma-1}\varepsilon(r)dr}{(y^{\gamma}-r^{\gamma})^{\mu}}+b(y)\int_{y}^{\beta}\frac{r^{\gamma-1}\varepsilon(r)dr}{(r^{\gamma}-y^{\gamma})^{\mu}}=I(y), \text{where } (0<\mu<1), (\alpha\le y\le\beta), \tag{2.5.8}$$

where the coefficients $a(y)$ and $b(y)$ do not vanish simultaneously.

In a special case when $\alpha=0, \beta=1$ and $\gamma=2$, by changing the variables, the Abel integral Eq. (2.5.8) reduces to

$$I(\sqrt{y})=\frac{a(\sqrt{y})}{2}\int_0^y \frac{\varepsilon(\sqrt{r})}{(y-r)^\mu}dr+\frac{b(\sqrt{y})}{2}\int_y^1 \frac{\varepsilon(\sqrt{r})}{(r-y)^\mu}dr, \quad 0\le y\le 1.$$

Which may be written as

$$I_1(y)=a_1(y)\int_0^y \frac{\eta(r)}{(y-r)^\mu}dr+b_1(y)\int_y^1 \frac{\eta(r)}{(r-y)^\mu}dr.$$

Now we introduce a fractional derivative of order $0<\beta<1$ inside the integrals involved in the above form of generalized Abel integral equation and solve it using the algorithm devolped so for. After introducing the fractional derivatives, the above integral equation takes the shape of following fractional integro-differential equation:

$$I_2(y)=a_1(y)\int_0^y \frac{D^\beta\eta(r)}{(y-r)^\mu}dr+b_1(y)\int_y^1 \frac{D^\beta\eta(r)}{(r-y)^\mu}dr, \;\; 0\le y\le 1, \text{with } \eta(0)=a. \tag{2.5.9}$$

Again approximating various terms involved in the above integral by their Legendre's scaling function approximations as

$$D^\beta\eta(r)=C^T\phi(r) \text{ and } \eta(0)=a=A^T\phi(r) \text{ and } I_2(y)=F^T\phi(y), \tag{2.5.10}$$

we have

$$\eta(r)=C^T I^{(\beta)}\phi(r)+A^T\phi(r). \tag{2.5.11}$$

From Eqs. (2.5.9) and (2.5.10), we get

$$F^T\phi(y)=a_1(y)\int_0^y \frac{C^T\phi(r)}{(y-r)^\mu}dr+b_1(y)\int_y^1 \frac{C^T\phi(r)}{(r-y)^\mu}dr, \quad 0\le y\le 1. \tag{2.5.12}$$

So

$$\int_0^y \frac{\phi(r)}{(y-r)^\mu}dr=P\phi(y) \text{ and } \int_y^1 \frac{\phi(r)}{(r-y)^\mu}dr=Q\phi(y), \tag{2.5.13}$$

where P and Q are the matrices taken as in type 1.

Grouping Eqs. (2.5.12) and (2.5.13) yields:

$$C^T = F^T(a_1(y)P + b_1(y)Q)^{-1}. \tag{2.5.14}$$

Then the approximate solution $\eta(r) = \varepsilon(\sqrt{r})$ for the generalized Abel fractional integro differential equation (2.5.9) is given by

$$\eta(r) = (F^T(a_1(y)P + b_1(y)Q)^{-1} I^{(\beta)} + A^T)\phi(r).$$

Type 3:

In this sub-section the third type of fractional integro differential equations with general form

$$D^\alpha y(t) = f(t) + \int_0^t \frac{D^\beta y(x)}{(t-x)^\gamma} dx, \quad 0<\alpha<1,\ 0<\beta<1,\ 0<\gamma<1 \text{ and } y(0)=a, \tag{2.5.15}$$

are considered for their numerical solutions.

If $D^\alpha y(t)$, $f(t)$ and $y(0)$ are approximated as

$$D^\alpha y(t) = C^T\phi(t),\ y(0) = a = A^T\phi(t) \text{ and } f(t) = F^T\phi(t), \tag{2.5.16}$$

then from Eq. (2.5.16), we have

$$y(t) = (C^T I^{(\alpha)} + A^T)\phi(t). \tag{2.5.17}$$

Using Eqs. (2.5.16) and (2.5.17), we get

$$\begin{aligned} C^T\phi(t) &= F^T\phi(t) + \int_0^t \frac{D^\beta((C^T I^{(\alpha)} + A^T)\phi(x))}{(t-x)^\gamma} dx \\ &= F^T\phi(t) + (C^T I^{(\alpha)} D^{(\beta)} + A^T D^{(\beta)}) \int_0^t \frac{\phi(x)}{(t-x)^\gamma} dx. \end{aligned}$$

So,

$$C^T = (F^T + A^T D^{(\beta)} P^{(\gamma)})(I - I^{(\alpha)} D^{(\beta)} P^{(\gamma)})^{-1}, \tag{2.5.18}$$

and the approximate solution of Eq. (2.5.15) is given by

$y(t) = (((F^T + A^T D^{(\beta)} P^{(\gamma)})(I - I^{(\alpha)} D^{(\beta)} P^{(\gamma)})^{-1}) I^{(\alpha)} + A^T)\phi(t)$ (using Eqs. (2.5.17) and (2.5.18)).

Type 4:

Type 4 deals with the singular fractional Volterra integro -differential equation of the type :

$$D^\alpha y(t) = f(t) + \int_0^t \frac{y(x)}{(t-x)^\gamma} dx, 0 < \gamma < 1, \alpha > 0 \text{ and } y^{(i)}(0) = d_i, i = 0,1,2,\ldots.,m-1, \tag{2.5.19}$$

where, $m-1 < \alpha \le m$ and $d_i\,(i = 0,1,2,\ldots\ldots.,m-1)$ describe the initial state $y(x)$.

Approximating $f(t)$ and the Riemann-Liouville differential $D^\alpha y(t)$ of order α as $F^T\phi(t)$ and $C^T\phi(t)$ respectively , we get

$$y(t) - \sum_{i=0}^{m-1} y^{(i)}(0^+)\frac{t^i}{i!} = C^T I^{(\alpha)}\phi(t). \tag{2.5.20}$$

If we replace $\sum_{i=0}^{m-1} y^{(i)}(0^+)\frac{t^i}{i!}$ by its Legendre's scaling functions as $\sum_{i=0}^{m-1} y^{(i)}(0^+)\frac{t^i}{i!} = G^T\phi(t)$, Eq. (2.5.20) becomes

$$y(t) = (C^T I^{(\alpha)} + G^T)\phi(t). \tag{2.5.21}$$

Thus

$$C^T\phi(t) = F^T\phi(t) + \int_0^t \frac{(C^T I^{(\alpha)} + G^T)\phi(x)}{(t-x)^\gamma} dx,$$

and hence

$$C^T = (F^T + G^T P^{(\gamma)})(I - I^{(\alpha)} P^{(\gamma)})^{-1}. \tag{2.5.22}$$

Now using Eqs. (2.5.21) and (2.5.22), the approximate solution of Eq. (2.5.19) is obtained as

$$y(t) = (((F^T + G^T P^{(\gamma)})(I - I^{(\alpha)} P^{(\gamma)})^{-1}) I^{(\alpha)} + G^T)\phi(t) .$$

Type 5 :

In type 5 we will combine type 3 and 4 to get a generalized fractional integro-differential equation of the form

$$D^{\alpha} y(t) = f(t) + \int_0^t \frac{D^{\beta} y(x)}{(t-x)^{\gamma}} dx, 1 \le \beta < \alpha < 3, 0 < \gamma < 1, y^{(i)}(0) = d_i, i = 0,1,..,m-1, \tag{2.5.23}$$

where $m-1 < \alpha \le m$ and $d_i\,(i = 0,1,..,m-1)$ describe the initial state of $y(x)$.

Approximating $f(t)$ and the Riemann-Liouville differential $D^{\alpha} y(t)$ of order α as $F^T \phi(t)$ and $C^T \phi(t)$ respectively , we get

$$y(t) - \sum_{i=0}^{m-1} y^{(i)}(0^+) \frac{t^i}{i!} = C^T I^{(\alpha)} \phi(t). \tag{2.5.24}$$

If we replace $\sum_{i=0}^{m-1} y^{(i)}(0^+) \frac{t^i}{i!}$ by its Legendre's scaling functions as $\sum_{i=0}^{m-1} y^{(i)}(0^+) \frac{t^i}{i!} = G^T \phi(t)$, Eq. (2.5.24) becomes

$$y(t) = (C^T I^{(\alpha)} + G^T)\phi(t). \tag{2.5.25}$$

Taking fractional derivative of order β on both side of Eq. (2.5.25), we get

$$D^{\beta} y(t) = (C^T I^{(\alpha)} D^{(\beta)} + G^T D^{(\beta)})\phi(t). \tag{2.5.26}$$

Using Eqs. (2.5.25) and (2.5.26) in Eq. (2.5.23), we obtain

$$C^T \phi(t) = F^T \phi(t) + \int_0^t \frac{(C^T I^{(\alpha)} D^{(\beta)} + G^T D^{(\beta)})\phi(x)}{(t-x)^{\gamma}} dx$$

$$C^T = (F^T + G^T D^{(\beta)} P^{(\gamma)})(I - I^{(\alpha)} D^{(\beta)} P^{(\gamma)})^{-1}. \tag{2.5.27}$$

Approximate solution of Eq. (2.5.23) is obtained by using Eqs. (2.5.25) and (2.5.27). Hence we have

$$y(t) = ((F^T + G^T D^{(\beta)} P^{(\gamma)})(I - I^{(\alpha)} D^{(\beta)} P^{(\gamma)})^{-1} I^{(\alpha)} + G^T)\phi(t).$$

2.6 Numerical Results

This section deals with the implementation of proposed numerical algorithms based on fractional operational matrices of integration and differentiation of Legendre's scaling function. The devolped algorithm is applied on different test functions for numerical solutions and its accuracy is investigated as compared to other known or exact solutions. A comparison between approximate solution obtained by the present algorithm and the exact known solutions is achieved through absolute error graphs. In these absolute error graphs $E1$ (blue line), $E2$ (red line) and $E3$ (green line) denote the absolute value of the difference of exact solutions and the approximate solutions for the different level of approxmations n =3, 6 and 11 respectively.

Example 2.6.1. Consider the Abel integral Eq. (2.5.2) with

$$\eta(r) = r^2 \text{ for } 0 \le r \le 1,$$

$$I_2(y) = \frac{2}{\Gamma\left(\frac{9}{4}\right)}\left[\frac{4}{7}(1-y)^{\frac{1}{2}} + \frac{\sqrt{\pi}\, y^{\frac{7}{4}}\,\Gamma\left(\frac{-7}{4}\right)}{\Gamma\left(\frac{-5}{4}\right)} + \frac{20}{21} y\, {}_2F_1\left(\frac{-3}{4},\frac{1}{2},\frac{1}{4},y\right)\right], \text{ for } 0 < y < 1.$$

With Initial condition $\eta(0) = 0$, $\beta = \frac{3}{4}$ and $b = \frac{1}{2}$.

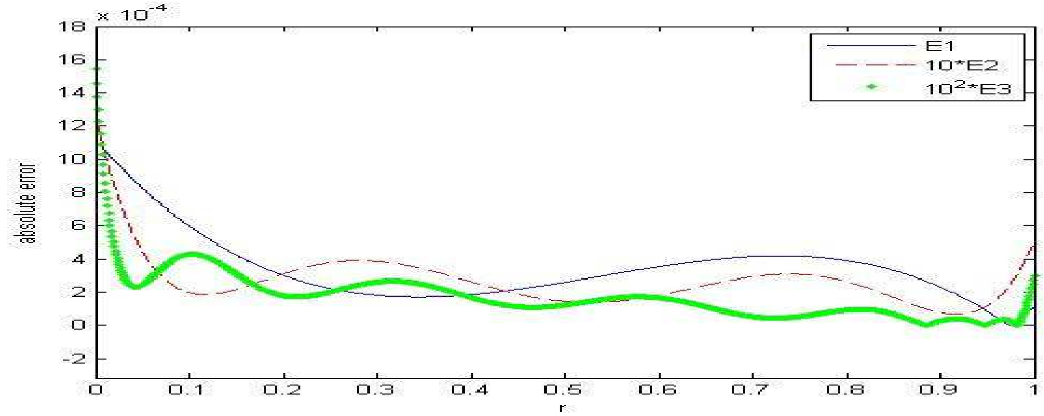

Figure 2.1.1 Comparison of the absolute errors for $n = 3,6$ and 11.

Example 2.6.2. Consider the generalised Abel integral Eq. (2.5.9) with $a_1(y) = y^2, b_1(y) = (1+y)$ and $\eta(r) = r$ for $0 \le r \le 1$.

$$I_2(y) = \frac{1}{\Gamma\left(\frac{9}{5}\right)}\left[\frac{\sqrt{\pi}\, y^{\frac{33}{10}}\,\Gamma\left(\frac{9}{5}\right)}{\Gamma\left(\frac{23}{10}\right)}\right] + \frac{(1+y)}{\Gamma\left(\frac{9}{5}\right)}\left[\frac{10}{13}(1-y)^{\frac{1}{2}} + \frac{\sqrt{\pi}\, y^{\frac{13}{10}}\,\Gamma\left(\frac{-13}{10}\right)}{\Gamma\left(\frac{-4}{5}\right)} + \frac{80}{39}y\ {}_2F_1\left(\frac{-3}{10},\frac{1}{2},\frac{7}{10},y\right)\right],$$

$0 < y < 1$, With Initial condition $\eta(0) = 0$, $\beta = \frac{1}{5}$ and $\mu = \frac{1}{2}$.

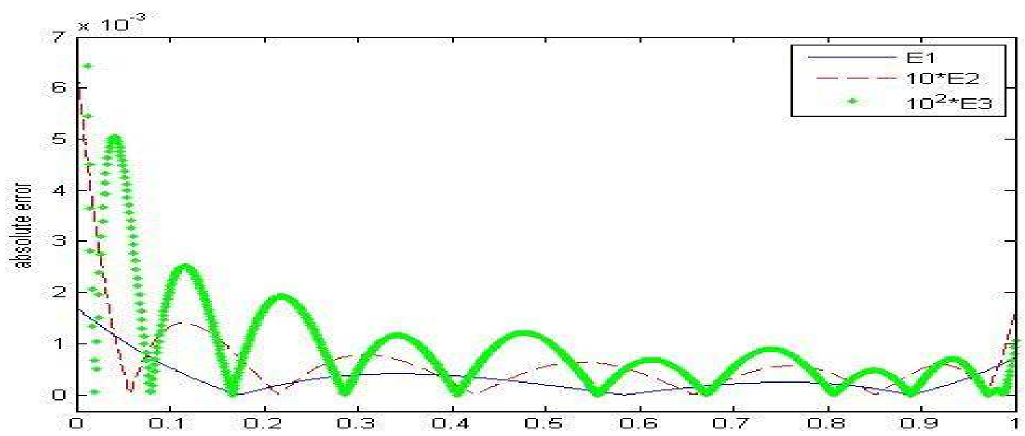

Figure 2.2.1 Comparison of the absolute errors for $n = 3,6$ and 11.

Example 2.6.3. For fractional integro-differential Eq. (2.5.15) consider

$$f(t)=\frac{6}{\Gamma\left(\frac{11}{3}\right)}t^{\frac{8}{3}}-\frac{6\sqrt{\pi}}{\Gamma\left(\frac{25}{6}\right)}t^{\frac{19}{6}} \text{ for } t>0,\ \alpha=\frac{1}{3},\ \beta=\frac{1}{3},\ \gamma=\frac{1}{2} \text{ with } y(0)=0,$$

with the exact solution $y(t)=t^3$.

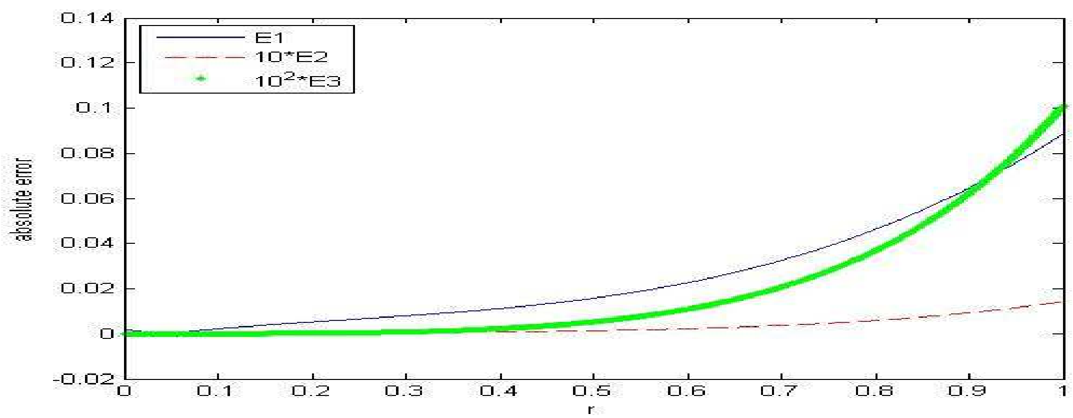

Figure 2.3.1 Comparison of the absolute errors for $n=3,6$ and 11.

Example 2.6.4. For Eq. (2.5.19) consider

$$f(t)=\frac{10}{\Gamma\left(\frac{11}{4}\right)}t^{\frac{7}{4}}+\frac{6}{\Gamma\left(\frac{7}{4}\right)}t^{\frac{3}{4}}-\frac{3}{40}t^{\frac{2}{3}}(40+9t(5t+8)), \quad ,\alpha=\frac{1}{4} \text{ and } \gamma=\frac{1}{3},$$

with initial condition $y(0)=2$. This has the exact solution $y(t)=5t^2+6t+2$.

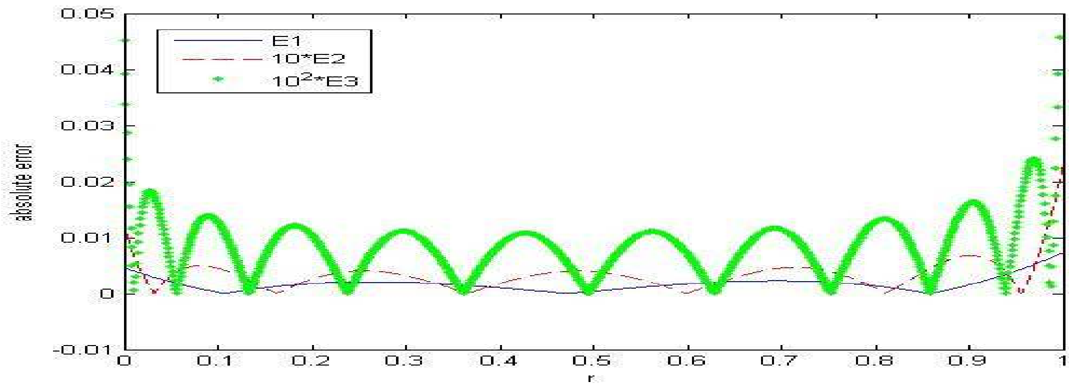

Figure 2.4.1 Comparison of the absolute errors for $n = 3,6$ and 11.

Example 2.6.5. For Eq. (2.5.2) consider

$$\beta = 0,\ b = \frac{1}{2},\quad \eta(r) = \frac{9}{4}\left(1-\sqrt{r}\right)^2\left(1+2\sqrt{r}\right),\ 0 \le r \le 1,$$ (Singh *et al.* (1012),

Buie *et al.* (1996), Ma *et al.* (2008)) and

$$I_2(y) = \frac{9}{8}(2-5y)(\sqrt{1-y}) + \frac{27}{8}y^2 \ln\left(\frac{1+\sqrt{1-y}}{\sqrt{y}}\right),\ 0 < y < 1.$$

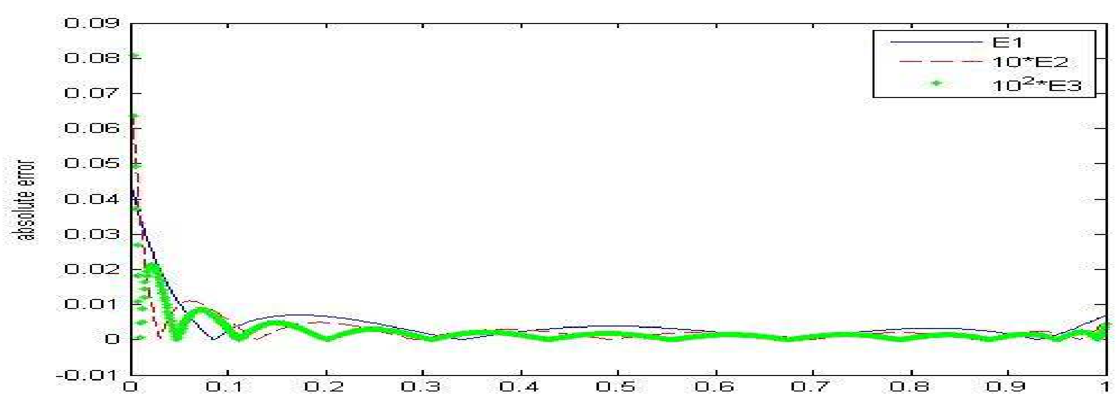

Figure 2.5.1 Comparison of the absolute errors for $n = 3,6$ and 11 .

Example 2.6.6. For Eq. (2.5.9) consider

$$\beta = 0,\ \mu = \frac{1}{2},\ a_1(y) = \frac{3}{4}e^{y},\ b_1(y) = e^{2y} + 1\big/\sqrt{2\pi},\ \eta(r) = \sin r \text{ and}$$

$$I_1(y) = \left[{}_1F_2\left(1; \frac{5}{4}; \frac{7}{4}; \frac{-y^2}{4}\right) e^{y} y^{\frac{3}{2}} + (e^{2y}+1) C\left(\frac{\sqrt{2-2y}}{\sqrt{\pi}}\right) \sin y + S\left(\frac{\sqrt{2-2y}}{\sqrt{\pi}}\right) \cos y \right],$$

(Dixit *et al.* (2012))where $C(z)$ and $S(z)$ are called Fresnel integrals, defined by

$$C(z) = \int_0^z \cos(\pi t^2/2)dt,\ S(z) = \int_0^z \sin(\pi t^2/2)dt \text{ respectively.}$$

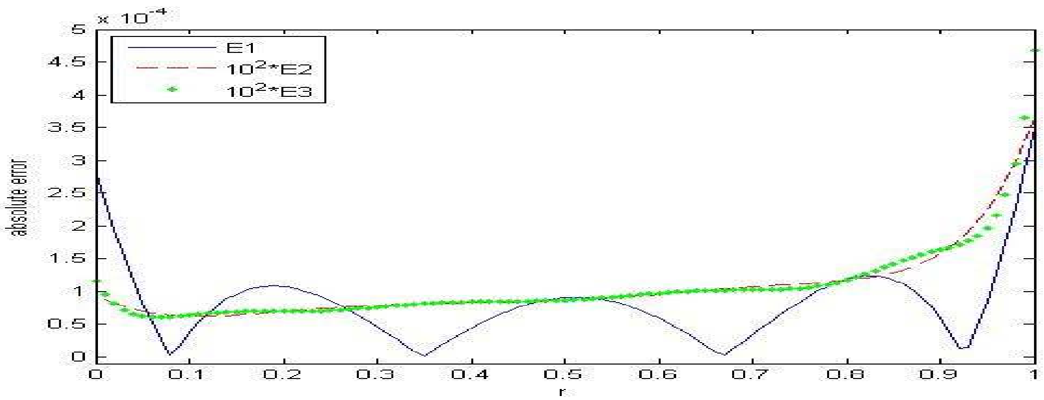

Figure 2.6.1 Comparison of the absolute errors for $n = 3,6$ and 11 .

Example 2.6.7. For Eq. (2.5.22) consider

$$f(t) = \sum_{k=0}^{\infty} \frac{t^{k+\frac{1}{2}}}{\Gamma(k+1.5)} - \sqrt{\pi}(e^{t}-1) \text{ for } t > 0,\ \alpha = \frac{5}{2},\ \beta = \frac{3}{2},\ \gamma = \frac{1}{2} \text{ and}$$

$$y(0) = y'(0) = y''(0) = 1,$$

with the exact solution $y(t) = e^t$.

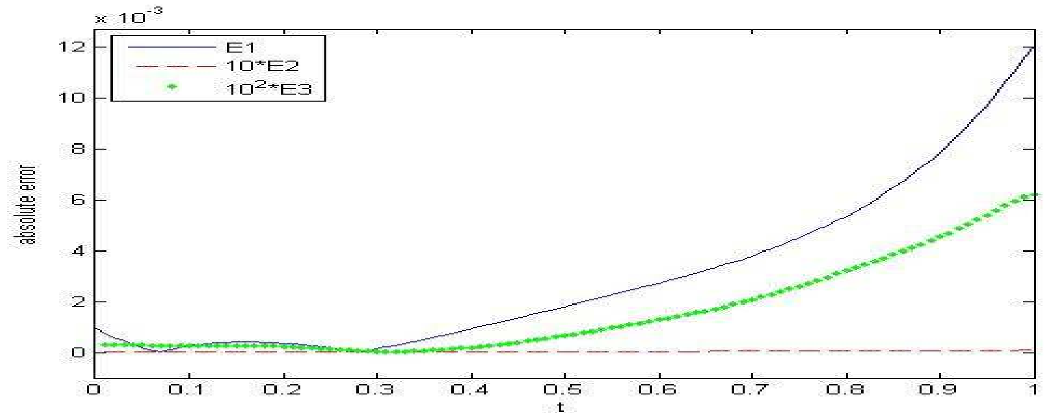

Figure 2.7.1 Comparison of the absolute errors for $n = 3, 6$ and 11 .

2.7 Error Table

In this section we discuss root mean square error and maximum absolute error for all the test functions.

Table 2.1.

Maximum absolute errors ($\boldsymbol{l}^{\infty} error$)

N	3	6	11
Exp1	1.11908×10^{-3}	1.27311×10^{-4}	1.54350×10^{-5}
Exp2	1.71404×10^{-3}	6.58027×10^{-4}	2.38992×10^{-4}
Exp3	8.85794×10^{-2}	1.42346×10^{-3}	1.01050×10^{-3}
Exp4	7.24676×10^{-3}	2.30460×10^{-3}	1.01050×10^{-3}
Exp5	4.61419×10^{-2}	5.37900×10^{-3}	1.61595×10^{-3}
Exp6	3.49705×10^{-4}	3.56555×10^{-6}	4.68059×10^{-6}
Exp7	1.20990×10^{-2}	9.79776×10^{-6}	6.18494×10^{-6}

Table 2.2.

Root mean square errors (l^2 *error*)

N	3	6	11
Exp1	4.00115×10^{-4}	3.14465×10^{-5}	2.36367×10^{-6}
Exp2	1.40377×10^{-5}	3.13334×10^{-6}	6.81666×10^{-7}
Exp3	1.08614×10^{-3}	1.51114×10^{-5}	1.00222×10^{-5}
Exp4	6.73370×10^{-5}	1.34529×10^{-5}	3.36631×10^{-6}
Exp5	7.02673×10^{-3}	4.23498×10^{-4}	8.97843×10^{-5}
Exp6	9.40023×10^{-6}	1.13097×10^{-7}	1.12763×10^{-7}
Exp7	4.13303×10^{-3}	4.42920×10^{-6}	2.30371×10^{-6}

2.8 Applications

The applications based on the fractional order singular Volterra Integro-differential equation are given as:

2.8.1. Equation for Magnetic Field Induction in Dielectric Media

Using the Maxwell equations, we obtain the equation for the magnetic field induction (Tarasov (2009)).

$$\frac{\partial^2 B(t,r)}{\partial t^2} = \frac{1}{\varepsilon_0 \mu}\nabla^2 B(t,r) + \frac{1}{\varepsilon_0}\frac{\partial}{\partial t} curl\, P(t,r) + \frac{1}{\varepsilon_0} curl\, j(t,r). \tag{2.8.1}$$

Where $B(t,r)$ is magnetic field, $P(t,r)$ is polarisation identity.

Now two cases arise depending upon frequency

Case1. For $\omega > \omega_p$, ω_p is low loss frequency.

In this case Eq. (2.8.1) becomes

$$\frac{1}{v^2}\frac{\partial^2 B(t,r)}{\partial t^2}+\frac{\chi_\alpha}{v^2}(D_t^{2-\alpha}B)(t,r)-\nabla^2 B(t,r)=\mu curl\ j(t,r), \tag{2.8.2}$$

where $0<\alpha<1$, $v^2=\dfrac{1}{\varepsilon_0\mu}$ and χ_α is a positive constant.

Case 2. For $\omega<\omega_p$, we obtain

$$\frac{1}{v_\beta^2}\frac{\partial^2 B(t,r)}{\partial t^2}-\frac{a_\beta}{v_\beta^2}(D_t^{2+\beta}B)(t,r)-\nabla^2 B(t,r)=\mu curl\ j(t,r), \tag{2.8.3}$$

where,

$$0<\beta<1,\ v_\beta^2=\frac{1}{\varepsilon_0\mu[1+\bar{\chi}(0)]},\ a_\beta=\frac{\chi_\beta}{[1+\bar{\chi}(0)]}.$$

Eqs. (2.8.2) and (2.8.3) are equation for the magnetic field induction. They can written in general form as

$$(D_t^\alpha B)(t,r)-\lambda_1(D_t^\beta B)(t,r)-\lambda_2\nabla^2 B(t,r)=f(t,r), \tag{2.8.4}$$

where $1\le\beta<\alpha<3$ and $f(t,r)=\mu\lambda_2\, curl\ j(t,r)$.

Eq. (2.8.4) yields Eq. (2.8.2) for $\alpha=2, 1<\beta<2, \lambda_1=-\chi_\alpha$ and $\lambda_2=v^2=\dfrac{1}{\varepsilon_0\mu}$.

Eq. (2.8.3) can be written in form (2.8.4) for $2<\alpha<3,\ \beta=2,\ \lambda_1=\dfrac{1}{a_\beta}=\dfrac{1+\bar{\chi}(0)}{\chi_\beta}$

and $\lambda_2=-\dfrac{v_\beta^2}{a_\beta}=\dfrac{-1}{\varepsilon_0\mu\chi_\beta}$. In Eq. (2.8.4) if we take $\lambda_2\to 0$, then (2.8.4) becomes

$$(D_t^\alpha B)(t,r)-\lambda_1(D_t^\beta B)(t,r)=f(t,r), \tag{2.8.5}$$

using definition of fractional derivative Eq. (2.8.5) can be written as

$$(D^\alpha B)(t,r)=\frac{\lambda_1}{\left\lceil m-\beta\right.}\int_0^t\frac{D^m B(x,r)}{(t-x)^\gamma}dx+f(t,r), \tag{2.8.6}$$

where $1\le\beta<\alpha<3, 0<\gamma<1$.

In Eq. (2.8.6) magnetic field is function of two variables but r is fixed. So Eq. (2.8.6) is a fractional singular integro differential equation.

2.8.2. The Fractional Langevin Equation (dynamics in one dimension for a Brownian particle)

According to the classical Langevin approach the dynamics in one dimension for a Brownian particle is described by Mainardi and Pironi (1996)

$$m\frac{dV}{dt} = F . \tag{2.8.7}$$

Where m is particle mass, $X = X(t), V = V(t)$ are the particle position and velocity and F is the force acting on the particle from molecules of the fluid surrounding the Brownian particle. The force F may be divided into two parts. The first part is fractional force given by

$$F_v = -\frac{m}{\sigma}V, \tag{2.8.8}$$

where $\dfrac{1}{\sigma}$ is the friction coefficient for unit mass.

Second part is called random force and is denoted by $R(t)$. Then Eq. (2.8.7) is written as a stochastic equation as

$$\frac{dV}{dt} = -\frac{1}{\sigma}V(t) + \frac{1}{m}R(t) = \frac{F_v}{m} + \frac{1}{m}R(t), \tag{2.8.9}$$

the fractional force is substituted as

$$F_v = -6\pi a\rho_f v\left\{V(t) + \frac{a}{\sqrt{\pi v}}\int_0^t \frac{dV(\tau)/d\tau}{\sqrt{t-\tau}}d\tau\right\}. \tag{2.8.10}$$

From Eqs. (2.8.9) and (2.8.10) we can write

$$\frac{dV}{dt} = -\frac{6\pi a\rho_f v}{m}\left\{V(t) + \frac{a}{\sqrt{\pi v}}\int_0^t \frac{dV(\tau)/d\tau}{\sqrt{t-\tau}}d\tau\right\} + \frac{1}{m}R(t). \tag{2.8.11}$$

Let $f(t) = \dfrac{1}{m}R(t) - \dfrac{6\pi a\rho_f v}{m}V(t)$ and $c = \dfrac{-6\pi a\rho_f va}{m\sqrt{\pi v}}$, then Eq. (2.8.11) become

$$\frac{dV}{dt} = c\int_0^t \frac{dV(\tau)/d\tau}{\sqrt{t-\tau}}d\tau + f(t), \tag{2.8.12}$$

The Eq. (2.8.12) is similar to Eq. (2.5.15) for particular case $\alpha = 1, \beta = 1$ and $\gamma = \frac{1}{2}$.

2.9 Conclusions

We have constructed the fractional order operational matrices of integration and differentiation for Legendre scaling basis function and used it to find the numerical solution of the fractional order integro-differential equation having fractional order derivative inside the integral. It is also noticed that maximum absolute error as well as root mean square error decrease as we increase the number of basis function n= 3 , 6 and 11, which can be seen in figures and in tables.