CERTAIN PSEUDO-DIFFERENTIAL OPERATORS AND WAVELET TRANSFORMS INVOLVING FRACTIONAL FOURIER AND FRACTIONAL HANKEL TRANSFORMS

Thesis submitted in partial fulfillment for

the Award of Degree

Doctor of Philosophy

by

Kush Kumar Mishra

DEPARTMENT OF MATHEMATICAL SCIENCES
INDIAN INSTITUTE OF TECHNOLOGY
(BANARAS HINDU UNIVERSITY)
VARANASI -221005
INDIA

Roll No: 18121017

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CERTIFICATE

It is certified that the work contained in the thesis titled "***Certain Pseudo-Differential Operators and Wavelet Transforms involving Fractional Fourier and Fractional Hankel Transforms***" by ***Kush Kumar Mishra*** has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

It is further certified that the student has fulfilled all the requirements of Comprehensive Examination, Candidacy and SOTA for the award of Ph.D. degree.

03.07.2023

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DECLARATION BY THE CANDIDATE

I, ***Kush Kumar Mishra***, certify that the work embodied in this thesis is my own bonafide work and carried out by me under the supervision of ***Prof. Santosh Kumar Upadhyay*** from ***July, 2018*** to ***May, 2023*** at the ***Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi***. The matter embodied in this thesis has not been submitted for the award of any other degree/diploma. I declare that I have faithfully acknowledged and given credits to the research workers wherever their works have been cited in my work in this thesis. I further declare that I have not willfully copied any other's work, paragraphs, text, data, results, *etc.*, reported in journals, books, magazines, reports dissertations, theses, *etc.*, or available at websites and have not included them in this thesis and have not cited as my own work.

Date: 03-07-2023

Place: Varanasi

Kush Kumar Mishra

(Kush Kumar Mishra)

CERTIFICATE BY THE SUPERVISOR

It is certified that the above statement made by the student is correct to the best of my/our knowledge.

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ACKNOWLEDGEMENTS

It has been the numerous blessings of Lord Shiva that have enabled my work to be successfully integrated. Several people contributed to the thesis and it cannot be attributed to one individual alone. There has been much blessing during my PhD journey, and acknowledging it is one of the most rewarding experiences. My sincere thanks to everyone who helped me with this work in so many different ways.

First of all, I would like to express my sincere thanks to my respected supervisor, ***Dr. Santosh Kumar Upadhyay, Professor, Department of Mathematical Sciences, IIT(BHU), Varanasi***, for his excellent guidance, continuous encouragement, patience and advice during the whole span of my Ph.D. Despite difficult circumstances, he provided adequate guidance and encouragement to help me become a professional. I was inspired and gained confidence from his inexhaustible capacity for working and positive outlook towards my work. My supervisor has been very supportive of my work, has developed my understanding of the subject matter, and has empowered me to work independently. I also learnt from him which is much more than one could have expected from the supervisor.

I express my cordial thanks to ***Prof. S. K. Pandey, Head of the Department of Mathematical Sciences, Dr. Ashok Ji Gupta, Convener, DPGC, of the Department of Mathematical Sciences*** for their supports throughout my research work. I also express my deep sense of gratitude to all faculty members of the department, especially ***Prof. K. N. Rai***, ***Prof. Shri Ram***, ***Prof. Rekha Srivastava*** and ***Prof. Tanmoy Som*** along with ***Prof. Subir Das, Prof. L. P. Singh, Prof. S. Mukhopadhyay, Prof. Murali Krishna Vemuri, Dr. Vineet Kumar Singh , Dr. R. K. Pandey, Dr. Anuradha Banerjee, Dr.***

Debdas Ghosh, Dr. Sunil Kumar, Dr. Rajeev, Dr. Lavanya Sivakumar, Dr. Aabhash K. Jha, Dr. Amit Kumar, Dr. Sheela Verma, Dr. Divya Goyal, Dr. Anoop Singh, Dr. Manish Kumar Khandelwal, Dr. Rakesh Arora and RPEC member ***Prof. K. K. Shukla, Department of Computer Science and Engineering*** for their constant moral supports, suggestions, and encouragement.

I would be very grateful to ***Prof. Hari. M. Srivastava, Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada*** for his valuable support and contibutions in my Ph.D. work.

I have special thanks to my fellow lab mates Dr. Komal Khatterwani, Dr. Jay Singh Maurya, Dr. Pragya Shukla, Mr. Mohd Sartaz, Mr. Sitaram Yadav, Mr. Amit, Mr. Manjay Pal and Mrs. Priyanka Balvant.

Further, I would like to extend my special thanks to my colleagues and, all the research scholars of the department for their moral supports.

I am also grateful to my institute, the Indian Institute of Technology (BHU), for providing necessary resources throughout my research. I express my thanks to all non teaching staff members of the department for their supports.

I gratefully acknowledge Council of Scientific and Industrial Research (HRDG), New Delhi, India for providing the fellowship in form of Junior Research fellowship and Senior Research fellowship (file no.: 09/1217(0043)/2018-EMR-I).

I am deeply indebted to my grandfather ***Late Shri Shobhnath Mishra***, my father ***Shri Prabhakant Mishra***, my mother ***Smt. Vijay Kumari***, my elder brother ***Shree Lav kumar***, my younger sisters ***Priti and Priyanka*** and all my cousins who always stood by my decisions and provided all kinds of supports, moral as well as financial. It was their love, care and patience which encouraged me to move on. Special love to my sweet niece ***Prachi, Radhika*** and nephew ***Prabal***. This acknowledgement would be incomplete if the name of great visionary ***Pt. Madan Mohan Malaviya*** is not mentioned, who made this divine centre of knowledge. Deepest regards to him.

Above all, praises and thanks to the Lord Shiva, for his showers of blessings throughout my research work, who has made everything possible.

Date:

Place: Varanasi **(Kush Kumar Mishra)**

Contents

Abbreviations

- $\mathbb{N}$, Set of natural numbers
- $\mathbb{N}_0$, Set of non-negative integers
- $\mathbb{N}_0^n$, Set of multi-indices
- $\mathbb{R}_+$ or I, Open interval $(0, \infty)$
- $\mathbb{R}$, Set of real numbers
- $\mathbb{R}^n$, Usual Euclidean space of dimension n
- $\mathbb{C}$, Set of complex numbers
- $D_x \equiv \frac{\partial}{\partial x}$, Partial derivative with respect to variable x
- *a.e.*, Almost everywhere
- R.H.S., Right hand side
- L.H.S., Left hand side
- $\mathcal{F}$, Fourier transform operator
- $x^\beta = x_1^{\beta_1} x_2^{\beta_2} \dots x_n^{\beta_n}$, for all $x \in \mathbb{R}^n$, where $\beta = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{N}_0^n$.
- $\prod_{j=1}^n |w_j|^{\frac{1}{\alpha_j}-1} = |w_1|^{\frac{1}{\alpha_1}-1} |w_2|^{\frac{1}{\alpha_2}-1} \dots |w_n|^{\frac{1}{\alpha_n}-1}$, where $w = (w_1, w_2, \dots, w_n) \in \mathbb{R}^n$
- $\alpha \in (0,1]^n$, where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}^n$ such that each $\alpha_j \in (0,1]$
- $[\alpha] = \prod_{j=1}^n \alpha_j = \alpha_1 \alpha_2 \dots \alpha_n$, for $\alpha \in (0,1]^n$
- $x \cdot y = \sum_{j=1}^n x_j \cdot y_j$ and $|x| = (\sum_{j=1}^n x_j^2)^{\frac{1}{2}}$,

  for $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ be any two vectors in $\mathbb{R}^n$

PREFACE

The pseudo-differential operator is an important tool, which is the generalization of the partial differential operator and useful to find the solution of the partial differential equation. Wavelet transform is an integral transform, whose kernel contains the translation and dilation in the time domain. A wavelet transform gives local as well as global information of a signal. This thesis consists of five chapters. In this thesis, we have considered different aspects, which are given below chapterwise.

Chapter 1 is introductory, which provides the historical background of the fractional Fourier transform, pseudo-differential operators, and wavelet transform. Definitions and properties of the Schwartz space, dual of Schwartz space, fractional operators, Lizorkin space, fractional Fourier transform, pseudo-differential operators, continuous fractional wavelet transform, Hankel transform and others are given.

Chapter 2 describes about the convolution property, Plancherel formula and continuity properties on $S(\mathbb{R}^n)$, and $S^{'}(\mathbb{R}^n)$ by using the n-dimensional fractional Fourier transform and its inversion formula. Boundedness of pseudo-differential operators on $S(\mathbb{R}^n)$, $S^{'}(\mathbb{R}^n)$, and Sobolev space are proved by exploiting the theory of the n-dimensional fractional Fourier transform. Applications of pseudo-differential operators are given in the Lizorkin space by using the Riemann-Liouville fractional derivative, and integral operators.

In Chapter 3, Abelian theorems for the fractional wavelet transform are obtained in classical and distributional sense both. An application and justification of Abelian theorems for the continuous fractional wavelet transform is given by using Mexican hat wavelet function.

In Chapter 4, the fractional Hankel transform is introduced by using the n-dimensional fractional Fourier transform of radial function. Parseval formula and various properties of the fractional Hankel convolution are discussed by using the technique of the fractional Hankel transform.

In Chapter 5, the fractional Bessel wavelet transform is introduced and obtained its inversion formula by exploiting the theory of the fractional Hankel transform. Boundedness properties and Calderon reproducing formula for the fractional Bessel wavelet transform are proved. Applications of the fractional Bessel wavelet transform associated with certain weighted Sobolev-type space are given. The time-invariant linear filter is expressed in the form of the fractional Bessel wavelet transform. With the help of aforesaid transform, the solution of the Fredholm integral equation is obtained.