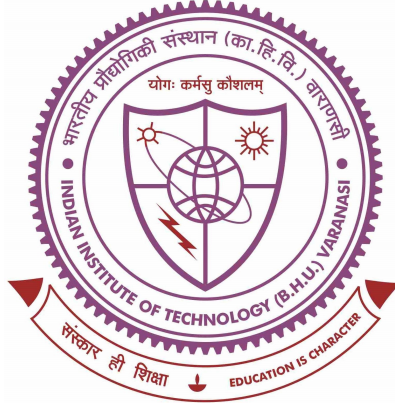


**Analysis of High-Order Methods for
Time-Fractional Partial Differential Equations
with Smooth and Non-smooth Solutions**



Thesis submitted in partial fulfillment

for the award of degree of

Doctor of Philosophy

by

Anshima Singh

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CERTIFICATE

It is certified that the work contained in this thesis titled “**Analysis of High-Order Methods for Time-Fractional Partial Differential Equations with Smooth and Non-smooth Solutions**” by **Anshima Singh** has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

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It is certified that the above statement made by the student is correct to the best of my knowledge.

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Anshima Singh

Dedicated
to
My Parents

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Symbols

\mathbb{C}	Set of complex numbers
\mathbb{R}	Set of real numbers
\mathbb{R}^+	Set of positive real numbers
\mathbb{Z}	Set of integers
\mathbb{Z}^+	Set of positive integers
\mathbb{N}	Set of natural numbers
Γ	Gamma function
$E_{\beta,\alpha}^\eta$	Mittag-Leffler function of three parameters
$E_{\beta,\alpha}$	Mittag-Leffler function of two parameters
E_β	Mittag-Leffler function of one parameter
α	Order of fractional derivatives
${}^C D_{t_0}^\alpha$	Generalized fractional derivative of Caputo type
${}^{CP} D_{t_0}^\eta$	Prabhakar fractional derivative of Caputo type
${}^C D_t^\alpha$	Caputo fractional derivative
$z(t)$	Scale function
$\omega(t)$	weight function
N_t	Discretization parameter in temporal direction
N_x, N_y	Discretization parameters in spatial direction
Δt	Uniform time step-size
Δt_n	Non-uniform time step-size

h, h_x, h_y	Uniform space step-size
\mathcal{O}	Landau symbol
$\ \cdot\ _\infty$	\mathcal{L}_∞ -norm
$\ \cdot\ , \ \cdot\ _2, \ \cdot\ _h$	\mathcal{L}_2 -norm
c	Generic positive constant, independent of discretization parameters