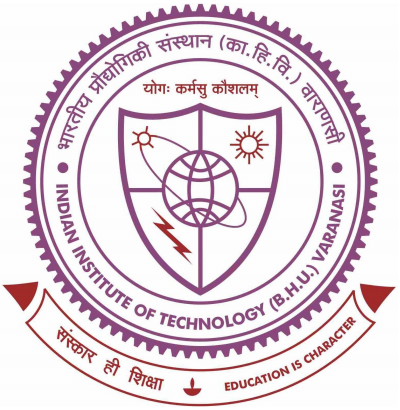
Analysis of High-Order Methods for Time-Fractional Partial Differential Equations with Smooth and Non-smooth Solutions



Thesis submitted in partial fulfillment

for the award of degree of

Doctor of Philosophy

by

Anshima Singh

Department of Mathematical Sciences

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CERTIFICATE

It is certified that the work contained in this thesis titled **"Analysis of High-Order Methods for Time-Fractional Partial Differential Equations with Smooth and Non-smooth Solutions"** by **Anshima Singh** has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

It is further certified that the student has fulfilled all the requirements of Comprehensive Examination, Candidacy and SOTA for the award of Ph.D. degree.

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DECLARATION BY THE CANDIDATE

I, **Anshima Singh**, certify that the work embodied in this thesis is my own bona fide work and carried out by me under the supervision of **Dr. Sunil Kumar** from **January 2019** to **December 2023** at the **Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi**. The matter embodied in this thesis has not been submitted for the award of any other degree/diploma. I declare that I have faithfully acknowledged and given credits to the research workers wherever their works have been cited in my work in this thesis. I further declare that I have not willfully copied any other's work, paragraphs, text, data, results, *etc.*, reported in journals, books, magazines, reports dissertations, theses, *etc.*, or available at websites and have not included them in this thesis and have not cited as my own work.

Anshima Singh

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It is certified that the above statement made by the student is correct to the best of my knowledge.

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Varanasi **Anshima Singh**

Dedicated

to

My Parents

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Symbols

$\mathbb{C}$	Set of complex numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{R}^+$	Set of positive real numbers
$\mathbb{Z}$	Set of integers
$\mathbb{Z}^+$	Set of positive integers
$\mathbb{N}$	Set of natural numbers
Γ	Gamma function
$E^{\eta}_{\beta,\alpha}$	Mittag-Leffler function of three parameters
$E_{\beta,\alpha}$	Mittag-Leffler function of two parameters
E_{β}	Mittag-Leffler function of one parameter
α	Order of fractional derivatives
${}^{C}_{t_0}D^{\alpha}_{t:[z,\omega]}$	Generalized fractional derivative of Caputo type
${}^{CP}_{t_0}D^{\eta}_{\beta,\alpha,\kappa}$	Prabhakar fractional derivative of Caputo type
${}^{C}_{t_0}D^{\alpha}_{t}$	Caputo fractional derivative
$z(t)$	Scale function
$\omega(t)$	weight function
N_t	Discretization parameter in temporal direction
N_x, N_y	Discretization parameters in spatial direction
Δt	Uniform time step-size
Δt_n	Non-uniform time step-size

h, h_x, h_y	Uniform space step-size
$\mathcal{O}$	Landau symbol
$\|.\|_\infty$	$\mathcal{L}_\infty$-norm
$\|.\|$, $\|.\|_2$, $\|.\|_h$	$\mathcal{L}_2$-norm
c	Generic positive constant, independent of discretization parameters