

CHAPTER - 6

AN APPROXIMATE METHOD FOR SOLVING NONLINEAR SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS

Chapter 6

An Approximate Method for Solving Nonlinear System of Fractional Differential Equations

6.1 Introduction

Fractional calculus is a branch of mathematics that deals with generalization of the well-known operations of differentiation and integration to arbitrary (real or complex) orders. Many papers on fractional calculus have been published during the second half of the 20th century. Recent developments on fractional calculus involve real world applications in science and engineering such as viscoelasticity (Bagley and Torvik (1983 a), (1983 b), Bagley and Torvik (1985)), bioengineering (Magin (2004)), biology (Robinson (1981)) and more can be found in Podlubny (1999) and Kilbas *et al.* (2006). The classical calculus deals with problems defined using integer order derivatives. However, in recent years, it has been demonstrated that many systems are described more accurately using fractional order derivatives. This makes fractional calculus richer than classical calculus. In recent years many numerical methods are studied and analyzed for fractional differential equations defined using fractional derivatives of Riemann Liouville or Caputo type. Some of these approximate methods are such as Adomian decompositions method (Odibat and Momani (2008), Momani and Odibat (2007), El-Wakil *et al.* (2006)), Variational iteration method (Sweilam *et al.* (2007 a), Das (2009)), Homotopy perturbation method (Momani and Odibat (2007), Sweilam *et al.* (2007 b)), Homotopy analysis method (Zurigat *et al.* (2010)), fractional differential transform method (Arikoglu and Ozkol (2007), Arikoglu and Ozkol (2009), Darania and Ebadian (2007), Erturk and Momani (2008), Erturk *et al.* (2008)), power series method (Odibat and Shawagfeh

(2007)) and other methods of importance (Li and Tao (2009), Kumar and Agrawal (2006)).

There has been growing interest to develop approximate method for fractional models using method of operational matrix of differentiation or integration. In 2010, Li *et al.* (2010 a) constructed Haar wavelet operational matrix of fractional integration using Block pulse functions and successfully applied to solve of special type of fractional differential equation. Further, Li (2010 b) used another operational matrix based on Chebyshev wavelet for the same problem. In 2011, Saadatmandi and Dehghan (2010) used the concept of orthogonal polynomial and constructed Legendre operational matrix of differentiation and then used it to solve such problems.

Bernstein polynomials and its importance are recently recognized by many researchers. It has been applied to many problems of science and engineering to get the numerical solutions. Bhatta (2008) has used modified Bernstein's polynomials to solve Kdv-Burgers equations. Garloff (2000) applied the Bernstein's approximation for control problems. Saadatmandi (2014) discussed application of the Bernstein's operational matrix of fractional differentiation for solving linear fractional differential equations. More recently, Bernstein approximation are used to solve generalized Abel equations (Pandey and Mandal (2010)), Lane-Emden equations (Pandey and Kumar (2012), Kumar *et al.* (2011)), and generalized isoperimetric fractional variational problems (Pandey and Agrawal (2015)). Alipour and Baleanu (2013) used Bernstein's polynomial matrices to solve system of fractional differential equations. Here, an attempt is made to apply the BOMFD (Bernstein operational matrix of fractional differentiation) and develop a numerical scheme for system of fractional nonlinear differential equations. The method is applied to solve only fractional system of physical importance such as 6.4.1) nonlinear system of fractional differential equations and 6.4.2)

sfractional stiff system of nonlinear differential equations and perform numerical investigations. However, it can be applied to many other fractional differential equations. Numerical results for these cases are discussed and shown through various figures for different values of the fractional order.

6.2 Bernstein Operational Matrix of Fractional Differentiation

The Bernstein polynomials of degree n , on interval $[0,1]$, are defined by

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}, \quad \text{for } i = 0, 1, 2, \dots, n. \quad (6.2.1)$$

There are $(n+1)$ n^{th} degree Bernstein basis polynomials forming a complete basis set $B(x) = \{B_{0,n}, B_{1,n}, \dots, B_{n,n}\}$ in the Hilbert space $L^2[0,1]$ and, if V_n is the space of consisting of all polynomials of degree less than or equal to n . Then any polynomial $f(x)$ in V_n can be expressed as,

$$f(x) = \sum_{i=0}^n \beta_i B_{i,n}(x) = \beta^T B(x), \quad (6.2.2)$$

where, $\beta = [\beta_1, \beta_2, \dots, \beta_n]^T$ and $\beta_i = \langle f, B_{i,n} \rangle$ are called Bernstein or Bezier coefficients. However, the Bernstein's basis polynomials can be generalized to cover any arbitrary interval $[a, b]$ as,

$$B_{i,n}(t) = \binom{n}{i} \frac{(x-a)^i (b-x)^{n-i}}{(b-a)^n}, \quad \text{for } i = 0, 1, 2, \dots, n. \quad (6.2.2A)$$

Theorem 6.2.1. Let $f : [0,1] \rightarrow \mathbb{R}$ is $(n+1)$ times continuously differentiable and $S_n = \text{Span}\{B_{0,n}, B_{1,n}, \dots, B_{n,n}\}$. If $\beta^T B(x)$ is the best approximation of f out of S_n then,

$$\|f - \beta^T B\|_{L^2[0,1]} = \frac{A}{\sqrt{(n+1)(2n+3)}}, \quad \text{where, } A = \max |f^{(n+1)}(x)| \text{ for } x \in [0,1].$$

The derivation of the Bernstein operational matrix of fractional differentiation is similar to the idea of derivation of Bernstein operational matrix of differentiation (Pandey and Kumar (2010)). To make the paper self contained, first we discuss in brief the derivation of the Bernstein's operational matrix of differentiation for the polynomial set $\phi(x)$ as follows:

$$\frac{d\phi(x)}{dx} = D^{(1)}\phi(x), \quad (6.2.3)$$

where $D^{(1)}$ is the $(n+1) \times (n+1)$ operational matrix of derivative. Let

$$\phi(x) = \begin{pmatrix} B_{0,n}(x) \\ B_{1,n}(x) \\ \cdot \\ \cdot \\ B_{n,n}(x) \end{pmatrix} = \begin{pmatrix} a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \\ b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n \\ \cdot \\ \cdot \\ k_0 + k_1 x + k_2 x^2 + \dots + k_n x^n \end{pmatrix} = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ b_0 & b_1 & b_2 & \dots & b_n \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ k_0 & k_1 & k_2 & \dots & k_n \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ \cdot \\ x^n \end{pmatrix},$$

where, $a_0, a_1, \dots, a_n, b_0, b_1, \dots, b_n, \dots, k_0, k_1, \dots, k_n$, are binomial coefficients.

$$\frac{d\phi(x)}{dx} = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ b_0 & b_1 & b_2 & \dots & b_n \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ k_0 & k_1 & k_2 & \dots & k_n \end{pmatrix} \frac{d}{dx} \begin{pmatrix} 1 \\ x \\ x^2 \\ \cdot \\ x^n \end{pmatrix} = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ b_0 & b_1 & b_2 & \dots & b_n \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ k_0 & k_1 & k_2 & \dots & k_n \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2x \\ \cdot \\ nx^{n-1} \end{pmatrix},$$

$$= \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ b_0 & b_1 & b_2 & \dots & b_n \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ k_0 & k_1 & k_2 & \dots & k_n \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ \cdot \\ x^n \end{pmatrix},$$

$$= \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ b_0 & b_1 & b_2 & \dots & b_n \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ k_0 & k_1 & k_2 & \dots & k_n \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ b_0 & b_1 & b_2 & \dots & b_n \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ k_0 & k_1 & k_2 & \dots & k_n \end{pmatrix}^{-1} \phi(x),$$

$$D^1 = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ b_0 & b_1 & b_2 & \dots & b_n \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ k_0 & k_1 & k_2 & \dots & k_n \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ b_0 & b_1 & b_2 & \dots & b_n \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ k_0 & k_1 & k_2 & \dots & k_n \end{pmatrix}^{-1},$$

From Eq. (6.2.3), it can be generalized for any $n \in N$ as,

$$\frac{d^n \phi(x)}{dx^n} = (D^{(1)})^n \phi(x) = D^{(n)} \phi(x), \quad n = 1, 2, 3, \dots \quad (6.2.4)$$

Now we present the operational matrix of fractional differentiation as follows. We consider the fractional derivative as the Caputo's derivative. The Caputo derivative of a function $f(x)$ is defined as,

$$D^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^n(\tau)}{(x-\tau)^{\alpha-n+1}} d\tau, & n-1 < \alpha < n ; \\ \frac{d^n}{dx^n} f(x), & \alpha = n \end{cases} \quad (6.2.5)$$

where n is a natural number.

Saadatmandi (2014) has presented the Bernstein's operational matrix of fractional differentiation and applied it to solve linear fractional differential equations. Here, we discuss in brief the formulation of the operational matrix of fractional differentiation (Saadatmandi (2014)).

From Eq. (6.2.1) can be rewritten (say, $b_{i,n}$) by expanding the term $(1-t)^{n-i}$ using binomial expansion as

$$b_{i,n}(x) = \sum_{k=0}^{n-i} (-1)^{(k)} \binom{n}{i} \binom{n-i}{k} x^{i+k},$$

$$b_{i,n}(x) = \sum_{k=i}^n (-1)^{(k-i)} \binom{n}{i} \binom{n-i}{k-i} x^k.$$

The fractional order derivative of an element of Bernstein polynomial can be computed as:

$$\begin{aligned} D^\alpha b_{i,n}(x) &= \sum_{k=i}^n (-1)^{(k-i)} \binom{n}{i} \binom{n-i}{k-i} (D^\alpha x^k) \\ &= \sum_{k=i}^n (-1)^{(k-i)} \binom{n}{i} \binom{n-i}{k-i} \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} (x^{k-\alpha}) \end{aligned} \quad , \text{where, } 0 < \alpha < 1. \quad (6.2.6)$$

Following the similar steps as presented in Saadatmandi (2014), now, we approximate $x^{k-\alpha}$ by $n+1$ terms of Bernstein polynomials series as:

$$x^{k-\alpha} \approx \sum_{j=0}^n d_{k,j} b_{j,n}(x),$$

where,

$$d_{k,j} = (2j+1) \int_0^1 x^{k-\alpha} b_{j,n}(x) dx. \quad (6.2.7)$$

We get ,

$$\begin{aligned} D^\alpha b_{i,n}(x) &= \sum_{k=i}^n (-1)^{(k-i)} \binom{n}{i} \binom{n-i}{k-i} \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} \sum_{j=0}^n d_{k,j} b_{j,n}(x), \\ &= \sum_{k=i}^n \sum_{j=0}^n (-1)^{(k-i)} \binom{n}{i} \binom{n-i}{k-i} \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} d_{k,j} b_{j,n}(x), \\ &= \left[\sum_{k=i}^n (-1)^{(k-i)} \binom{n}{i} \binom{n-i}{k-i} \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} \{d_{k,0} b_{0,n} + \dots + d_{k,n} b_{n,n}\} \right] \varphi(x) \\ &= \left[\sum_{k=i}^n (-1)^{(k-i)} \binom{n}{i} \binom{n-i}{k-i} \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} d_{k,0} + \dots + \sum_{k=i}^n (-1)^{(k-i)} \binom{n}{i} \binom{n-i}{k-i} \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} d_{k,n} \right] \varphi(x), \end{aligned}$$

$$D^\alpha \varphi(x) = \begin{bmatrix} \sum_{k=0}^n (-1)^{(k-i)} \binom{n}{i} \binom{n-i}{k-i} \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} d_{k,0} & \cdots & \sum_{k=0}^n (-1)^{(k-i)} \binom{n}{i} \binom{n-i}{k-i} \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} d_{k,n} \\ \vdots & \cdots & \vdots \\ \sum_{k=n}^n (-1)^{(k-i)} \binom{n}{i} \binom{n-i}{k-i} \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} d_{k,0} & \cdots & \sum_{k=n}^n (-1)^{(k-i)} \binom{n}{i} \binom{n-i}{k-i} \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} d_{k,n} \end{bmatrix} \varphi(x).$$

$$D^\alpha \approx \begin{bmatrix} \sum_{k=0}^n (-1)^{(k-i)} \binom{n}{i} \binom{n-i}{k-i} \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} d_{k,0} & \cdots & \sum_{k=0}^n (-1)^{(k-i)} \binom{n}{i} \binom{n-i}{k-i} \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} d_{k,n} \\ \vdots & \cdots & \vdots \\ \sum_{k=n}^n (-1)^{(k-i)} \binom{n}{i} \binom{n-i}{k-i} \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} d_{k,0} & \cdots & \sum_{k=n}^n (-1)^{(k-i)} \binom{n}{i} \binom{n-i}{k-i} \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} d_{k,n} \end{bmatrix}. \quad (6.2.8)$$

In general, we can define, $\frac{d^\alpha \varphi(x)}{dx^\alpha} = D^\alpha \varphi(x)$, for, $0 < \alpha < 1$ where D^α is defined by Eq. (6.2.8). In the next section, we present numerical scheme for solving the nonlinear system of fractional differential equations based on the operational matrix of fractional differentiation.

6.3 Outline of the Method

In this section, we discuss the numerical scheme for solving the system of fractional differential equations using the operational matrix of fractional derivative of Bernstein's polynomials. We consider a case of the general system of fractional differential equations as follow

$$\left. \begin{aligned} D^{\alpha_1} y_1(x) &= f_1(x, y_1(x), y_2(x), y_3(x)) \\ D^{\alpha_2} y_2(x) &= f_2(x, y_1(x), y_2(x), y_3(x)) \\ D^{\alpha_3} y_3(x) &= f_3(x, y_1(x), y_2(x), y_3(x)) \end{aligned} \right\}, \quad (6.3.1)$$

$$\text{or, } D^{\alpha_i} = f_i(x, Y(x)), \quad (6.3.2)$$

where, $Y(x)=[y_1(x), y_2(x), y_3(x)]^T$, $i=1,2,3$ and $0 < x \leq 1, 0 < \alpha_i \leq 1$, with initial conditions $y_i(0)=a_i$ and D^α denotes the Caputo derivative defined in Eq. (6.2.5). For solving Eqs. (6.3.1), the unknowns $y_1(x)$, $y_2(x)$ and $y_3(x)$ are approximated in terms of the Bernstein's polynomials as,

$$y_1(x) \approx \sum_{i=0}^n c_i b_{i,n} = C^T \varphi(x), y_2(x) \approx \sum_{i=0}^n g_i b_{i,n} = G^T \varphi(x) \text{ and } y_3(x) \approx \sum_{i=0}^n u_i b_{i,n} = U^T \varphi(x), \quad (6.3.3)$$

and hence,

$$D^\alpha y_1(x) \approx C^T D^\alpha \varphi(x), D^\alpha y_2(x) \approx G^T D^\alpha \varphi(x) \text{ and } D^\alpha y_3(x) \approx U^T D^\alpha \varphi(x), \quad (6.3.4)$$

using Eqs. (6.3.3)-(6.3.4), the residual $R_{n,i}(x)$ for Eq. (6.3.1) can be expressed as:

$$\left. \begin{aligned} R_{n,1}(x) &\approx C^T D^{\alpha_1} \varphi(x) - f_1(x, C^T \varphi(x), G^T \varphi(x), U^T \varphi(x)) \\ R_{n,2}(x) &\approx C^T D^{\alpha_2} \varphi(x) - f_2(x, C^T \varphi(x), G^T \varphi(x), U^T \varphi(x)) \\ R_{n,3}(x) &\approx C^T D^{\alpha_3} \varphi(x) - f_3(x, C^T \varphi(x), G^T \varphi(x), U^T \varphi(x)) \end{aligned} \right\}. \quad (6.3.5)$$

Using typical tau method, we generate system of nonlinear equations as,

$$\langle R_{n,1}(x), b_{i,n}(x) \rangle = \int_0^1 R_{n,1}(x) b_{i,n}(x) dx = 0, \quad i=0,1,\dots,n-2, \quad (6.3.6A)$$

$$\langle R_{n,2}(x), b_{i,n}(x) \rangle = \int_0^1 R_{n,2}(x) b_{i,n}(x) dx = 0, \quad i=0,1,\dots,n-2, \quad (6.3.7B)$$

$$\langle R_{n,3}(x), b_{i,n}(x) \rangle = \int_0^1 R_{n,3}(x) b_{i,n}(x) dx = 0, \quad i=0,1,\dots,n-2. \quad (6.3.8C)$$

Eqs. (6.3.6A)-(6.3.8C) with initial conditions generate $3.(n+1)$ equations. These equations can be solved using any standard method and unknown coefficients C^T , G^T and U^T are obtained and hence, $y_1(x)$, $y_2(x)$ and $y_3(x)$ are calculated. In the next section, we consider different cases to perform the

numerical investigations. Error analysis for the nonlinear system of fractional differential equations will be presented in a future work.

6.4 Results and Discussions

Here, we apply the operational matrix of the fractional differentiation of Bernstein's polynomials, explained in Section 6.3, for solving some system of nonlinear fractional differential equations having physical importance. We test the discussed numerical scheme for different values of the fractional order α and also varying the number of the Bernstein's polynomials n . Two case studies are presented and numerical results are discussed. In first case, nonlinear system of fractional differential equations having exact solution is considered and numerical investigations are performed and in second case, the proposed method has been applied to get the approximate solution of the fractional stiff system of nonlinear differential equations. Numerical results obtained of the respective cases are also discussed.

Case 6.4.1. Application to Nonlinear System of Fractional Differential Equations

In this case, nonlinear system of fractional differential equations (Zurigat (2010)) is considered as follows:

$$\begin{aligned} D^\alpha u &= u, \\ D^\beta v &= 2u^2, \\ D^\gamma w &= 3uv, \end{aligned} \tag{6.4.1}$$

for $0 < \alpha, 0 < \beta \leq 1$ and $0 < \gamma \leq 1$, with initial conditions $u(0)=1, v(0)=1, w(0)=0$.

The exact solution of this system, when $\alpha = \beta = \gamma = 1$, is given by,

$$\begin{pmatrix} u(x) \\ v(x) \\ w(x) \end{pmatrix} = \begin{pmatrix} e^x \\ e^{2x} \\ e^{3x-1} \end{pmatrix}. \tag{6.4.2}$$

Here, the method developed in Section 6.3 is applied to approximate the unknown solutions in terms of Bernstein's polynomials as,

$$\begin{aligned} u(x) &\approx \sum_{i=0}^n c_i b_{i,n} = C^T \varphi(x), \quad v(x) \approx \sum_{i=0}^n g_i b_{i,n} = G^T \varphi(x) \text{ and} \\ w(x) &\approx \sum_{i=0}^n h_i b_{i,n} = H^T \varphi(x). \end{aligned} \quad (6.4.3)$$

From Eq. (6.4.3) and Eq. (6.4.1), the residuals, $R_{n,1}(x)$, $R_{n,2}(x)$ and $R_{n,3}(x)$ are calculated as:

$$R_{n,1}(x) \approx C^T D^\alpha \varphi(x) - C^T \varphi(x),$$

$$R_{n,2}(x) \approx G^T D^\beta \varphi(x) - 2(C^T \varphi(x))^2,$$

$$R_{n,3}(x) \approx H^T D^\gamma \varphi(x) - 3(C^T \varphi(x))(G^T \varphi(x)).$$

Applying typical tau method, the system of linear equations are generated as,

$$\langle R_{n,1}(x), b_{i,n}(x) \rangle = \int_0^1 R_{n,1}(x) b_{i,n}(x) dx = 0, \quad i = 0, 1, \dots, n-2, \quad (6.4.4)$$

$$\langle R_{n,2}(x), b_{i,n}(x) \rangle = \int_0^1 R_{n,2}(x) b_{i,n}(x) dx = 0, \quad i = 0, 1, \dots, n-2, \quad (6.4.5)$$

$$\langle R_{n,3}(x), b_{i,n}(x) \rangle = \int_0^1 R_{n,3}(x) b_{i,n}(x) dx = 0, \quad i = 0, 1, \dots, n-2. \quad (6.4.6)$$

Eqs. (6.4.4), (6.4.5), and Eq. (6.4.6) with the initial condition have been solved using the presented method and the approximate solutions for $u(x)$, $v(x)$ and $w(x)$ are obtained for $n=5$. Absolute errors between the exact and approximated solution are shown in Figure 6.4.1. The plot of exact and approximated solutions for the unknowns $u(x)$, $v(x)$ and $w(x)$ are shown in the "Figures. 6.4.2, 6.4.3 and 6.4.4" respectively. " Figures 6.4.5, 6.4.6 and

6.4.7'' present the approximated solutions for different values of pair of fractional orders such as $\alpha = \beta = 1$, $\alpha = \beta = 0.9$ and $\alpha = \beta = 0.7$.

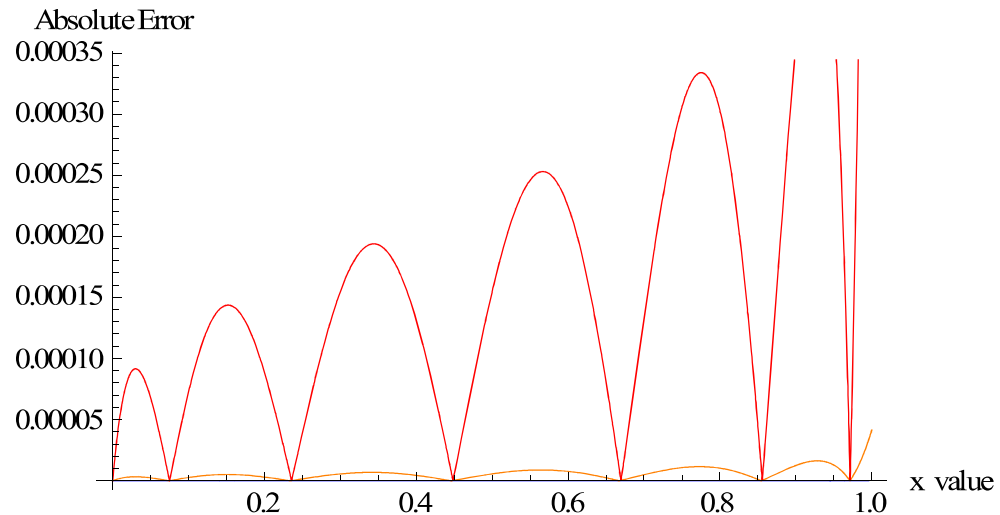


Figure 6.4.1 Plot of absolute errors between exact solution and approximate solution for $\alpha = \beta = 1$, where blue line represents absolute error of $u(x)$, orange line shows absolute error for $v(x)$ and red line indicates absolute error for $w(x)$.

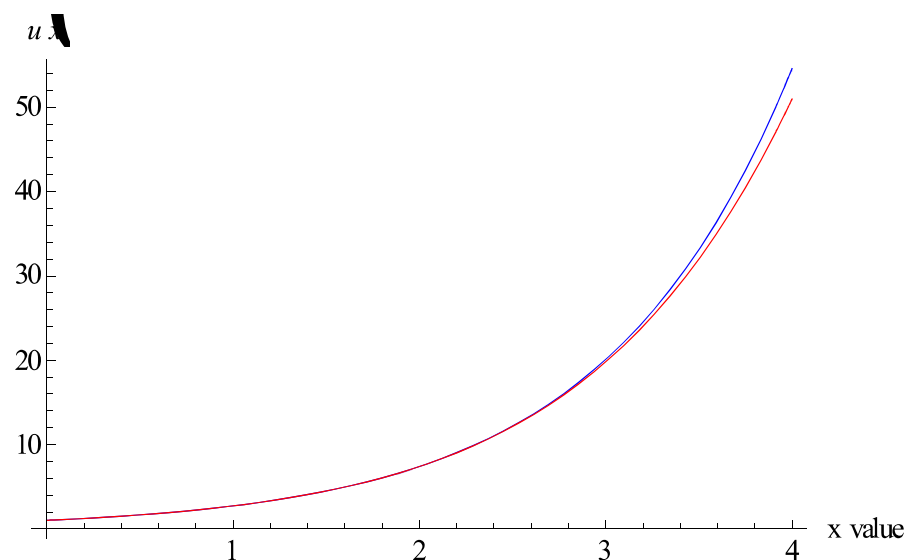


Figure 6.4.2 Plot of exact solution (blue line) and proposed solution (red line) of $u(x)$ for $\alpha = \beta = 1$.

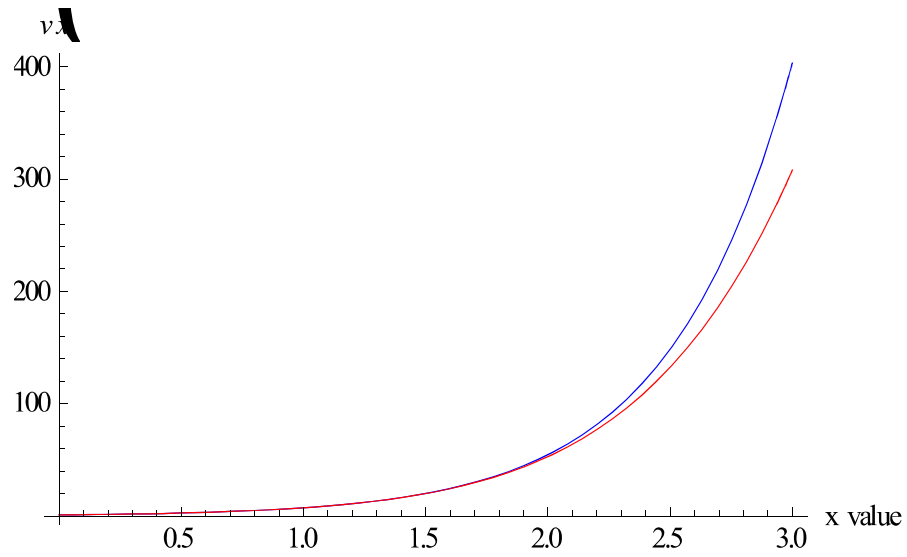


Figure 6.4.3 Plot of exact solution (blue line) and proposed solution (red line) of $v(x)$ for $\alpha = \beta = 1$.

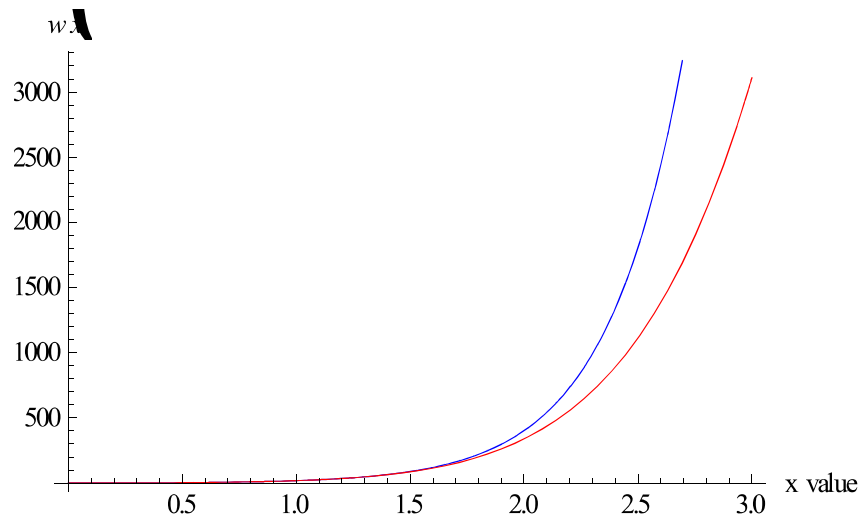


Figure 6.4.4 Plot of exact solution (blue line) and proposed solution (red line) of $w(x)$ for $\alpha = \beta = 1$.

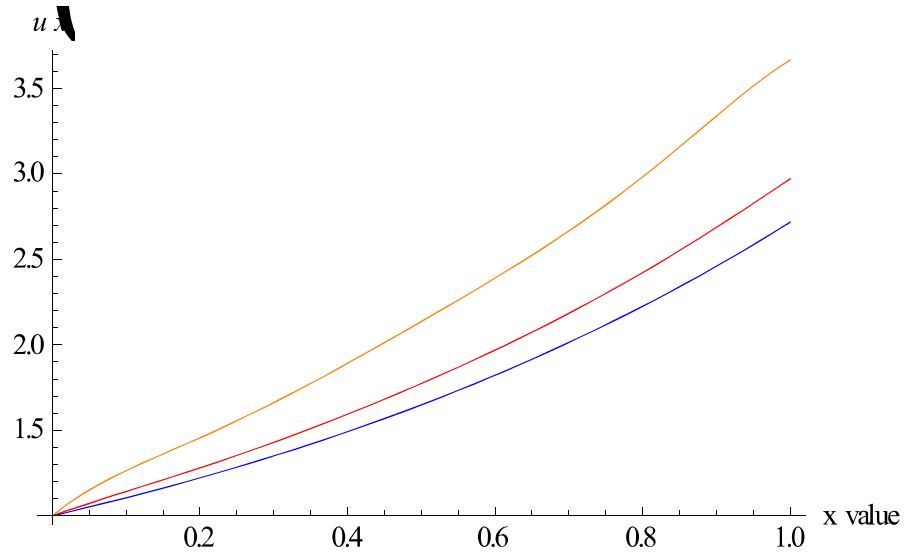


Figure 6.4.5 Approximate solution plot of $u(x)$ for $\alpha = \beta = 1$ (blue line), $\alpha = \beta = 0.9$ (orange line) and for $\alpha = \beta = 0.7$ (red line).

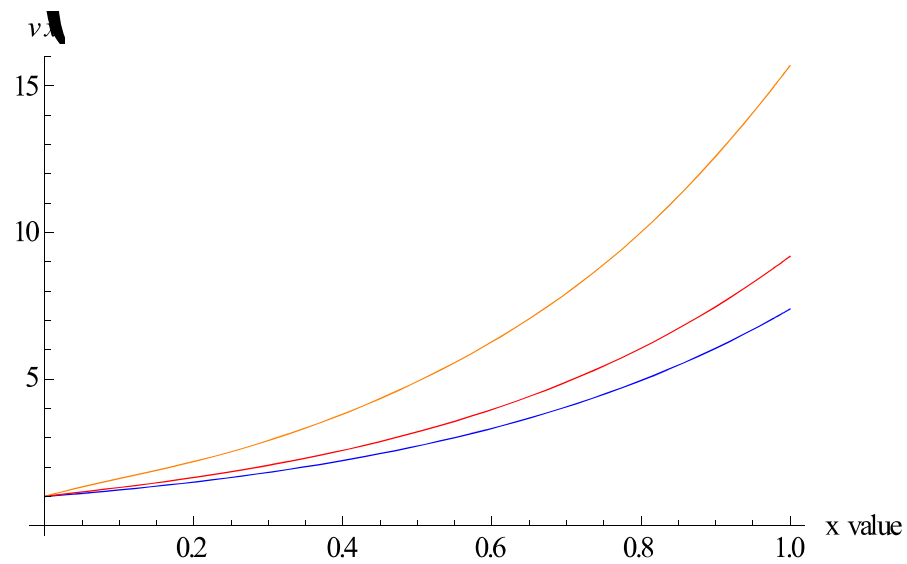


Figure 6.4.6 Approximate solution plot of $v(x)$ for $\alpha = \beta = 1$ (blue line), $\alpha = \beta = 0.9$ (orange line) and for $\alpha = \beta = 0.7$ (red line).

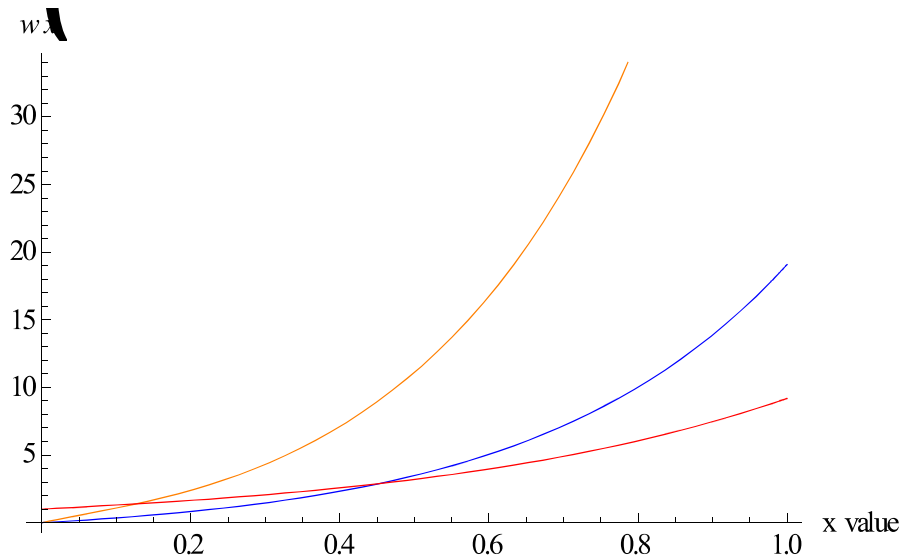


Figure 6.4.7 Approximate solution plot of $w(x)$ for $\alpha = \beta = 1$ (blue line), $\alpha = \beta = 0.9$ (orange line) and for $\alpha = \beta = 0.7$ (red line).

Case 6.4.2. Application to Fractional Stiff System of Nonlinear Differential Equations

Here, in this case the method presented in Section 6.3 is applied to solve a fractional stiff system of nonlinear differential equations. When the solution of a system contains components which change at significantly different rates to give changes in the independence variable, the system is said to be “stiff” (Hairer and Wanner (1996), Lapidus and Schiesser (1976)).

In practice to solve stiff problems, the choice of the solution steps is critical. Large steps will lose some fast changing properties of the system, while small steps will introduce too many round-off errors and cause numerical instability. The situation becomes much more complicated when stiffness is coupled with nonlinearity. In Carroll (1993) presented an exponentially fitted scheme for solving the stiff systems of initial-value problems. Here, we consider the following fractional stiff system of nonlinear differential equation (Wu and Xia (2001)):

$$\begin{aligned} D^\alpha u &= -1002 u + 1000 v^2, \\ D^\alpha v &= u - v - v^2, \end{aligned} \quad (6.4.7)$$

where, $0 < \alpha \leq 1$, and the initial conditions are $u(0)=1$ and $v(0)=1$. The exact solution of the system for $\alpha = 1$, is

$$\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \begin{pmatrix} e^{-2x} \\ e^{-x} \end{pmatrix}. \quad (6.4.8)$$

To solve this stiff system, we apply the scheme discussed in Section 6.3. The unknowns $u(x)$ and $v(x)$ are approximated in terms of the Bernstein's polynomials with $n = 5$ and the approximate solutions are obtained. Absolute errors of the approximate solutions and exact solutions are plotted in Figure 6.4.8 respectively for $u(x)$ and $v(x)$. And corresponding exact and approximated solutions are presented in Figure 6.4.9 and Figure 6.4.10 for $\alpha = \beta = 1$.

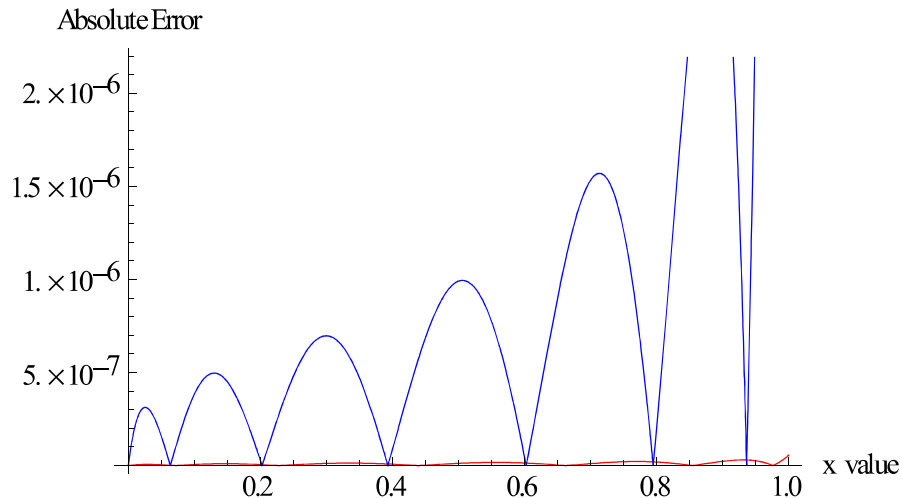


Figure 6.4.8 Plot of absolute errors between exact solution and approximate solution for $\alpha = \beta = 1$, where red line represents absolute error of $u(x)$ and blue line shows absolute error for $v(x)$.

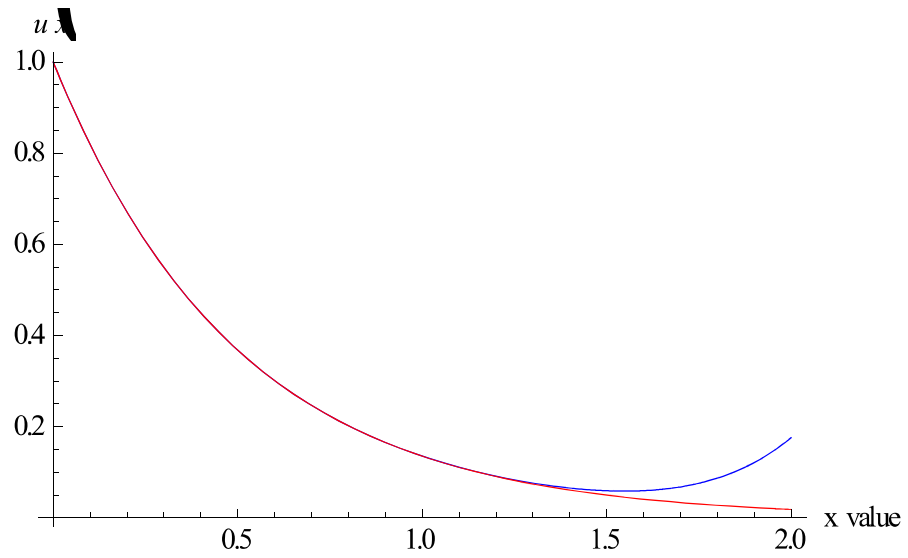


Figure 6.4.9 Plot of exact solution (red line) and proposed solution (blue line) of $u(x)$ for $\alpha = \beta = 1$.

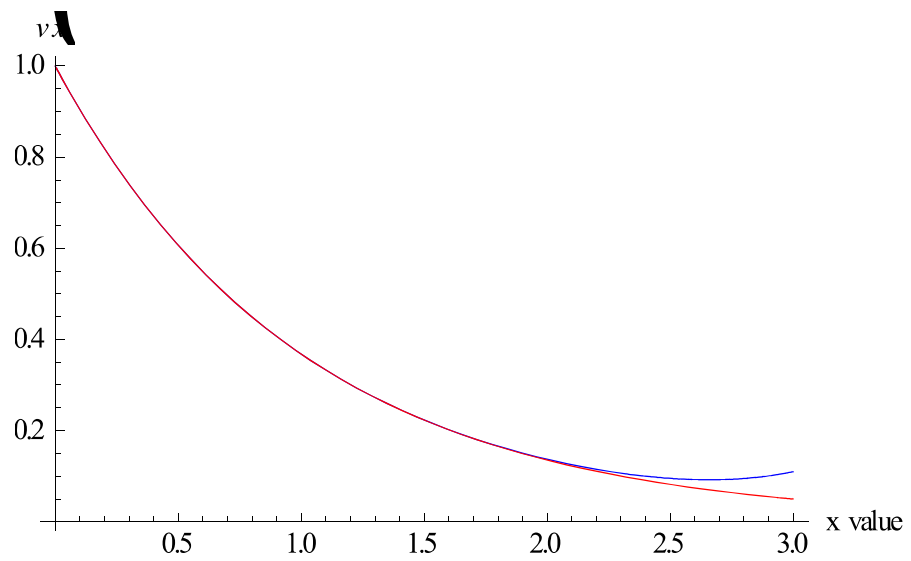


Figure 6.4.10 Plot of exact solution (red line) and proposed solution (blue line) of $v(x)$ for $\alpha = \beta = 1$.

6.5 Conclusions

In this article author discussed about the operational matrix of fractional differential equation of Bernstein polynomials. By using this method a nonlinear system of fractional differential equation and fractional stiff system of nonlinear differential equations are solved. For various values of fractional order obtained results are showed by figures. It can observe that newly developed works more efficiently in terms of error and achieves high accuracy even for small number of Bernstein polynomial. Yet as number of Bernstein polynomial increase the approximate solution will rapidly approach to exact solution. Here in this study, the proposed scheme for the two cases and further a researcher may be utilize it for other fractional models.