Chapter 4

Controller and Observer design using generalized Contraction theory

4.1 Introduction

Classical contraction approach witnesses a difficulty in proving the largest eigenvalue of the symmetric part of the Jacobian matrix to be negative definite (as it employs distance function between trajectories as a scalar function), since the matrix contains nonlinear terms, particularly for the class of large scale highly nonlinear systems. In order to overcome these difficulties, a generalized contraction theory based on the framework of vector distances known as vector contraction analysis is introduced in this chapter. In this, the concept of vector-valued distance function simplifies the problem with the help of a comparison system and vector differential inequalities. In fact, results are derived by comparing the relative distances of the comparison system with that of the auxiliary system. The comparison system is chosen with the property that it is quasi-monotone non-decreasing and contracting.

Moreover, in this chapter, we extend the proposed analysis to design controllers and observers for nonlinear systems in particular chaotic systems. In fact, we estimate the control input and observer gain in such a way that the comparison system thus obtained follows the indicated properties. The proposed approach of observer design avoids the usage of virtual system concepts in partial contraction theory as in [45]. Moreover, the proposed approach does not require the construction of the Lyapunov function as it is difficult to construct the Lyapunov function without any idea of its structure. We consider two examples namely duffing system and Chua's circuit for illustration.

The further part of this chapter is structured as follows. Section 4.2 provides the main results of convergence analysis using vector-based contraction theory. The exploitation of this convergence result to design controller and observer is elaborated in Section 4.3. Section 4.4 discusses the simulation results of the undertaken examples. At last, brief conclusions are entitled in Section 4.5.

4.2 Convergence analysis via vector contraction analysis

Vector contraction analysis attempts to relax the conditions of standard contraction analysis to achieve the exponential convergence of any pair of trajectories of the system. We note that the derivative of the scalar squared distance between a pair of neighboring trajectories of the large-scale nonlinear system need not be strictly negative definite. In such cases, the convergence analysis is observed with the help of a *comparison system*. Vector contraction analysis performs the convergence by comparing the relative distances of the trajectories of the original nonlinear dynamical system and the comparison system. In this study, we propose a few comparison results towards convergence analysis with the help of a notion of a *vector-valued norm* as defined below.

4.2.1 A vector-valued norm

We define a vector-valued norm as a function $\|\cdot\|_v : \mathbb{R}^n \to \mathbb{R}^m$ given by

$$\|\delta x\|_v = \sqrt{P\ (\mathrm{diag}(\delta x))^2 \mathbb{1}}, \tag{4.1}$$

where P is a nonzero real matrix $(p_{ij})_{m\times n}$ with all p_{ij} non-negative and $m \leq n$. Note that for $\delta x = (\delta x_1, \delta x_2, \cdots, \delta x_n)^\top \in \mathbb{R}^n$,

$$(\mathrm{diag}(\delta x))^2 = \begin{bmatrix} \delta x_1^2 & 0 & \cdots & 0 \\ 0 & \delta x_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \delta x_n^2 \end{bmatrix}, (\mathrm{diag}(\delta x))^2 \mathbb{1} = \begin{bmatrix} \delta x_1^2 \\ \delta x_2^2 \\ \vdots \\ \delta x_n^2 \end{bmatrix}$$

Thus, explicitly, (4.1) can be written as

$$\|\delta x\|_v = \begin{bmatrix} D_1(\delta x) \\ D_2(\delta x) \\ \vdots \\ D_m(\delta x) \end{bmatrix} := \begin{bmatrix} \sqrt{\sum_{j=1}^n p_{1j}\delta x_j^2} \\ \sqrt{\sum_{j=1}^n p_{2j}\delta x_j^2} \\ \vdots \\ \sqrt{\sum_{j=1}^n p_{mj}\delta x_j^2} \end{bmatrix}.$$

It is important to mention here that the terminology 'vector norm' is often used in the literature. Thus, 'vector norm' gives a *scalar-valued norm*. However, note that (4.1) introduces *vector-valued norm* for a vector. The following points can be observed from the definition of vector-valued norm $\|\cdot\|_v$.

(i) Every component of $\|\delta x\|_v$, i.e., $D_i(\delta x)$ for each $i = 1, 2, \cdots, m$, is a (scalar) norm in $\mathbb{R}^n$ (for proof, see the proof of Theorem 4.1).

(ii) $\|\delta x\|_v$ reduces to a (scalar-valued) norm in $\mathbb{R}^n$ when P is a matrix of order $n \times 1$.

(iii) $\|\delta x\|_v$ reduces to the usual Euclidean norm in $\mathbb{R}^n$ when P is a matrix of order $n \times 1$ with all entries 1.

These three points show that the vector-valued norm $\|\delta x\|_v$ is a true generalization of the notion of a norm in $\mathbb{R}^n$.

In the Theorem 4.1, we show that $\|\cdot\|_v$ follows all the properties of a *norm* and $\|\cdot\|_v^2$ possesses the *convexity* and *locally-Lipschitzian* properties. We further show that $\|\cdot\|_v^2$ is differentiable and then compute its derivative.

Theorem 4.1 *(a)* (Norm property). *The vector-valued norm $\|\cdot\|_v : \mathbb{R}^n \to \mathbb{R}^m$ defined in (4.1) has the following properties:*

(i) $\|\delta x\|_v = 0 \in \mathbb{R}^m$ *iff* $\delta x = 0 \in \mathbb{R}^n$,

(ii) $\|c\delta x\|_v = |c|\ \|\delta x\|_v$ *for any* $c \in \mathbb{R}$ *and* $\delta x \in \mathbb{R}^n$, *and*

(iii) $\|\delta x + \delta y\|_v \leq\leq \|\delta x\|_v + \|\delta y\|_v$ *for all* $\delta x, \delta y \in \mathbb{R}^n$.

(b) (Convexity property). *Let* $F(\delta x) = \|\delta x\|_v^2$, $\delta x \in \mathbb{R}^n$. *Then, for any* $\delta x, \delta y \in \mathbb{R}^n$ *and* $\lambda \in [0, 1]$,

$$F(\lambda\delta x + (1-\lambda)\delta y) \ \leq\leq\ \lambda F(\delta x) + (1-\lambda)F(\delta y).$$

(c) (Locally-Lipschitzian property). *Let* $F(\delta x) = \|\delta x\|_v^2$, $\delta x \in \mathbb{R}^n$. *Then, for any compact set* $\Omega \subset \mathbb{R}^n$ *there exists a constant* $k \in \mathbb{R}$ *such that*

$$\|F(\delta x) - F(\delta y)\| \leq k\|\delta x - \delta y\|^2 \quad \textit{for all} \quad \delta x, \delta y \in \Omega.$$

(d) (Differentiability). *The function* $F(\delta x) = \|\delta x\|_v^2$ *for* $\delta x \in \mathbb{R}^n$ *is (Fréchet) differentiable in* $\mathbb{R}^n$ *and for any* $\delta x \in \mathbb{R}^n$, *the Fréchet derivative of* F *is*

$$F'(\delta x) = 2\ P\ \delta x.$$

Proof of (a) (Norm Property).

(i) *This property follows from the fact that for each* $i = 1, 2, \cdots, m$

$$D_i(\delta x) = 0 \Longleftrightarrow (\delta x_1, \delta x_2, \cdots, \delta x_n) = (0, 0, \cdots, 0).$$

(ii) *From the expression of* $D_i(\delta x)$, *we note that* $D_i(c\delta x) = |c| D_i(\delta x)$ *for each* $i = 1, 2, \cdots, m$. *Hence,* $\|c\delta x\|_v = |c|\|\delta x\|_v$.

(iii) *For each* $i = 1, 2, \cdots, m$, *the* i*-th component* $\|\delta x\|_v$ *has the property that* $D_i(\delta x + \delta y) \leq D_i(\delta x) + D_i(\delta y)$ *for all* $\delta x, \delta y \in \mathbb{R}^n$. *Since every* $p_{ij} \geq 0$, *we get from the Cauchy-Schwarz inequality that*

$$\begin{aligned} \sum_{j=1}^{n} p_{ij}|\delta x_j||\delta y_j| &= \sum_{j=1}^{n} \left|\sqrt{p_{ij}}\ \delta x_j\right|\ \left|\sqrt{p_{ij}}\ \delta y_j\right| \\ &\leq \sqrt{\sum_{j=1}^{n} \sqrt{p_{ij}}\ \delta x_j^2} \sqrt{\sum_{j=1}^{n} \sqrt{p_{ij}}\ \delta y_j^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} (D_i(\delta x + \delta y))^2 &= \sum_{j=1}^{n} p_{ij}\ (\delta x_j + \delta y_j)^2 \\ &\leq \sum_{j=1}^{n} p_{ij}\ \left(\delta x_j^2 + 2\delta x_j \delta y_j + \delta y_j^2\right) \\ &\leq \sum_{j=1}^{n} p_{ij}\ \delta x_j^2 + 2\ \sum_{j=1}^{n} p_{ij}\ |\delta x_j|\ |\delta y_j| \\ &+ \sum_{j=1}^{n} p_{ij}\ \delta y_j^2 \\ &\leq \left(\sqrt{\sum_{j=1}^{n} p_{ij}\ \delta x_j^2} + \sqrt{\sum_{j=1}^{n} p_{ij}\ \delta y_j^2}\right)^2. \end{aligned}$$

Hence, $D_i(\delta x+\delta y)\le D_i(\delta x)+D_i(\delta y)$.

Proof of (b) (Convexity property). The proof is followed from the fact that the i-th component function $D_i(\delta x)$*, being a norm in* $\mathbb{R}^n$*, a convex function on* $\mathbb{R}$*, for each* $i=1,2,\cdots,n$.

Proof of (c) (Locally-Lipschitzian property). For any $i\in\{1,2,\cdots,n\}$*, the i-th component function of* $F(\delta x)$ *is* $(D_i(\delta x))^2=\sum_{j=1}^n p_{ij}\delta x_j^2$*. Note that the Hessian matrix of* $(D_i(\delta x))^2$ *is the diagonal matrix diag* $(p_{i1},p_{i2},\cdots,p_{in})$ *which is positive semi-definite as every* $p_{ij}\ge 0$*. Hence,* $(D_i(\delta x))^2$ *is a convex function on* $\mathbb{R}^n$*. By the result in [51], for each* $i=1,2,\cdots,m$*, there exists a constant* $k_i>0$ *such that*

$$\left|(D_i(\delta x))^2-(D_i(\delta y))^2\right|\le k_i\|\delta x-\delta y\| \quad \text{for all} \quad \delta x,\delta y\in\Omega.$$

Hence, for any $\delta x,\delta y$ *in* Ω,

$$\begin{aligned}\|F(\delta x)-F(\delta y)\| &= \sqrt{\sum_{j=1}^n\left((D_1(\delta x))^2-(D_1(\delta y))^2\right)^2}\\ &= \sqrt{k_1^2+k_2^2+\cdots+k_m^2}\ \|\delta x-\delta y\|.\end{aligned}$$

The result follows by letting $k=\sqrt{k_1^2+k_2^2+\cdots+k_m^2}$.

Proof of (d) (Differentiability). Let $h=(h_1,h_2,\cdots,h_n)^\top$*. The result is followed by the following limit:*

$$\begin{aligned}&\lim_{h\to 0}\frac{1}{\|h\|}\|F(\delta x+h)-F(\delta x)-2\ P\ \delta x\|\\ =&\lim_{h\to 0}\frac{1}{\|h\|}\left\|P(\delta x+h)^2-P(\delta x)^2-2\ P\ \delta x\right\|\\ =&\lim_{h\to 0}\frac{1}{\|h\|}\left\|P\left[h_1^2,h_2^2,\cdots,h_n^2\right]^\top\right\|\\ =&\lim_{h\to 0}\left\|P\left[\frac{h_1^2}{\|h\|},\frac{h_2^2}{\|h\|},\cdots,\frac{h_n^2}{\|h\|}\right]^\top\right\|\\ =&\ 0.\end{aligned}$$

□

The notion of norm $\|\cdot\|_v$, defined by (4.1), evidently, induces a vector distance between a pair of points $x,\ x+\delta x\in\mathbb{R}^n$ as follows:

$$\|x+\delta x-x\|_v=\sqrt{P\ (\mathrm{diag}(\delta x))^2\mathbb{1}}.$$

In the rest of the article, we assume that P is a nonzero matrix, as the case of P being zero matrix is uninteresting for the relative distance of two trajectories.

4.2.2 Main comparison results

In this section, we derive some results on the comparison of solutions for the system (1.4) and the solution of an auxiliary (comparison) system $\dot{u} = \phi(t, u, x)$, where ϕ possesses certain quasi-monotone property.

Theorem 4.2 *Consider the system (1.4) and a function* $\phi \in C\left[\mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^n,\ \mathbb{R}^n\right]$, $(t, u, x) \mapsto \phi(t, u, x)$, *which is quasi-monotone non-decreasing in* $u \in \mathbb{R}^n$. *Suppose for any solution* $\delta\psi(t, x_0, \delta x_0)$ *on* $t \geq t_0$ *of the system (1.4)*

$$2P\ diag(\delta\psi)(diag(h(t, \delta\psi, \psi)))\mathbb{1} << \phi\left(t, P(diag(\delta\psi))^2\mathbb{1}, \psi\right)$$

for a matrix $P = (p_{ij})_{n\times n}$ *with* $p_{ij} \geq 0$ *for all* $i, j = 1, 2, \cdots, n$. *Further, suppose for* $t \geq t_0 > 0$, *there exists* [1] *a maximal solution* [2] $R(t)$ *of*

$$\dot{u} = \phi(t, u, x),\ \ u(t_0) = u_0 \geq\geq 0. \tag{4.2}$$

Then, any solution $\delta\psi(t, x_0, \delta x_0)$ *of (1.4) on* $t \geq t_0$ *which satisfies* $P\ (diag(\delta x_0))^2\mathbb{1} << u_0$ *has the property that*

$$P\ (diag(\delta\psi(t)))^2\mathbb{1} << R(t)\ \ \text{for all}\ \ t \geq t_0.$$

From the above, the following conclusion holds:
If $R(t) \to 0$ *as* $t \to \infty$, *then* $\|\delta\psi(t)\| \to 0$ *as* $t \to \infty$, *which implies that all trajectories of the original dynamical system (1.1) converge with respect to one another as* $t \to \infty$.

Proof: Consider

$$D(t) = P(\mathrm{diag}(\delta\psi(t)))^2\mathbb{1}\ \ \text{for}\ t \geq t_0.$$

Suppose the component functions of $D(t)$ and $R(t)$ be $D_i(t)$ and $R_i(t)$, respectively, $i = 1, 2, \cdots, n$. Evidently, if the i-th row of the matrix P be $p_i^\top$, then $D_i(t) = p_i^\top(\mathrm{diag}(\delta\psi(t)))^2\mathbb{1}$ for each $i = 1, 2, \cdots, n$. As $D(t_0) = P(\mathrm{diag}(\delta\psi(t_0)))^2\mathbb{1} << u_0 = R(t_0)$, and $D(t)$ and $R(t)$ are two continuous functions, there exists $\delta_1 > 0$ such that $D(t) << R(t)$ for all $t \in [t_0, t_0 + \delta_1)$. Construct a set

$$S := \bigcup_{i=1}^{n} \{t \in [t_0, \infty) | R_i(t) \leq D_i(t)\}\,. \tag{4.3}$$

[1] The existence of a maximal solution is evident from Theorem 1.3.1 of [95].

[2] A solution $R(t)$ of the system (1.1) is called a maximal solution if for every solution $x(t)$ of (1.1), $x(t) \leq\leq R(t)$ for all $t \geq t_0$.

We prove that S is an empty set. Then, the proof will be complete.

If possible let S be not empty. Then, S, being a nonempty and bounded below set, has an infimum. Let $\tau = \inf S$. We note that the set S is closed, since $R(t)$ and $D(t)$ are continuous functions on $[t_0, \infty)$. Therefore, $\tau \in S$ and hence there exists j in $\{1, 2, \cdots, n\}$ such that $D_j(\tau) = R_j(\tau)$. Moreover, $p_i^\top (\mathrm{diag}(\delta\psi(\tau)))^2 \mathbb{1} = D_i(\tau) \geq R_i(\tau)$ for all $i = 1, 2, \cdots, j-1, j+1, \cdots, n$. Therefore, due to quasi-monotone non-decreasing property of the function ϕ, we obtain

$$\phi_j\left(\tau, P\ (\mathrm{diag}(\delta\psi(\tau)))^2 \mathbb{1}, \psi(\tau)\right) \geq \phi_j(\tau, R(\tau), \psi(\tau)). \tag{4.4}$$

Again, since $D(t_0) << R(t_0)$, we have $\tau \neq t_0$ and hence $\tau > t_0$. By the definition of τ, there exists $\delta_2 > 0$ such that $D_j(t) < R_j(t)$ for all $t \in (\tau - \delta_2, \tau) \subset [t_0, t_0 + \delta_1)$. Therefore,

$$\begin{aligned} \dot{D}_j(\tau) &= \lim_{\xi \to 0-} \frac{D_j(\tau + \xi) - D_j(\tau)}{\xi} \geq \lim_{\xi \to 0-} \frac{R_j(\tau + \xi) - R_j(\tau)}{\xi} \\ &= \dot{R}_j(\tau) \end{aligned} \tag{4.5}$$

By the assumption

$$2P\ \mathrm{diag}(\delta\psi)(\mathrm{diag}(h(t, \delta\psi, \psi)))\mathbb{1} << \phi(t, P(\mathrm{diag}(\delta\psi))^2\mathbb{1}, \psi),$$

we obtain

$$\begin{aligned} \phi_j(\tau, R(\tau), \psi(\tau)) &> 2p_j^\top \mathrm{diag}(\delta\psi(\tau))\mathrm{diag}(h(\tau, \delta\psi(\tau), \psi(\tau))) \\ &> \phi_j(\tau, R(\tau), \psi(\tau)), \end{aligned}$$

which is a contradiction. Hence, the set S is empty, and

$$\begin{aligned} R_i(t) &> p_i^\top (\mathrm{diag}(\delta\psi(t))^2 \mathbb{1} \text{ for all } i = 1, 2, \cdots, n, \\ \text{i.e., } P\ (\mathrm{diag}(\delta\psi(t)))^2\mathbb{1} &<< R(t), \quad \text{for all } t \geq t_0. \end{aligned} \tag{4.6}$$

Hence, the conclusion follows from (4.6). □

The following example is provided to better illustrate the above proposed theory.

Example 4.3 *Consider a non-autonomous nonlinear system*

$$\left.\begin{aligned} \dot{x}_1 &= -\sin^4(t)x_1^5 - \cos^2(t)x_1 + 2\cos^2(t)x_2 - 10x_1 + x_3 \\ \dot{x}_2 &= 6x_1 - 20x_2 - 2t^2x_2^5 - \sin^2(t)x_2 + 2\sin^2(t)x_3 + x_4 \\ \dot{x}_3 &= 6x_1 + 2x_2 - 15x_3 - 4\cos^4(t)x_3^7 + x_4 + 5x_5 \\ \dot{x}_4 &= x_1 + 2x_2 - 3x_4 - \sin^2(t)x_4^5 - 5t^2x_4^5 + x_5 \\ \dot{x}_5 &= x_1 + 4x_2 - 10x_5 + x_3 - \sin^2(t)x_5^3 - \cos^4(t)x_5^7 \end{aligned}\right\} \tag{4.7}$$

where $x = [x_1, x_2, x_3, x_4, x_5]^\top$ *is a state vector. We noticed that it is quite complex to construct Lyapunov candidate function to show the stability of the above non-autonomous nonlinear system (4.7). Moreover, using the classical contraction, it is quite tedious to prove the largest eigenvalue of the Jacobian matrix of the system (4.7) to be negative definite for exponential convergence of this system as this matrix is large and contains the time-varying nonlinear terms.*

Hence, we use the approach of vector-based contraction which provides an easy and efficient way to prove the convergence of such type of large-scale nonlinear systems. The squared vector distance function assuming P *as diag*$(\mathbb{1})$ *is defined as:* $\|\delta x\|_v^2 = [\delta x_1^2, \delta x_2^2, \delta x_3^2, \delta x_4^2, \delta x_5^2]^\top$. *The rate of change of this squared vector distance along the virtual dynamics of the system (4.7) is given by*

$$\begin{aligned}
\frac{d}{dt}\delta x_1^2 &= 2\delta x_1 \delta \dot{x}_1 \\
&= (-20 - 10\sin^4(t)x_1^4 - 2\cos^2(t))\delta x_1^2 + 4\cos^2(t)\delta x_1\delta x_2 + 2\delta x_1 \delta x_3 \\
&< -19\delta x_1^2 + 2\delta x_2^2 + \delta x_3^2 \\
\frac{d}{dt}\delta x_2^2 &= 12\delta x_1\delta x_2 + (-2\sin^2(t) - 20t^2x_2^4 - 40)\delta x_2^2 + 2\delta x_2\delta x_4 + 4\sin^2(t)\delta x_2 \delta x_3 \\
&< 6\delta x_1^2 - 33\delta x_2^2 + 2\delta x_3^2 + \delta x_4^2 \\
\frac{d}{dt}\delta x_3^2 &= 12\delta x_1\delta x_3 + 4\delta x_2\delta x_3 + (-56\cos^4(t)x_3^6 - 30)\delta x_3^2 + 2\delta x_3\delta x_4 + 10\delta x_3\delta x_5 \\
&< 6\delta x_1^2 + 2\delta x_2^2 - 16\delta x_3^2 + \delta x_4^2 + 5\delta x_5^2 \\
\frac{d}{dt}\delta x_4^2 &= 2\delta x_1\delta x_4 + (-10\sin^2(t)x_4^4 - 50t^2x_4^4 - 6)\delta x_4^2 + 2\delta x_4\delta x_5 + 4\delta x_2\delta x_4 \\
&< \delta x_1^2 + 2\delta x_2^2 - 2\delta x_4^2 + \delta x_5^2 \\
\frac{d}{dt}\delta x_5^2 &= 2\delta x_1\delta x_5 + (-6\sin^2(t)x_5^2 - 14\cos^4(t)x_5^6 - 20)\delta x_5^2 + 2\delta x_3\delta x_5 + 8\delta x_2\delta x_5 \\
&< \delta x_1^2 + 4\delta x_2^2 + \delta x_3^2 - 14\delta x_5^2
\end{aligned} \tag{4.8}$$

The comparison system obtained from equations (4.8) is

$$\left.\begin{aligned}
\dot{w}_1 &= -19w_1 + 2w_2 + w_3 \\
\dot{w}_2 &= 6w_1 - 33w_2 + 2w_3 + w_4 \\
\dot{w}_3 &= 6w_1 + 2w_2 - 16w_3 + w_4 + 5w_5 \\
\dot{w}_4 &= w_1 + 2w_2 - 2w_4 + w_5 \\
\dot{w}_5 &= w_1 + 4w_2 + w_3 - 14w_5
\end{aligned}\right\} \tag{4.9}$$

quasi-monotone non-decreasing (off-diagonal elements non-negative) and exponentially convergent (largest eigenvalue of the symmetric part of its Jacobian matrix is negative definite). Hence, from Theorem 4.2, the original system (4.7) is exponentially convergent which is shown by the simulation result in Fig. 4.1 with $x(0) = [5, 3, 1, -5, -8]$.

From above, it can be observed that it is easy to prove the largest eigenvalue of the Jacobian of this comparison system to be negative definite (since this system is linear) to prove the original system to be contracting.

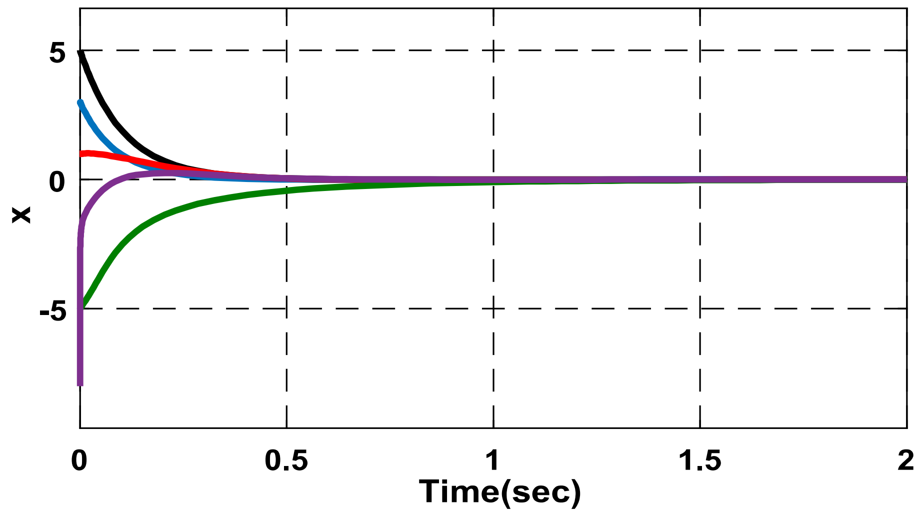

Figure 4.1: State evolutions of system (4.7) with time

Remark 1 *It is important to discuss that how to choose the function* ϕ *in (4.2) (Theorem 4.2). Let us see some examples. Let us suppose that the system (4.2) is a linear system, then function* $\phi = Au$*, where* A *is some square matrix and* u *is some vector of appropriate dimension. If* ϕ *is quasi-monotone non-decreasing, then all the off-diagonal elements of* A *must be non-negative (see system (4.9)). Now, let us suppose that system (4.2) is a nonlinear function. If* ϕ *is quasi-monotone non-decreasing then for* $1 \leq i \leq n$ *and all* $1 \leq j \leq n, j \neq i, x_i = y_i, x_j \leq y_j$ *implies that* $\phi_i(t, x, z) \leq \phi_i(t, y, z)$ *for the i-th component of* $\phi(t, \cdot)$ *and for each* t *and* z*. This quasi-monotone non-decreasing property (similar in some sense to positivity property) is an essential condition to be satisfied by the comparison system (4.2) because we need to compare the solution of this comparison system with the solution of the system (1.4) and in order to make the comparison between solutions feasible, the solutions of the comparison system should be non-negative for the non-negative initial conditions.*

In linear cases, the estimation of derivative of $\|\delta x\|_v^2$ as a function of t and $P\ (\mathrm{diag}(\delta\psi(t)))^2\mathbb{1}$ is more natural. The following corollary is in that direction. Let us suppose the linear system with $f(t,x) = M(t)x$ in (1.1), where $M(t)$ is a time-varying matrix with appropriate dimension, then its variational system is as follows:

$$\delta\dot{x} = M(t)\delta x = \bar{h}(t, \delta x) \tag{4.10}$$

where $\bar{h} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ and t is the time.

Corollary 1 *Consider a function $\phi \in C\,[\mathbb{R}_{\geq 0} \times \mathbb{R}^n,\ \mathbb{R}^n]$, $(t,u) \mapsto \phi(t,u)$ to be quasi-monotone non-decreasing in $u \in \mathbb{R}^n$. Suppose for any solution $\delta\psi(t, \delta x_0)$ on $t \geq t_0$ of the system (4.10)*

$$2P\ diag(\delta\psi)(diag(\bar{h}(t,\delta\psi)))\mathbb{1}\ <<\ \phi\left(t, P\ (diag(\delta\psi))^2\mathbb{1}\right)$$

for a matrix $P = (p_{ij})_{n\times n}$ with $p_{ij} \geq 0$ for all $i,j = 1,2,\cdots,n$. Further, assume for $t \geq t_0 > 0$, there exists a maximal solution $R(t)$ of

$$\dot{u} = \phi(t,u),\ u(t_0) = u_0 \geq\geq 0.$$

Then, any solution $\delta\psi(t, \delta x_0)$ of (4.10) on $t \geq t_0$ with $P\ (diag(\delta x_0))^2\mathbb{1} << u_0$ has the property that

$$P\ (diag(\delta\psi))^2\mathbb{1} << R(t)\ \ \textit{for all}\ \ t \geq t_0.$$

Proof: The proof is similar to that of Theorem 4.2. □

Theorem 4.4 *Let K be a pointed closed convex cone in $\mathbb{R}^n$. Consider the system (1.4) and a function $\phi \in C\,[\mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^n,\ \mathbb{R}^n]$, $(t,u,x) \mapsto \phi(t,u,x)$, which is quasi-monotone non-decreasing in $u \in \mathbb{R}^n$ with respect to K. Suppose for any solution $\delta\psi(t, x_0, \delta x_0)$ on $t \geq t_0$ of the variational system (1.4)*

$$2P diag(\delta\psi)\, diag(h(t,\delta\psi,\psi))\mathbb{1} <<_K \phi\left(t, P\ (diag(\delta\psi))^2\mathbb{1}, \psi\right)$$

for a nonzero matrix $P = (p_{ij})_{n\times n}$ with $p_{ij} \geq 0$ for all $i,j = 1,2,\cdots,n$. Further, suppose that for $t \geq t_0 > 0$, there exists a maximal solution $R(t)$ of

$$\dot{u} = \phi(t,u),\ u(t_0) = u_0 \geq\geq 0.$$

Then, any solution $\delta\psi(t, x_0, \delta x_0)$ of (1.4) on $t \geq t_0$ with $P\ (diag(\delta x_0))^2 \mathbb{1} <<_K u_0$ has the property that

$$P\ (diag(\delta\psi))^2 \mathbb{1} <<_K R(t) \quad for\ all \quad t \geq t_0. \tag{4.11}$$

From (4.11), a similar conclusion as in Theorem 4.2 is also followed here in the framework of cone.

Proof: The proof is similar to that of Theorem 4.2. It is exactly the same till the equation (4.3). Then the rest of the part is an appropriate modification of the inequalities w.r.t. the partial ordering induced by K. □

We now again consider the system (1.1) and assume that it has a finite equilibrium solution $\bar{x}$. Suppose the squared vector distance of a solution x, of the system, from $\bar{x}$ is given by

$$\|x - \bar{x}\|_v^2 = P\ (\text{diag}(x - \bar{x}))^2 \mathbb{1},$$

where P is a nonzero real matrix $(p_{ij})_{n\times n}$ where all p_{ij}'s are non-negative. Obviously, $\|x - \bar{x}\|_v^2$ is a vector in $\mathbb{R}^n$. In the following, we denote the i-th row of P by $p_i^\top$. Suppose $C_0 \in \mathbb{R}^n$ be the squared distance between the initial data x_0 and the equilibrium solution $\bar{x}$. We denote $\|\delta x\|_v^2$ for the squared virtual displacement of x from $\bar{x}$. Then, evidently, $\|\delta x_0\|_v^2 = C_0$. From (1.1), we get the following differential relation

$$\begin{aligned}\frac{d}{dt}\left(\|\delta x\|_v^2\right) &= \frac{d}{dt}\begin{bmatrix} p_1^\top\ (\text{diag}(\delta x))^2 \mathbb{1} \\ \vdots \\ p_n^\top\ (\text{diag}(\delta x))^2 \mathbb{1} \end{bmatrix} \\ &= 2\ P\ \text{diag}(\delta x)\ (\text{diag}(\delta \dot{x}) \mathbb{1}).\end{aligned}$$

Therefore under the assumption in Theorem 4.2, we have,

$$\frac{d}{dt}\left(\|\delta x\|_v^2\right) \ << \ \phi(t, \|\delta x\|_v^2, x). \tag{4.12}$$

For finding the properties of solution of the above inequality, we take the comparison system

$$\dot{u} = \phi(t, u, x), \qquad u(t_0) = u_0 >> \|\delta x_0\|_v^2 = C_0. \tag{4.13}$$

Further, if $R(t)$ is the maximal solution of the above equation, then a solution of (4.12), follows from Theorem 4.2 satisfies, $\|\delta x(t)\|_v^2 << R(t)$. If $R(t)$ is exponentially convergent, by component-wise integration, we have

$$\|\delta x\|_v \ << \ C_0^{\frac{1}{2}} \exp\left(-\lambda t\right), \tag{4.14}$$

where λ is the convergence rate. Then (4.14) shows that the virtual vector distance $\|\delta x\|_v$ is lesser than $C_0^{\frac{1}{2}}$ and it converges exponentially to zero as $t \to \infty$, which implies that all trajectories converge to the equilibrium point $\bar{x}$ and as a result, the system is asymptotically stable.

4.3 Controller and Observer design using vector-based Contraction theory

In this section, we exploit the proposed theory for the design of controller and observer for a class of nonlinear systems. The strategy for designing control law followed by the observer design procedure is discussed in Theorem 4.5 and Theorem 4.6.

4.3.1 Controller design

We discuss the results of control design to solve a tracking problem and then we generalize results for the stabilization problem in the following theorem.

Theorem 4.5 *Let us consider the system with feedback control τ as*

$$\dot{x} = f(t, x, \tau), \quad y = H(t, x)$$

where $x \in \mathbb{R}^n$ is a state vector, $\tau \in \mathbb{R}^m$ is a control input vector, $y \in \mathbb{R}^p$ is an output vector, $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a nonlinear smooth vector field, and $H : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^p$ is a differentiable function. Now, our aim is that output $y(t)$ tracks the signal $r(t)$, hence, we describe the tracking error signal as $e(t) = y(t) - r(t)$ and consider the error dynamics as

$$\dot{e} = h(t, e, \tau), \tag{4.15}$$

where $h : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is a smooth nonlinear vector field. If there exists the control $\tau = g(e)$, $g : \mathbb{R}^p \to \mathbb{R}^m$ with $g(0) = 0$ such that derivative of squared vector valued norm $\|\delta e\|_v : \mathbb{R}^p \to \mathbb{R}^q$ follows

$$\frac{d}{dt}(\|\delta e\|_v^2) \ << \ \phi\left(t, P\ diag(\delta e)^2 \mathbb{1}, e\right) \qquad \forall\ t \geq t_0,$$

for a matrix $P = (p_{ij})_{q\times n}$ *with* $p_{ij} \geq 0$ *and function* $\phi \in C\left[\mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^n,\ \mathbb{R}^q\right]$ *follows the quasi-monotonicity non-decreasing and contracting property. Then the tracking error becomes zero with the control law* τ*, i.e. output tracks the reference trajectory.*

Proof: The virtual dynamics of the system (4.15) at fixed time t with the control $\tau = g(e)$ is obtained as

$$\delta\dot{e} = \Big(\frac{\partial h}{\partial e} + \frac{\partial h}{\partial g}\frac{\partial g}{\partial e}\Big)\delta e, \tag{4.16}$$

Consider the vector valued norm defined by (4.1) for the system (4.15). Then, its derivative along the system trajectories (4.16) is

$$\frac{d}{dt}(\|\delta e\|_v^2) = 2P\ \mathrm{diag}(\delta e)\ \mathrm{diag}((\frac{\partial h}{\partial e} + \frac{\partial h}{\partial g}\frac{\partial g}{\partial e})\delta e)\mathbb{1},$$

The control $g(e)$ is designed in such a way that above vector valued norm follows

$$\frac{d}{dt}(\|\delta e\|_v^2)\ << \phi\left(t, P\ \mathrm{diag}(\delta e)^2\mathbb{1}, e\right) \quad \forall\ t\ \geq t_0,$$

with a matrix $P = (p_{ij})_{q\times n}, p_{ij} \geq 0$ and function ϕ follows the quasi-monotonicity non-decreasing and contracting property. The rest part of the proof is same as that of Theorem 4.2. Hence, it is proved that the system trajectories track the actual trajectory with the control law τ. □

In a similar way, the derived results can be applied for the stabilization problem by defining the system dynamics in place of error dynamics.

4.3.2 Observer design

In the following procedure of observer design we do not need the concept of virtual system and partial contraction approach as explored by many researchers [45, 110]. We investigate the novel procedure of observer design using a vector framework based contraction approach. Let us consider the dynamical system

$$\dot{x} = f(x), \tag{4.17}$$

where $x \in \mathbb{R}^n$ denotes the state vector and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear vector field which is continuously differentiable with system output as

$$y = g(x), \tag{4.18}$$

where $y \in \mathbb{R}^q$ represents the output vector and $g : \mathbb{R}^n \to \mathbb{R}^q$ is assumed to be smooth. We assume that the system (4.17) is globally detectable. The structure of the observer for the undertaken system (4.17) can be defined as

$$\dot{\hat{x}} = f(\hat{x}) + K(y - \hat{y}), \tag{4.19}$$

where $\hat{x} \in \mathbb{R}^n$ represents the estimated states, matrix K is the observer gain matrix to be chosen appropriately and the output $\hat{y}$ is represented as

$$\hat{y} = g(\hat{x}).$$

Our aim is to find the gain matrix K in such a way that error of estimation $e(t) = x(t) - \hat{x}(t)$ converges exponentially to zero. Note that Theorem 4.2 cannot be directly applicable to this system as we do not require system (4.17) to be contracting. Therefore, we provide the following results through the dynamics of the error of estimation.

Theorem 4.6 *Suppose that there exists the gain matrix K such that the squared vector-valued norm derivative of the system (4.19) follows*

$$\frac{d}{dt}\|\delta\hat{x}\|_v^2 \;<<\; \phi\left(P\;(diag(\delta\hat{x})^2)\mathbb{1}, \hat{x}\right) \qquad \forall\, t \geq t_0, \tag{4.20}$$

for a nonzero matrix $P = (p_{ij})_{q\times n}$ with $p_{ij} \geq 0$, and function $\phi \in C\left[\mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^n,\ \mathbb{R}^q\right]$ satisfies the quasi-monotonicity non-decreasing and contracting property, then the error of estimation $e(t) = x(t) - \hat{x}(t)$ converges exponentially to zero, i.e, for some constants $b > 0$ and $L > 0$

$$\|e(t)\| \leq Le^{b(t-t_0)}\|e(t_0)\|.$$

Proof: The virtual dynamics of system (4.19) is defined as

$$\delta\dot{\hat{x}} = \left(\frac{\partial f}{\partial \hat{x}} - K\frac{\partial g}{\partial \hat{x}}\right)\delta\hat{x}, \tag{4.21}$$

Consider the vector-valued norm defined by (4.1) for the system (4.19). Hence, the squared vector-valued norm derivative along the system trajectories (4.21) is given by

$$\frac{d}{dt}(\|\delta\hat{x}\|_v^2) = 2P\ \mathrm{diag}\delta\hat{x}\ (\mathrm{diag}\delta\dot{\hat{x}})\mathbb{1},$$

The matrix K is designed such that the above equality transforms to the following inequality

$$\frac{d}{dt}(\|\delta\hat{x}\|_v^2) \;<<\; \phi\left(P\;(\mathrm{diag}(\delta\hat{x})^2)\mathbb{1}, \hat{x}\right) \qquad \forall\, t \geq t_0, \tag{4.22}$$

for a matrix $P = (p_{ij})_{q\times n}$ with $p_{ij} \geq 0$ and function ϕ as quasi-monotone non-decreasing and contracting. Hence, from Theorem 4.2, the distance between any pair of trajectories $\|\delta\hat{x}\|$ of the estimated system (4.19) converges exponentially to zero as the comparison system (obtained from the inequality (4.22)) trajectory converges to zero (the comparison system is contracting). This means that, $\forall\, t \geq t_0$ we have

$$\|\hat{x}_1(t) - \hat{x}_2(t)\| \leq \; Le^{-b(t-t_0)}\|\hat{x}_1(t_0) - \hat{x}_2(t_0)\|, \tag{4.23}$$

for two solutions $\hat{x}_1(t)$ and $\hat{x}_2(t)$. Furthermore, note that $x(t)$ is a particular solution of (4.19) as (4.17) and (4.19) differ in only correction term which vanishes when $x(t)$ is the solution of the (4.17). Hence, $\hat{x}_2(t)$ is replaced with $x(t)$, also $\hat{x}_1(t)$ is denoted as $\hat{x}(t)$, therefore above equation (4.23) becomes

$$\|e(t)\| = \|x(t) - \hat{x}(t)\| \leq Le^{-b(t-t_0)}\|e(t_0)\|.$$

Therefore, it is proved that the error of estimation vanishes exponentially to zero. □

Remark 4.7 *The control τ or K matrix can be intuitively selected in such a way that the comparison system obtained has non-negative off-diagonal elements (quasi-monotone non-decreasing) and its Jacobian must be negative definite (contracting).*

4.4 Simulation examples

Example 4.8 *Consider the model*

$$\left.\begin{aligned} \frac{d}{dt}x_i =& -\rho_i x_i^3 - 3x_i + \sigma, \quad i = 1, 2, \ldots, n \\ \frac{d}{dt}\sigma =& \sum_{i=1}^{n} a_i x_i - (p+1)\sigma, \end{aligned}\right\} \tag{4.24}$$

where $\rho_i > 0$ and $p > 0$. The squared vector-valued norm, given by (4.1), assuming the matrix P as diag($\mathbb{1}$) is defined as: $\|\delta x\|_v^2 = [\delta x_1^2, \delta x_2^2, \ldots, \delta x_n^2, \delta\sigma^2]^\top$. The virtual dynamics of the system (4.24) is given by

$$\left.\begin{aligned} \delta\dot{x}_i =& \; -3\rho_i x_i^2 \delta x_i - 3\delta x_i + \delta\sigma \\ \delta\dot{\sigma} =& \sum_{i=1}^{n} a_i \delta x_i - (p+1)\delta\sigma. \end{aligned}\right\} \tag{4.25}$$

The rate of change of squared vector-valued norm along the system trajectories (4.25),

$$\begin{aligned}\frac{d}{dt}(\delta x_i^2) &= (-6\rho_i x_i^2 - 6)\delta x_i^2 + 2\delta x_i \delta\sigma\\ \frac{d}{dt}(\delta\sigma^2) &= \sum_{i=1}^{n} 2a_i \delta\sigma\delta x_i - 2(p+1)\delta\sigma^2.\end{aligned} \tag{4.26}$$

Using the inequality

$$\delta x_i \delta\sigma \leq \frac{\delta x_i^2}{2} + \frac{\delta\sigma^2}{2}, \tag{4.27}$$

Equation (4.26) gives,

$$\begin{aligned}\frac{d}{dt}(\delta x_i^2) &< -5\delta x_i^2 + \delta\sigma^2.\\ \frac{d}{dt}(\delta\sigma^2) &\leq \sum_{i=1}^{n} a_i \delta x_i^2 - \Big(2(p+1) - \sum_{i=1}^{n} a_i\Big)\delta\sigma^2.\end{aligned} \tag{4.28}$$

With the help of (4.28), we consider the comparison system

$$\begin{aligned}\frac{d}{dt}w_i &= -5w_i + w_{n+1}\\ \frac{d}{dt}w_{n+1} &= \sum_{i=1}^{n} a_i w_i - \Big(2(p+1) - \sum_{i=1}^{n} a_i\Big)w_{n+1}.\end{aligned} \tag{4.29}$$

The system (4.29) is quasi-monotone non-decreasing in w and also convergent if $2(p+1) > \sum_{i=1}^{n} a_i, a_i > 0$. Therefore, the trajectories of the original system converge to each other. The simulation result is shown in Fig. 4.2 with $p = 3, a_i = 1, \rho_i = 2, i = 1, \ldots, 4,$ and $x(0) = [1, 4, -5, -2, 6]$.

Example 4.9 *In this example, we consider the problem in Example 4.8 for $n = 1$. Now, let us suppose that we obtain the following comparison system with $\bar{\rho}_1$ and $\bar{\rho}_2$ as constant parameters*

$$\left.\begin{aligned}\frac{d}{dt}w_1 &= -\bar{\rho}_1 w_1 + \frac{w_2}{\bar{\rho}_1}\\ \frac{d}{dt}w_2 &= |a_1|w_1 - \left(2(p+1) - \frac{|a_1|}{\bar{\rho}_1} - \frac{|a_2|}{\bar{\rho}_2}\right)w_2\end{aligned}\right\} \tag{4.30}$$

which is not quasi-monotone non-decreasing since the coefficient of w_2 may possibly be negative. For instance, if we take $a_1 = 1, a_2 = 0, \bar{\rho}_1 = -\frac{1}{2}, \bar{\rho}_2 = \frac{1}{2}$ and $p = 1$. The function on the right-hand side of (4.30) becomes

$$F(w_1, w_2) = \begin{bmatrix}\frac{1}{2}w_1 - 2w_2\\ w_1 - 4w_2\end{bmatrix}.$$

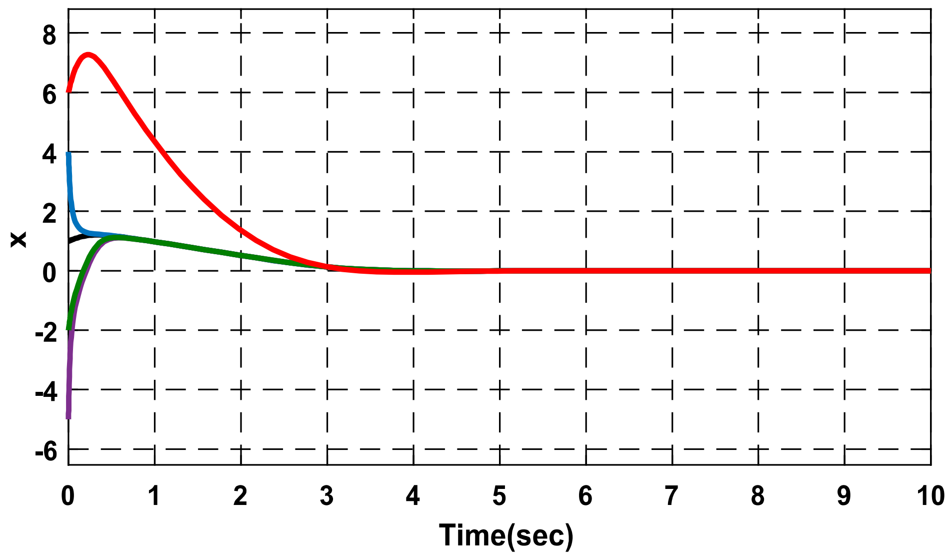

Figure 4.2: Evolution of the states of the system (4.24)

Note that this F is not quasi-monotone non-decreasing w.r.t. usual component-wise ordering in $\mathbb{R}^2$. However, F is quasi-monotone non-decreasing w.r.t the cone $K = \{(w_1, w_2) \in \mathbb{R}^2_+ | w_2 \leq w_1 \leq 3w_2\}$ since

(i) *on the boundary $w_1 = w_2$, taking $\phi = (1, -1) \in K^*$, we see that, $\langle \phi, (w_1, w_1) \rangle = \langle (1, -1), (w_1, w_1) \rangle = 0$ and*

$$\langle \phi, F(w_1, w_2) \rangle = \langle (1, -1), (\frac{1}{2}w_1 - 2w_1, w_1 - 4w_1) \rangle = \frac{3}{2}w_1 \geq 0, \forall\ w_1 \geq 0 \text{ and}$$

(ii) *on the boundary $w_1 = 3w_2$, taking $\phi = (1, -3)$, we see that $\langle \phi, (3w_2, w_2) \rangle = 0$ and*

$$\langle \phi, F(3w_2, w_2) \rangle = 1(\frac{3}{2}w_2 - 2w_2) - 3(3w_2 - 4w_2) = \frac{5}{2}w_2 \geq 0, \quad \forall\ w_2 \geq 0.$$

Example 4.10 *Consider the nonlinear dynamical system representing two duffing systems [109]*

$$\begin{aligned} \dot{r}_1 &= s_1 \\ \dot{s}_1 &= cr_1 + ds_1 - r_1^3 + k\cos(0.8t), \end{aligned} \tag{4.31}$$

$$\begin{aligned} \dot{r}_2 &= s_2 \\ \dot{s}_2 &= cr_2 + ds_2 - r_2^3 + k\cos(2t) + \tau. \end{aligned} \tag{4.32}$$

where c, d, k are known parameters and τ is the control input. Our main task is to design control such that the two duffing systems synchronize. Subtraction of (4.31) from (4.32) provides the error dynamics

$$\begin{aligned}\dot{e}_r &= e_s \\ \dot{e}_s &= ce_r + de_s - e_r(e_r^2 + 3r_1e_r + 3r_1^2) + k(\cos(2t) - \cos(0.8t)) + \tau,\end{aligned} \tag{4.33}$$

where $e_r = r_2 - r_1$; $e_s = s_2 - s_1$. The main aim is to design control τ so that error dynamics (4.33) trajectories converge to the origin. We use backstepping technique and vector contraction approach to design the control τ. Starting with the first subsystem

$$\dot{e}_r = e_s,$$

where e_s acts as a control input and we start with input $e_s = \psi(e_r)$ to stabilize the origin $e_r = 0$. With $e_s = \psi(e_r) = -e_r$, we obtain the subsystem as $\dot{e}_r = -e_r$, which is globally asymptotic stable. For further steps, we use the deviation variable as

$$z = e_s - \psi(e_r) = e_s + e_r,$$

to transform the system into the structure as

$$\begin{aligned}\dot{e}_r =& -e_r + z \\ \dot{z} =& (d+1)z + (c-d-1)e_r - e_r^3 - 3r_1e_r^2 - 3r_1^2e_r + k\cos(2t) - k\cos(0.8t) + \tau,\end{aligned} \tag{4.34}$$

The virtual dynamics of the system (4.34) is

$$\begin{aligned}\delta\dot{e}_r =& -\delta e_r + \delta z \\ \delta\dot{z} =& (d+1+\Delta_z\tau)\delta z + (c-d-1+\Delta_{e_r}\tau - 3e_r^2 - 6r_1e_r \\ & - 3e_r^2)\delta e_r + (-2k\sin(2t) + 0.8k\sin(0.8t) + \Delta_t\tau)\delta t,\end{aligned} \tag{4.35}$$

Let the squared vector-valued norm, defined by (4.1), assuming the matrix P as diag$(\mathbb{1})$ be: $\|\delta e\|_v^2 = [\delta e_r^2, \delta z^2]^\top$. Then, the squared vector valued norm derivative along the system trajectories (4.35)

$$\begin{aligned}\frac{d}{dt}(\delta e_r^2) =& -2\delta e_r^2 + 2\delta e_r\delta z \\ \leq& -\delta e_r^2 + \delta z^2, \\ \frac{d}{dt}(\delta z^2) =& 2(d+1+\Delta_z\tau)\delta z^2 + 2((c-d-1) - 3e_r^2 - 6r_1e_r - 3r_1^2 + \Delta_{e_r}\tau)\delta e_r\delta z \\ \leq& (2d+2+2\Delta_z\tau + (c-d-1) - 3e_r^2 - 6r_1e_r - 3r_1^2 + \Delta_{e_r}\tau)\delta z^2 \\ & + [(c-d-1) - 3e_r^2 - 6r_1e_r - 3r_1^2 + \Delta_{e_r}\tau]\delta e_r^2,\end{aligned}$$

We obtain the following comparison system

$$\begin{aligned}\dot{w}_1 &= -w_1 + w_2 \\ \dot{w}_2 &= 2w_1 - 4w_2,\end{aligned} \tag{4.36}$$

with the control input as

$$\begin{aligned}\Delta_{e_r}\tau &= 3e_r^2 + 6r_1 e_r + 3r_1^2 - c + d + 3 \\ \Delta_z \tau &= -(d+4) \\ \Delta_t \tau &= 2k\sin(2t) - 0.8k\sin(0.8t),\end{aligned}$$

After integrating the above equations, we obtain the following control input

$$\tau(z, e_r, t) = e_r^3 + 3r_1 e_r^2 + 3r_1^2 e_r + (d-c)e_r + 3e_r - (d+4)z - k\cos(2t) + k\cos(0.8t).$$

The comparison system obtained with the control input has off-diagonal entries positive that means it is quasi-monotone non-decreasing and its Jacobian is negative definite hence it is contracting. Therefore, two duffing systems synchronize with the calculated control input τ *as shown through the simulation results in Fig. 4.3 with* $c = 1.8, d = -0.1$, $k = -1.1$, $r(0) = [1, 2]$ *and* $s(0) = [1, 2]$.

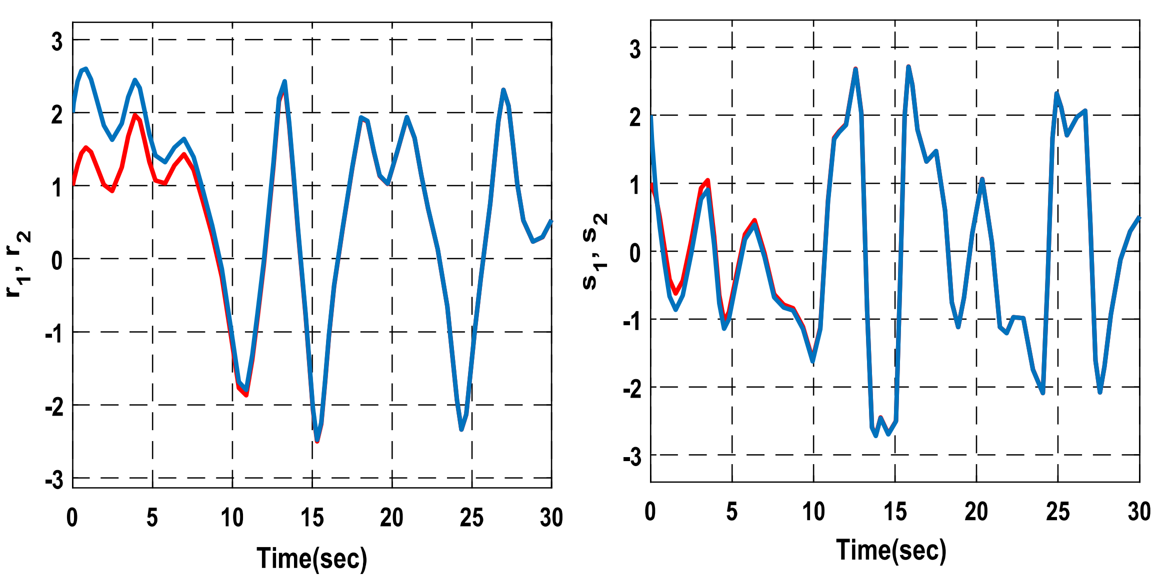

Figure 4.3: Evolution of the states of system (4.31) (4.32) with time

Example 4.11 *Consider the system of Chua's circuit [45] consisting of the cubic non-linearity*

$$\begin{aligned}\dot{x}_1 &= ax_2 - aqx_1^3 - acx_1\\ \dot{x}_2 &= x_1 - x_2 + x_3\\ \dot{x}_3 &= -bx_2,\end{aligned} \tag{4.37}$$

where a, b, c and q are system parameters and $x \in \mathbb{R}^3$ is the state vector. Let the measured outputs of Chua's circuit be $y_1 = x_1$ and $y_2 = x_3$. Our main task is to design an observer with y_1 and y_2 as measures. The structure of the observer is formulated as

$$\begin{aligned}\dot{\hat{x}}_1 &= a\hat{x}_2 - aq\hat{x}_1^3 - acx_1 + L_1(x_1 - \hat{x}_1)\\ \dot{\hat{x}}_2 &= \hat{x}_1 - \hat{x}_2 + \hat{x}_3 + L_2(x_1 - \hat{x}_1)\\ \dot{\hat{x}}_3 &= -b\hat{x}_2 + L_3(x_3 - \hat{x}_3),\end{aligned} \tag{4.38}$$

The virtual dynamics of the above system (4.38) is given by

$$\begin{aligned}\delta\dot{\hat{x}}_1 &= a\delta\hat{x}_2 - 3aq\hat{x}_1^2\delta\hat{x}_1 - ac\delta\hat{x}_1 - L_1\delta\hat{x}_1\\ \delta\dot{\hat{x}}_2 &= \delta\hat{x}_1 - L_2\delta\hat{x}_1 - \delta\hat{x}_2 + \delta\hat{x}_3\\ \delta\dot{\hat{x}}_3 &= -b\delta\hat{x}_2 - L_3\delta\hat{x}_3,\end{aligned} \tag{4.39}$$

Let the squared vector-valued norm, defined by (4.1), assuming the matrix P as diag($\mathbb{1}$) be: $\|\delta\hat{x}\|_v^2 = [\delta\hat{x}_1^2, \delta\hat{x}_2^2, \delta\hat{x}_3^2]^\top$. Then, the squared vector-valued norm derivative along (4.39) is calculated as

$$\begin{aligned}\frac{d}{dt}(\delta\hat{x}_1^2) &= (-6aq\hat{x}_1^2 - 2ac - 2L_1)\delta\hat{x}_1^2 + 2a\delta\hat{x}_1\delta\hat{x}_2\\ &\leq (-2ac - 2L_1 + |a|)\delta\hat{x}_1^2 + |a|\delta\hat{x}_2^2,\\ \frac{d}{dt}(\delta\hat{x}_2^2) &= 2(1 - L_2)\delta\hat{x}_1\delta\hat{x}_2 - 2\delta\hat{x}_2^2 + 2\delta\hat{x}_2\delta\hat{x}_3\\ &\leq |1 - L_2|\delta\hat{x}_1^2 + (|1 - L_2| - 1)\delta\hat{x}_2^2 + \delta\hat{x}_3^2,\\ \frac{d}{dt}(\delta\hat{x}_3^2) &= -2b\delta\hat{x}_2\delta\hat{x}_3 - 2L_3\delta\hat{x}_3^2\\ &\leq |b|\delta\hat{x}_2^2 + (|b| - 2L_3)\delta\hat{x}_3^2,\end{aligned}$$

In order to obtain the comparison system as quasi-monotone non-decreasing (off-diagonal elements must be non-negative) and contracting, we select the gains as $L_1 =$

10, $L_2 = 1$ and $L_3 = 8$. Now, the comparison system obtained is

$$\dot{w}_1 = (-2ac - 20 + a)w_1 + aw_2$$
$$\dot{w}_2 = \ - w_2 + w_3$$
$$\dot{w}_3 = bw_2 + (b - 16)w_3.$$

Further we consider the values of constants as $a = 9.5, b = 5, c = -1/7$ and $q = 4/63$. Hence, original dynamical system (4.38) contracts as well. This means that the estimated states converge to the actual states as shown in Fig. 4.4 with initial conditions $x(0) = [-1, -0.8, 0.2]$ and $\hat{x}(0) = [-2, -0.4, 1]$. Fig. 4.4 also shows that the estimation errors $e_1 = x_1 - \hat{x}_1, e_2 = x_2 - \hat{x}_2, e_3 = x_3 - \hat{x}_3$ converge to zero exponentially.

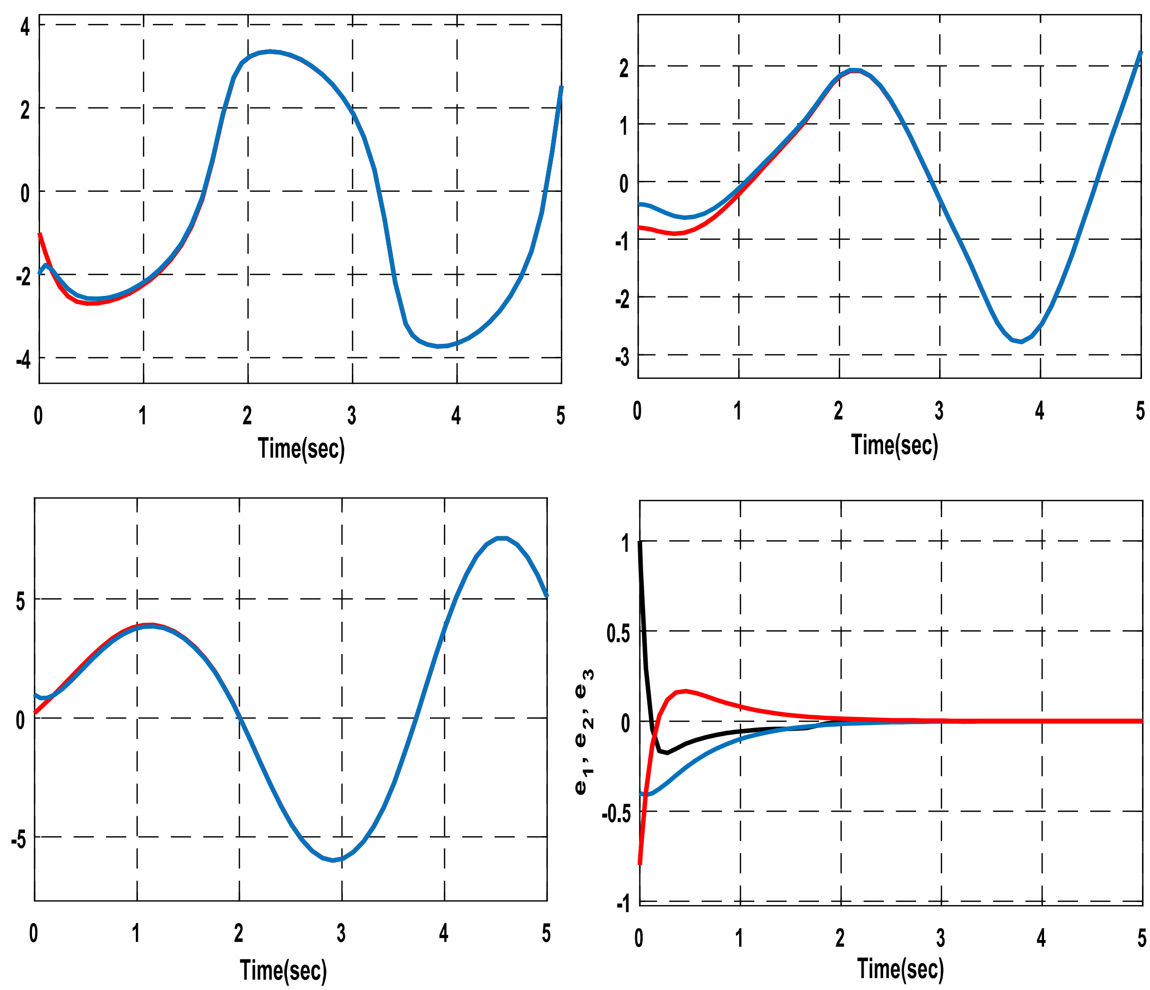

Figure 4.4: Response of the estimated states along the actual states and the error of the system (4.37) with time

4.5 Conclusion

Further results on contraction theory utilizing the notion of vector distances for addressing the convergence of trajectories of nonlinear dynamical systems have been presented in this chapter. In particular, we derived comparison results employing the quasi-monotonicity property of the function for proving convergence of the original dynamical system by comparing the solutions of the auxiliary system and the original system without much less strict conditions. In addition, in order to overcome the component-wise inequalities of vectors, the results are also derived in the framework of the cone ordering. Furthermore, the proposed results have been exploited for the design of controller and state observer. General results have been derived to estimate the control input and observer gains through a suitable selection of the comparison system. The theoretical results are illustrated through the example of controller design to synchronize two duffing systems followed by the observer design problem of Chua's circuit. Moreover, simulation results are shown to observe the proposed outcomes.