Chapter 4

On Bivariate Fractal Approximation

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The present chapter explores the approximation perspective relative to fractal dimension of a function and its partial derivatives.

The chapter is structured as follows. In Section 4.1, we prove some results regarding dimension preserving approximation. In Section 4.2, we define some multi-valued mappings which are defined with the help of bivariate α-fractal functions, and establish some properties of them.

4.1 Dimension Preserving Approximation of Bivariate Functions

Firstly, we mention the following result required for this chapter:

Let us denote the class of Y-valued Lipschitz functions on X by $\mathcal{L}ip(X,Y)$, where (X, d_X) is a compact metric space and $(Y, \|.\|_Y)$ is a normed linear space. Note that this space is a dense subset of $\mathcal{C}(X,Y)$ with respect to the supremum norm.

In view of Lipschitz invariance property of dimension, one may conclude that the upcoming theorem holds for all aforementioned dimensions.

Theorem 4.1.1. *Let* $\dim(X) \leq \beta \leq \dim(X) + \dim(Y)$. *Then the set* $\mathcal{S}_\beta := \{f \in \mathcal{C}(X,Y) : \dim(Gr(f)) = \beta\}$ *is dense in* $\mathcal{C}(X,Y)$.

Proof. Let $f \in \mathcal{C}(X,Y)$ and $\epsilon > 0$. Then using the density of $\mathcal{L}ip(X,Y)$ in $\mathcal{C}(X,Y)$, there exists g in $\mathcal{L}ip(X,Y)$ such that

$$\|f - g\|_{\infty,Y} < \frac{\epsilon}{2}.$$

Further, we consider a non-vanishing function $h \in \mathcal{S}_\beta$. Taking $h_* = g + \frac{\epsilon}{2\|h\|_{\infty,Y}} h$, it immediately gives

$$\|g - h_*\|_{\infty,Y} \leq \frac{\epsilon}{2}.$$

This together with Lemma 1.2.3 implies that $\dim(Gr(h_*)) = \dim(Gr(h)) = \beta$. Hence, we have $h_* \in \mathcal{S}_\beta$ and

$$\|f - h_*\|_{\infty,Y} \leq \|f - g\|_{\infty,Y} + \|g - h_*\|_{\infty,Y} < \epsilon.$$

Thus the proof of theorem is complete. □

To the best of our knowledge, the univariate version of the next theorem is well-known, however, we could not find a proof of theorem in bivariate setting. Hence, we write a detailed proof of it.

Theorem 4.1.2. *Let $\{f_k\}$ be a sequence of differentiable functions on $\square$. Assume that for some $(x_0, y_0) \in \square$, the sequences $\big\{f_k(x_0, .)\big\}$ and $\big\{f_k(., y_0)\big\}$ converges uniformly on $[c, d]$ and $[a, b]$ respectively. If $\{D^{(1,1)} f_k\}$ converges uniformly on $\square$, then $\{f_k\}$ converges uniformly on $\square$ to a function f, and*

$$D^{(1,1)} f(\boldsymbol{x}) = \lim_{k\to\infty} D^{(1,1)} f_k(\boldsymbol{x}),$$

for every $\boldsymbol{x} \in \square$.

Proof. Let $\epsilon > 0$. Since $\{D^{(1,1)} f_k\}$ converges uniformly, there exists $N_1 \in \mathbb{N}$ such that

$$|D^{(1,1)} f_k(\boldsymbol{x}) - D^{(1,1)} f_m(\boldsymbol{x})| < \frac{\epsilon}{4(b-a)(d-c)}, \quad \forall\ \boldsymbol{x} \in \square,\ \ k, m \geq N_1.$$

By the mean-value theorem, see, for instance, [88, Theorem 9.40], we have

$$\begin{aligned}
&\big|f_k(x+h, y+k) - f_m(x+h, y+k) - f_k(x+h, y) + f_m(x+h, y)\\
&\quad - f_k(x, y+k) + f_m(x, y+k) + f_k(x, y) - f_m(x, y)\big|\\
&= hk\ \big|D^{(1,1)}(f_k - f_m)(t, s)\big|\\
&\leq hk \max_{(t,s)\in\square} \big|D^{(1,1)} f_k(t, s) - D^{(1,1)} f_m(t, s)\big|\\
&\leq \frac{\epsilon}{4(b-a)(d-c)} hk\\
&\leq \frac{\epsilon}{4}.
\end{aligned} \tag{4.1.1}$$

By the hypothesis for $(x_0, y_0) \in \square$, one can choose $N_0\ (> N_1)\ \in \mathbb{N}$ such that

$$|f_k(x_0, y) - f_m(x_0, y)| < \frac{\epsilon}{4} \quad \forall\ k, m \geq N_0$$

and

$$|f_k(x, y_0) - f_m(x, y_0)| < \frac{\epsilon}{4} \quad \forall\ k, m \geq N_0.$$

Now using the above estimates and substituting $x + h$ with x_0 and $y + k$ with y_0 in Equation (4.1.1), we have

$$\begin{aligned}
|f_k(x, y) - f_m(x, y)| &\leq \frac{\epsilon}{4} + |f_k(x, y_0) - f_m(x, y_0)| + |f_k(x_0, y) - f_m(x_0, y)| \\
&\quad + |f_k(x_0, y_0) - f_m(x_0, y_0)| \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} \\
&= \epsilon,
\end{aligned}$$

for every $(x, y) \in \square$ and $k, m \geq N_0$. This immediate confirms the uniform convergence of $\{f_k\}$. The rest part of the proof follows by routine calculations. This completes the proof. $\square$

Lemma 4.1.3. *Let $f : I \to \mathbb{R}$ be a Lipschitz map and $g : J \to \mathbb{R}$ a continuous function. If the mapping $h : \square \to \mathbb{R}$ is defined by*

$$h(x, y) = f(x) + g(y),$$

then

$$\dim_H(Gr(h)) = \dim_H(Gr(g)) + 1.$$

Proof. Proof follows by defining a bi-Lipschitz mapping from $Gr(h)$ to the set $\{(x, y, g(y)) : x \in I,\ y \in J\}$. $\square$

Here, let us recall some dimensional results for univariate functions. Mauldin and Williams [89] considered the following class of functions:

$$W_b(x) := \sum_{n=-\infty}^{\infty} b^{-\alpha n}[\phi(b^n x + \theta_n) - \phi(\theta_n)],$$

where θ_n is an arbitrary real number, ϕ is a periodic function with period one and $b > 1$, $0 < \alpha < 1$. They showed that for a large enough b there exists a constant $C > 0$ such that $\dim_H(Gr(W_b))$ is bounded below by $2 - \alpha - (C/\ln b)$.

Further, Shen [59] has proved some dimensional results for the following class of functions:

$$f^{\phi}_{\lambda,b}(x) := \sum_{n=0}^{\infty} \lambda^n \phi(b^n x),$$

where $b \geq 2$ and ϕ is a real-valued, $\mathbb{Z}$-periodic, non-constant, C^2-function defined on $\mathbb{R}$. For instance, he has proved that there exists a constant K_0 depending on ϕ and b such that if $1 < \lambda b < K_0$ then

$$\dim_H(Gr(f^{\phi}_{\lambda,b})) = 2 + \frac{\log \lambda}{\log b}.$$

For $f \in \mathcal{C}^{1,1}(\square)$, we get $\dim(Gr(f)) = 2$. However, no conclusion can be drawn for dimensions of its partial derivatives. This is evident from the following example: let Weierstrass-type nowhere differentiable continuous function $W : I \to \mathbb{R}$ be as in [59] with $1 \leq \dim(Gr(W)) \leq 2$. Now we define $h : \square \to \mathbb{R}$ by

$$h(x, y) = W(x) + y.$$

Here, by Lemma 4.1.3, we obtain $2 \leq \dim(Gr(h)) = \dim(Gr(W)) + 1 \leq 3$. Then for the function f defined by

$$f(x,y) := \int_a^x \int_c^y h(t,s)dtds,$$

we have $\dim(Gr(f)) = 2$ and $2 \le \dim(Gr(D^{(1,1)}f)) = \dim(Gr(h)) \le 3$.

Theorem 4.1.4. *Let $f \in \mathcal{C}^{1,1}(\square)$ and $2 \le \beta \le 3$. Then we have a sequence $\{f_k\}$ in $\mathcal{C}^{1,1}(\square)$ such that $\dim(Gr(D^{(1,1)}f_k)) = \beta$ and $f_k \to f$ uniformly on $\square$.*

Proof. In view of Theorem 4.1.1, there exists a sequence $\{g_k\}$ in $\mathcal{C}(\square)$ such that $\dim(Gr(g_k)) = \beta$ and $g_k \to D^{(1,1)}f$ uniformly on $\square$. Further, let us consider a function $f_k : \square \to \mathbb{R}$ defined by

$$f_k(x,y) := \int_a^x \int_c^y g_k(t,s)dtds.$$

Then $D^{(1,1)}f_k = g_k$ and $D^{(1,1)}f_k \to D^{(1,1)}f$ uniformly. Next, we have that $f_k(a,y) \to 0$ and $f_k(x,c) \to 0$ uniformly on J and I respectively. Now Theorem 4.1.2 completes the proof. $\square$

Theorem 4.1.5. *Let $f \in \mathcal{C}(\square)$ with $f(\boldsymbol{x}) \ge 0 \ \forall \ \boldsymbol{x} \in \square$. Then for a given $\epsilon > 0$, there exists $g \in \mathcal{S}_\beta$ satisfying the following:*

$$g(\boldsymbol{x}) \ge 0 \ \forall \ \boldsymbol{x} \in \square \ \textit{and} \ \|f - g\|_\infty < \epsilon.$$

Proof. Let $\epsilon > 0$. Then Theorem 4.1.1 yields an element $h \in \mathcal{S}_\beta$ such that

$$\|f - h\|_\infty < \frac{\epsilon}{2}.$$

We define

$$g(\boldsymbol{x}) := h(\boldsymbol{x}) + \frac{\epsilon}{2}, \ \forall \ \boldsymbol{x} \in \square.$$

Then by Lemma 1.2.3, $g \in \mathcal{S}_\beta$, and by routine calculations, we get

$$g(\boldsymbol{x}) = h(\boldsymbol{x}) - f(\boldsymbol{x}) + f(\boldsymbol{x}) + \frac{\epsilon}{2} \geq -\|f-h\|_\infty + f(\boldsymbol{x}) + \frac{\epsilon}{2} > f(\boldsymbol{x}) \geq 0.$$

Furthermore, one has

$$\|f-g\|_\infty \leq \|f-h\|_\infty + \|h-g\|_\infty < \epsilon.$$

Thus the proof of theorem is complete. □

Theorem 4.1.6. *Let $f : \square \to \mathbb{R}$ be a (m,n)-convex function such that $f(a,y) = f(x,c) = 0,\ \forall\ x \in I,\ y \in J$. Then for $\epsilon > 0$, there exists (m,n)-convex function g such that $D^{(m,n)}g \in \mathcal{S}_\beta$ and $\|f-g\|_\infty < \epsilon$.*

Proof. Let $\epsilon > 0$. Then using Theorem 4.1.1, there exists $h \in \mathcal{S}_\beta$ such that $\|D^{(m,n)}f - h\| < \frac{\epsilon}{(b-a)^m(d-c)^n}$. By choosing

$$g(x,y) := \int_a^x \int_c^y \cdots \int_a^{x_{m-1}} \int_c^{y_{n-1}} h(x_m, y_n)dx_m dy_n \dots dx_1 dy_1,$$

we have

$$\|f-g\| = \sup_{(x,y)\in\square} \left\{ \left| f - \int_a^x \int_c^y \cdots \int_a^{x_{m-1}} \int_c^{y_{n-1}} h(x_m, y_n)dx_m dy_n \dots dx_1 dy_1 \right| \right\} < \epsilon,$$

proving the assertion. □

Theorem 4.1.7. *Let $f \in \mathcal{C}(\square)$. Then for $\epsilon > 0$ there exists $g \in \mathcal{S}_\beta$ such that*

$$g(\boldsymbol{x}) \leq f(\boldsymbol{x})\ \forall\ \boldsymbol{x} \in \square \text{ and } \|f-g\|_\infty < \epsilon.$$

Proof. Since $f \in \mathcal{C}(\square)$ and $\epsilon > 0$, Theorem 4.1.1 generates a member $h \in \mathcal{S}_\beta$ such that

$$\|f - h\|_\infty < \frac{\epsilon}{2}.$$

Choose $g(\boldsymbol{x}) := h(\boldsymbol{x}) - \frac{\epsilon}{2}, \quad \forall \; \boldsymbol{x} \in \square$. Then

$$g(\boldsymbol{x}) = h(\boldsymbol{x}) - f(\boldsymbol{x}) + f(\boldsymbol{x}) - \frac{\epsilon}{2} \leq \|f - h\|_\infty + f(\boldsymbol{x}) - \frac{\epsilon}{2} < f(\boldsymbol{x}).$$

Furthermore,

$$\|f - g\|_\infty \leq \|f - h\|_\infty + \|h - g\|_\infty < \epsilon,$$

establishing the proof. □

Now we aim to show the existence of the best one-sided approximation. Let $\beta \in [2, 3]$, and define

$$\mathcal{C}_\beta(\square) := \{f \in \mathcal{C}(\square) : \overline{\dim}_B(Gr(f)) \leq \beta\}.$$

In view of [72, Proposition 3.4], recall that $\mathcal{C}_\beta(\square)$ is a normed linear space. Let $\{g_1, g_2, \dots, g_n\}$ be a linearly independent subset of $\mathcal{C}_\beta(\square)$. Furthermore, for a bounded below and Lebesgue integrable function $f : \square \to \mathbb{R}$, we define

$$\mathcal{Y}_n^\beta(f) := \Big\{h \in \text{span}\{g_1, g_2, \dots, g_n\} : h(\boldsymbol{x}) \leq f(\boldsymbol{x}) \; \forall \; \boldsymbol{x} \in \square\Big\}.$$

Theorem 4.1.7 guarantees the non-emptyness of $\mathcal{Y}_n^\beta(f)$. A function $h_f \in \mathcal{Y}_n^\beta(f)$ is said to be a best one-sided approximation from below to f on $\square$ if

$$\int_\square h_f(\boldsymbol{x}) \; d\boldsymbol{x} = \sup\Big\{\int_\square h(\boldsymbol{x}) \; d\boldsymbol{x} : h \in \mathcal{Y}_n^\beta(f)\Big\},$$

where $\int_\square g(\boldsymbol{x}) \; d\boldsymbol{x} = \int_a^b \int_c^d g(\boldsymbol{x}) \; d\boldsymbol{x}$. In a similar way, we define best one-sided approximations from above. We state the next theorem for one-sided approximation from below. Though a similar result can be proved in terms of one-sided approximation from above, see, for instance, [58, 66].

Theorem 4.1.8. *For a bounded below and integrable function $f : \square \to \mathbb{R}$, there exists a member in $\mathcal{Y}_n^\beta(f)$ of best one-sided approximant from below to f on $\square$.*

Proof. Let $\{h_m\}$ be a sequence in $\mathcal{Y}_n^\beta(f)$ such that

$$\int_\square h_m(\boldsymbol{x})\, d\boldsymbol{x} \to A \quad \text{as} \quad m \to \infty, \tag{4.1.2}$$

where $A = \sup\left\{ \int_\square h(\boldsymbol{x})\, d\boldsymbol{x} : h \in \mathcal{Y}_n^\beta(f)\right\}$. Then using an appropriate constant $M_* > 0$, we have

$$\begin{aligned}\int_\square |h_m(\boldsymbol{x})|\, d\boldsymbol{x} \le \int_\square &\left| h_m(\boldsymbol{x}) - \frac{A}{(b-a)(d-c)}\right| d\boldsymbol{x} \\ &+ \int_\square \frac{A}{(b-a)(d-c)}\, d\boldsymbol{x} \le M_*.\end{aligned}$$

Let us recall a well-known fact that a closed and bounded subset of a normed linear space is compact if and only if it is finite-dimensional space. Since $\mathcal{Y}_n^\beta(f)$ is a subset of finite-dimensional linear space, it follows that every closed ball of radius M_* in $\mathcal{Y}_n^\beta(f)$ will be compact. Therefore, using the above inequality, there exist a subsequence $\{h_{m_k}\}$ and a function h in $\mathcal{Y}_n^\beta(f)$ such that the sequence $\{h_{m_k}\}$ converges to h in $\mathcal{L}^1(\square)$. Recall a basic functional analysis result that every norm is equivalent on a finite-dimensional linear space. Now from the finite-dimensionality of $\mathcal{Y}_n^\beta(f)$, it follows that the sequence $\{h_{m_k}\}$ also converges to h uniformly. Further, since $h_m(\boldsymbol{x}) \le f(\boldsymbol{x})$, $\forall\, \boldsymbol{x} \in \square$, and $h_{m_k} \to h$ uniformly, we get $h(\boldsymbol{x}) \le f(\boldsymbol{x})$, $\forall\, \boldsymbol{x} \in \square$. Thus $h \in \mathcal{Y}_n^\beta(f)$. Now by (4.1.2), we have

$$\int_\square h(\boldsymbol{x})\, d\boldsymbol{x} = \lim_{k\to\infty} \int_\square h_{m_k}(\boldsymbol{x})\, d\boldsymbol{x} = A,$$

completing the proof. □

Remark 4.1.1. *We have the following.*

$$B_{m,n}(f)(\boldsymbol{x}) = \frac{1}{(b-a)^m(d-c)^n}\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}(x-a)^i(b-x)^{m-i} (y-c)^j(d-y)^{n-j}f\Big(a+\frac{i(b-a)}{m}, c+\frac{j(d-c)}{n}\Big).$$

Theorem 4.1.9. *The fractal operator $\mathcal{F}^{\alpha}_{m,n}:\mathcal{C}(\square)\to\mathcal{C}(\square)$ is a topological isomorphism.*

Proof. Using Equation (1.11.5) and Remark 4.1.1, one gets

$$\big\|f-\mathcal{F}^{\alpha}_{m,n}(f)\big\|_\infty \leq \|\alpha\|_\infty\big\|\mathcal{F}^{\alpha}_{m,n}(f)-B_{m,n}f\big\|_\infty \leq \|\alpha\|_\infty\big\|\mathcal{F}^{\alpha}_{m,n}(f)\big\|_\infty + \|\alpha\|_\infty\|f\|_\infty.$$

Since $\|\alpha\|_\infty < 1$, the previous lemma yields that the fractal operator $\mathcal{F}^{\alpha}_{m,n}$ is a topological isomorphism. □

Remark 4.1.2. *The above theorem may strengthen item-4 of [43, Theorem 3.2]. To be precise, item-4 tells that $\mathcal{F}^{\alpha}_{m,n}$ is a topological isomorphism if $\|\alpha\|_\infty < \big(1+\|I-B_{m,n}\|\big)^{-1}$, which is more restricted than the assumption considered in the above theorem, that is, $\|\alpha\|_\infty < 1$.*

The techniques used in the following proof of the theorem are borrowed from Chand et al. [26].

Theorem 4.1.10. *Let $f\in\mathcal{C}(\square)$ be such that $f(\boldsymbol{x})\geq 0,\ \forall\ \boldsymbol{x}\in\square$. Then for $\epsilon>0$, $\beta\in[2,3]$, and for $\alpha\in\mathcal{C}(\square)$ satisfying $\|\alpha\|_\infty<1$, we have an α-fractal function $g^{\alpha}_{\Delta,B_{m,n}}\in\mathcal{S}_\beta$ satisfying*

$$g^{\alpha}_{\Delta,B_{m,n}}(\boldsymbol{x})\geq 0,\quad \forall\ \boldsymbol{x}\in\square \quad \text{and } \|f-g^{\alpha}_{\Delta,B_{m,n}}\|_\infty<\epsilon.$$

Proof. Note that the Bernstein operator $B_{m,n}$ fixes the constant function 1, that is, $B_{m,n}(1) = 1$, where $1(\boldsymbol{x}) = 1$ on $\square$. Consider $\alpha \in \mathcal{C}(\square)$ such that $\|\alpha\|_\infty < 1$. From Equation (1.11.5), we deduce

$$\|g^{\alpha}_{\Delta,B_{m,n}} - g\|_\infty \leq \|\alpha\|_\infty \|g^{\alpha}_{\Delta,B_{m,n}} - B_{m,n}g\|_\infty, \ \forall \ g \in \mathcal{C}(\square).$$

Choose $g = 1$, then the above inequality gives

$$\|1^{\alpha}_{\Delta,B_{m,n}} - 1\|_\infty \leq \|\alpha\|_\infty \|1^{\alpha}_{\Delta,B_{m,n}} - 1\|_\infty,$$

and this further yields $\|1^{\alpha}_{\Delta,B_{m,n}} - 1\|_\infty = 0$. Therefore, $1^{\alpha}_{\Delta,B_{m,n}} = 1$, that is, $\mathcal{F}^{\alpha}_{m,n}(1) = 1$.

Let $f \in \mathcal{C}(\square)$ such that $f(\boldsymbol{x}) \geq 0$, and let $\beta \in [2,3]$. Then for each $\epsilon > 0$, using Theorem 4.1.1 and Theorem 4.1.9, there exists a function $h^{\alpha}_{\Delta,B_{m,n}} \in \mathcal{S}_\beta$ such that

$$\|f - h^{\alpha}_{\Delta,B_{m,n}}\|_\infty < \frac{\epsilon}{2}, \quad \text{where } \mathcal{F}^{\alpha}_{m,n}(h) = h^{\alpha}_{\Delta,B_{m,n}}.$$

Define $g^{\alpha}_{\Delta,B_{m,n}}(\boldsymbol{x}) = h^{\alpha}_{\Delta,B_{m,n}}(\boldsymbol{x}) + \frac{\epsilon}{2}$ for all $\boldsymbol{x} \in \square$. Since $\mathcal{F}^{\alpha}_{m,n}(1) = 1$,

$$g^{\alpha}_{\Delta,B_{m,n}}(\boldsymbol{x}) = h^{\alpha}_{\Delta,B_{m,n}}(\boldsymbol{x}) + \frac{\epsilon}{2}1(\boldsymbol{x}) = h^{\alpha}_{\Delta,B_{m,n}}(\boldsymbol{x}) + \frac{\epsilon}{2}1^{\alpha}(\boldsymbol{x}).$$

Further, since $\mathcal{F}^{\alpha}_{m,n}$ is a linear operator, we have

$$g^{\alpha}_{\Delta,B_{m,n}} = h^{\alpha}_{\Delta,B_{m,n}} + \frac{\epsilon}{2}1^{\alpha} = \mathcal{F}^{\alpha}_{m,n}(h + \frac{\epsilon}{2}1).$$

Moreover,

$$\begin{aligned} g^{\alpha}_{\Delta,B_{m,n}}(\boldsymbol{x}) &= h^{\alpha}_{\Delta,B_{m,n}}(\boldsymbol{x}) + \frac{\epsilon}{2} \\ &= h^{\alpha}_{\Delta,B_{m,n}}(\boldsymbol{x}) + \frac{\epsilon}{2} - f(\boldsymbol{x}) + f(\boldsymbol{x}) \\ &\geq f(\boldsymbol{x}) + \frac{\epsilon}{2} - \|h^{\alpha}_{\Delta,B_{m,n}} - f\|_{\infty} \\ &\geq 0. \end{aligned}$$

Further, we get

$$\begin{aligned} \|f - g^{\alpha}_{\Delta,B_{m,n}}\|_{\infty} &\leq \|f - h^{\alpha}_{\Delta,B_{m,n}}\|_{\infty} + \|h^{\alpha}_{\Delta,B_{m,n}} - g^{\alpha}_{\Delta,B_{m,n}}\|_{\infty} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon, \end{aligned}$$

completing the proof. □

4.2 Some Results for Multi-valued Mappings

Theorem 4.2.1. *The multi-valued mapping $\mathcal{W}^{\alpha}_{\Delta} : \mathcal{C}(\square) \rightrightarrows \mathcal{C}(\square)$ defined by*

$$\mathcal{W}^{\alpha}_{\Delta}(f) = \{f^{\alpha}_{\Delta,B_{m,n}} : m,\ n \in \mathbb{N}\}$$

is a Lipschitz process.

Proof. Using the linearity of $\mathcal{F}^{\alpha}_{m,n}$, we have

$$\mathcal{W}^{\alpha}_{\Delta}(\lambda f) = \{(\lambda f)^{\alpha}_{\Delta,B_{m,n}} : m,\ n \in \mathbb{N}\} = \lambda \mathcal{W}^{\alpha}_{\Delta}(f),\ \forall\ f \in \mathcal{C}(\square),\ \lambda > 0.$$

Again by linearity of $\mathcal{F}^{\alpha}_{m,n}$, it is clear that $\mathcal{W}^{\alpha}_{\Delta}(0) = \{0\}$. Therefore, $\mathcal{W}^{\alpha}_{\Delta}$ is a process.

Let $f, g \in \mathcal{C}(\square)$. On applying Equation (1.11.5), we have

$$\left|f^{\alpha}_{\Delta,B_{m,n}}(\boldsymbol{x}) - g^{\alpha}_{\Delta,B_{m,n}}(\boldsymbol{x})\right| \leq \|f-g\|_\infty + \|\alpha\|_\infty \|f^{\alpha}_{\Delta,B_{m,n}} - g^{\alpha}_{\Delta,B_{m,n}}\|_\infty$$
$$+ \|\alpha\|_\infty \|B_{m,n}(g) - B_{m,n}(f)\|_\infty,$$

for any $\boldsymbol{x} \in \square$. Further, we deduce

$$\|f^{\alpha}_{\Delta,B_{m,n}} - g^{\alpha}_{\Delta,B_{m,n}}\|_\infty \leq \frac{1+\|\alpha\|_\infty \|B_{m,n}\|}{1-\|\alpha\|_\infty}\|f-g\|_\infty.$$

Using $\|B_{m,n}\| = 1$,

$$\|f^{\alpha}_{\Delta,B_{m,n}} - g^{\alpha}_{\Delta,B_{m,n}}\|_\infty \leq \frac{1+\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\|f-g\|_\infty.$$

Consequently, we have

$$\mathcal{W}^{\alpha}_{\Delta}(g) \subseteq \mathcal{W}^{\alpha}_{\Delta}(f) + \frac{1+\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\ \|f-g\|_\infty U_{\mathcal{C}(\square)},$$

proving the Lipschitz property of $\mathcal{W}^{\alpha}_{\Delta}$, and hence, we complete the proof. □

Remark 4.2.1. *For the multi-valued mapping $\mathcal{W}^{\alpha}_{\Delta}$, let us first note the following:*

1. *By linearity of $\mathcal{F}^{\alpha}_{m,n}$, we have $\mathcal{W}^{\alpha}_{\Delta}(0) = \{0\}$.*
2. *If $\alpha \neq 0$, $m \neq k$ then $f^{\alpha}_{\Delta,B_{m,n}} \neq f^{\alpha}_{\Delta,B_{k,l}}$, hence $\mathcal{W}^{\alpha}_{\Delta} : \mathcal{C}(\square) \rightrightarrows \mathcal{C}(\square)$ is not single-valued.*

In view of the above points, Theorems 1.12.3-1.12.4 produce that the mapping $\mathcal{W}^{\alpha}_{\Delta} : \mathcal{C}(\square) \rightrightarrows \mathcal{C}(\square)$ is not convex.

Theorem 4.2.2. *Let $\triangle$ be a fixed net and $m,n \in \mathbb{N}$. Then the multi-valued mapping $\mathcal{T}^{\Delta}_{m,n} : \mathcal{C}(\square) \rightrightarrows \mathcal{C}(\square)$ defined by*

$$\mathcal{T}^{\Delta}_{m,n}(f) = \{f^{\alpha}_{\Delta,B_{m,n}} : \alpha \in \mathcal{C}(\square) \text{ such that } \|\alpha\|_\infty < 1\}$$

is a process and lower semicontinuous.

Proof. Let $f \in \mathcal{C}(\square)$ and $\lambda > 0$,

$$\begin{aligned}\lambda \mathcal{T}^{\Delta}_{m,n}(f) =& \lambda \{ f^{\alpha}_{\triangle, B_{m,n}} : \alpha \in \mathcal{C}(\square) \text{ such that } \|\alpha\|_\infty < 1 \} \\ =& \{ \lambda f^{\alpha}_{\triangle, B_{m,n}} : \alpha \in \mathcal{C}(\square) \text{ such that } \|\alpha\|_\infty < 1 \} \\ =& \mathcal{T}^{\Delta}_{m,n}(\lambda f).\end{aligned}$$

Moreover, using linearity of fractal operator, we have $f^{\alpha}_{\triangle, B_{m,n}} = 0$, whenever $f = 0$. That is, $0 \in \mathcal{T}^{\Delta}_{m,n}(0)$. Therefore, $\mathcal{T}^{\Delta}_{m,n}$ is a process. Now it remains to show that $\mathcal{T}^{\Delta}_{m,n}$ is lower semicontinuous. For this, let $f \in \mathcal{C}(\square)$, $f^{\alpha}_{\triangle, B_{m,n}} \in \mathcal{T}^{\Delta}_{m,n}(f)$ and a sequence $\{f_k\}$ in $\mathcal{C}(\square)$ such that $f_k \to f$. Since the fractal operator is continuous, we have $(f_k)^{\alpha}_{\triangle, B_{m,n}} \to f^{\alpha}_{\triangle, B_{m,n}}$. It is clear that $(f_k)^{\alpha}_{\triangle, B_{m,n}} \in \mathcal{T}^{\Delta}_{m,n}(f_k)$. Therefore, the result follows. □

Remark 4.2.2. *One may see that $\mathcal{T}^{\Delta}_{m,n}$ is not convex through the following explanation. Let $f, g \in \mathcal{C}(\square)$. Then we have*

$$\begin{aligned}\mathcal{T}^{\Delta}_{m,n}(f+g) =& \{ (f+g)^{\alpha}_{\triangle, B_{m,n}} : \alpha \in \mathcal{C}(\square) \textit{ such that } \|\alpha\|_\infty < 1 \} \\ =& \{ f^{\alpha}_{\triangle, B_{m,n}} + g^{\alpha}_{\triangle, B_{m,n}} : \alpha \in \mathcal{C}(\square) \textit{ such that } \|\alpha\|_\infty < 1 \} \\ \subseteq& \{ f^{\alpha}_{\triangle, B_{m,n}} + g^{\beta}_{\triangle, B_{m,n}} : \alpha, \beta \in \mathcal{C}(\square) \textit{ such that } \|\alpha\|_\infty < 1, \|\beta\|_\infty < 1 \} \\ =& \{ f^{\alpha}_{\triangle, B_{m,n}} : \alpha \in \mathcal{C}(\square) \textit{ such that } \|\alpha\|_\infty < 1 \} \\ &+ \{ g^{\beta}_{\triangle, B_{m,n}} : \beta \in \mathcal{C}(\square) \textit{ such that } \|\beta\|_\infty < 1 \} \\ \subseteq& \mathcal{T}^{\Delta}_{m,n}(f) + \mathcal{T}^{\Delta}_{m,n}(g).\end{aligned}$$

Theorem 4.2.3. *Let $\triangle$ be a fixed net and $m, n \in \mathbb{N}$. Then the multi-valued mapping $\mathcal{T}^{\Delta}_{m,n} : \mathcal{C}(\square) \rightrightarrows \mathcal{C}(\square)$ defined by*

$$\mathcal{T}^{\Delta}_{m,n}(f) = \{f^{\alpha}_{\triangle, B_{m,n}} : \|\alpha\|_\infty \leq q < 1\},$$

is Lipschitz with Lipschitz constant $l = \frac{1+q}{1-q}$, and satisfies the following:

$$\|\mathcal{T}^{\Delta}_{m,n}\| \leq 1 + \frac{q}{1-q}\|Id - B_{m,n}\|.$$

Proof. Let $f, g \in \mathcal{C}(\square)$. Then Equation (1.11.5) yields

$$\begin{aligned} \left|f^{\alpha}_{\triangle, B_{m,n}}(\boldsymbol{x}) - g^{\alpha}_{\triangle, B_{m,n}}(\boldsymbol{x})\right| =& \|f - g\|_\infty + \|\alpha\|_\infty \|f^{\alpha}_{\triangle, B_{m,n}} - g^{\alpha}_{\triangle, B_{m,n}}\|_\infty \\ &+ \|\alpha\|_\infty \|B_{m,n} g - B_{m,n} f\|_\infty, \end{aligned}$$

for every $\boldsymbol{x} \in \square$. Further, we deduce

$$\|f^{\alpha}_{\triangle, B_{m,n}} - g^{\alpha}_{\triangle, B_{m,n}}\| \leq \frac{1 + \|\alpha\|_\infty \|B_{m,n}\|}{1 - \|\alpha\|_\infty}\|f - g\|_\infty.$$

Since $\|\alpha\|_\infty \leq q$ and $\|B_{m,n}\| = 1$, we get

$$\|f^{\alpha}_{\triangle, B_{m,n}} - g^{\alpha}_{\triangle, B_{m,n}}\| \leq \frac{1+q}{1-q}\|f - g\|.$$

Choosing $l = \frac{1+q}{1-q}$, we have

$$\mathcal{T}^{\Delta}_{m,n}(g) \subset \mathcal{T}^{\Delta}_{m,n}(f) + l \ \|f - g\|_\infty U_{\mathcal{C}(\square)},$$

proving first part of theorem. For the other part, we have

$$\begin{aligned}
\|\mathcal{T}^{\Delta}_{m,n}\| &= \sup_{f\in\mathcal{C}(\Box)} \frac{d(0,\mathcal{T}^{\Delta}_{m,n}(f))}{\|f\|_\infty} \\
&= \sup_{f\in\mathcal{C}(\Box)} \inf_{f^{\alpha}_{\triangle,B_{m,n}}\in\mathcal{T}^{\Delta}_{m,n}(f)} \frac{\|f^{\alpha}_{\triangle,B_{m,n}}\|}{\|f\|} \\
&\leq \sup_{f\in\mathcal{C}(\Box)} \Big(1+\frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\|Id-B_{m,n}\|\Big) \\
&\leq \sup_{f\in\mathcal{C}(\Box)} \Big(1+\frac{q}{1-q}\|Id-B_{m,n}\|\Big) \\
&=1+\frac{q}{1-q}\|Id-B_{m,n}\|,
\end{aligned}$$

completing the proof. □

Theorem 4.2.4. *For a fixed admissible scale vector* α *and* $m,n\in\mathbb{N}$, *the multi-valued mapping* $\mathcal{V}^{\alpha}_{m,n}:\mathcal{C}(\Box)\rightrightarrows\mathcal{C}(\Box)$ *defined by*

$$\mathcal{V}^{\alpha}_{m,n}(f)=\{f^{\alpha}_{\triangle,B_{m,n}}:\textit{all possible net }\triangle\}$$

is a process and lower semicontinuous.

Proof. Let $f\in\mathcal{C}(\Box)$ and $\lambda>0$. Then we have

$$\begin{aligned}
\lambda\mathcal{V}^{\alpha}_{m,n}(f) =&\lambda\{f^{\alpha}_{\triangle,B_{m,n}}:\text{all possible net }\triangle\}\\
=&\{\lambda f^{\alpha}_{\triangle,B_{m,n}}:\text{all possible net }\triangle\}\\
=&\{(\lambda f)^{\alpha}_{\triangle,B_{m,n}}:\text{all possible net }\triangle\}\\
=&\mathcal{V}^{\alpha}_{m,n}(\lambda f).
\end{aligned}$$

The third equality follows from the fact that the fractal operator $\mathcal{F}^{\alpha}_{m,n}$ is a linear operator. Moreover, using linearity of the fractal operator, we have $f^{\alpha}_{\triangle,B_{m,n}}=0$, whenever $f=0$. That is, $0\in\mathcal{V}^{\alpha}_{m,n}(0)$. Therefore, $\mathcal{V}^{\alpha}_{m,n}$ is a process. Now for the lower semicontinuity of $\mathcal{V}^{\alpha}_{m,n}$, let $f \in \mathcal{C}(\square)$, $f^{\alpha}_{\triangle,B_{m,n}} \in \mathcal{V}^{\alpha}_{m,n}(f)$ and a sequence $\{f_k\}$ converges to f in $\mathcal{C}(\square)$. Since the fractal operator is continuous, we have $\big(f_k\big)^{\alpha}_{\triangle,B_{m,n}} \to f^{\alpha}_{\triangle,B_{m,n}}$. By definition of $\mathcal{V}^{\alpha}_{m,n}$, $(f_k)^{\alpha}_{\triangle,B_{m,n}} \in \mathcal{V}^{\alpha}_{m,n}(f_k)$. Hence, the lower semicontinuity of $\mathcal{V}^{\alpha}_{m,n}$ follows. □

Theorem 4.2.5. *The multi-valued function* $\Phi : [\dim(X), \dim(X) + \dim(Y)] \to \mathcal{C}(X,Y)$ *defined by*

$$\Phi(\beta) := \{f \in \mathcal{C}(X,Y) : \dim(Gr(f)) = \beta\}$$

is lower semicontinuous.

Proof. Let U be an open set of $\mathcal{C}(X,Y)$. In the light of Theorem 4.1.1, that is, $\Phi(\alpha) = \mathcal{S}_\alpha$ is a dense subset of $\mathcal{C}(X,Y)$, we obtain

$$\mathcal{S}(\alpha) \cap U \neq \emptyset, \ \forall \ \alpha \in [\dim(X), \dim(X) + \dim(Y)].$$

Now by the definition of lower semicontinuity, the result follows. □

Remark 4.2.3. *Note that the multi-valued mapping* Φ *is not closed. To show this, let* $f \in \mathcal{C}(X,Y)$ *with* $\dim(Gr(f)) > \dim(X)$. *Consider a sequence of Lipschitz functions* $\{f_k\}$ *converging to* f *uniformly. It is obvious that* $\dim(Gr(f_k)) = \dim(X)$. *Now we have* $\big(\dim(X), f_k\big) \to \big(\dim(X), f\big)$ *as* $n \to \infty$. *Using* $\big(\dim(X), f_k\big) \in Gr(\Phi)$ *and* $\big(\dim(X), f_k\big) \to \big(\dim(X), f\big)$ *with* $\dim(Gr(f)) > \dim(X)$, *we get the result.*

4.3 Conclusions

In this chapter, we studied dimension preserving approximation for bivariate functions. Furthermore, we discussed multi-valued operators associated with bivariate α-fractal functions.

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